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ON SOME GENERALIZATIONS OF THE MALMQUIST THEOREM

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To the memory of Johannes Malmquist

1. Introduction.

We shall be concerned with the existence of non-rational meromorphic solutions of certain first order non-linear differential equations (abbreviated DE) and the limitations which the existence of such solutions impose on the form of the equation. The prototype for this range of ideas is a theorem proved by Johannes Malmquist [12] in 1913:

If $R(z,w)$ is a rational function of z and of w and if the DE

$$w' = R(z,w) \tag{1.1}$$

has a meromorhic transcendental solution then (1.1) is a rational Riccati equation, i.e.

$$w'(z) = A_0(z)+A_1(z)w(z)+A_2(z)[w(z)]^2 \tag{1.2}$$

where the A's are rational functions of z.

This theorem has inspired a large literature. An alternate proof and an extension to more general equations were given by Kôsaku Yosida in 1933 [22] using the Nevanlinna value distribution theory. A systematic examination of the implications of this theory for linear and non-linear DE's was undertaken by Hans Wittich [14–18] in the 1950's. In particular, for equation (1.1) he found that Malmquist's assumptions implies that the solution is of finite Nevanlinna order and the proximity functions

$$m(r,\infty;w) = O[\log r], \quad m(r,\infty;w') = O[\log r]\,. \tag{1.3}$$

He based the polar statistics on the notion of polar neighborhood, a concept which goes back to Pierre Boutroux [4, pp. 47–54]. These concepts were made more precise by the present author [7 and Chapters 4, 11 and 12 of 8] who showed that the polar statistics could be based on an elementary geometric problem: if overlapping is not al-

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lowed, what is the closest packing of small circular disks in a large disk or on its rim?

Very far-reaching generalizations of the Malmquist–Yosida theorems were given by A. A. Gol'dberg [6] in 1956. See further Sections 4 and 7 below.

Generalizations with a different mode of approach appeared in the early 1970's. Ilpo Laine [9–11] and Chung-Chun Yang [19–21] studied non-linear DE's where the coefficients are meromorphic functions of lower order than the considered solution so that for each coefficient $C_j(z)$ in the numerator or denominator of $R(z,w)$

$$T[r; C_j(z)] = o\{T[r; w(z)]\} . \tag{1.4}$$

This leads to generalizations of the Malmquist–Yosida theorem and also of Yosida's extension to DE's

$$(w')^n = R(z,w) \tag{1.5}$$

with the reduced equation (below called a hyper-Riccati DE)

$$(w')^n = \sum_{j=0}^{N} A_j(z)w^j, \quad n+1 \leqq N \leqq 2n . \tag{1.6}$$

The poles of such meromorphic solutions are (if located at non-singular points of the DE) all of the same order α which is a divisor of n and $N=(1+1/\alpha)n$.

2. Remarks and preliminary observations.

These interesting generalizations raise a number of questions some of which will be answered in the present paper. It is assumed that the DE, (1.1) or (1.5), has a solution meromorphic in the finite plane. This is reasonable if the coefficients are entire functions, but if some are transcendental meromorphic, then the DE has infinitely many fixed singular points which cluster at infinity (poles of the C_j's or zeros of the coefficients of the highest powers of w in the numerator and the denominator of $R(z,w)$.) The special Lamé equation

$$w' = w^2-[m(m+1)\wp(z)+B] \tag{2.1}$$

shows that such a profusion of singularities is still compatible with the existence of meromorphic solutions. It should be noted that condition (1.4) is *not* satisfied by this equation. The trivial example

$$w' = \tfrac{1}{2}\pi \sec^2(\tfrac{1}{2}\pi z)[1+w^2] \quad \text{with } w(z)=\tan[\tan(\tfrac{1}{2}\pi z)] \tag{2.2}$$

shows what may happen with meromorphic coefficients. Here the fixed singularities are double poles at the odd integers and are clusterpoints of the poles of the solution.

The author likes to know to what extent the methods used by Wittich and himself can cope with the more general cases studied by Gol'dberg, Laine and Yang. Having once emphasized the importance of the finiteness of the order of the solutions [7], he now wants to show the unimportance of finiteness under suitable restrictions on the coefficients. Actually they may be so general that only locally meromorphic solutions can exist.

For $n=1$ the equation is

$$w' = \frac{P(z,w)}{Q(z,w)} \tag{2.3}$$

with

$$P(z,w) = \sum_{j=0}^{p} P_j(z)w^j, \quad Q(z,w) = \sum_{k=0}^{q} Q_k(z)w^k . \tag{2.4}$$

Throughout the study of this case the coefficients will be holomorphic in the open unit disk. Further restrictions will be imposed later.

We note first that if (2.3) has a solution with a pole at a point $z=z_0$, not a fixed singularity of the equation, then the pole is necessarily simple and $p=q+2$. For suppose that

$$w(z) = a(z-z_0)^{-\alpha}[1+o(1)] \tag{2.5}$$

then

$$w'(z) = -\alpha a(z-z_0)^{-\alpha-1}[1+o(1)] , \tag{2.6}$$

$$R[z,w(z)] = a^{p-q}\frac{P_p(z_0)}{Q_q(z_0)}(z-z_0)^{-(p-q)\alpha}[1+o(1)] . \tag{2.7}$$

Since z_0 is not a fixed singularity, neither $P_p(z_0)$ nor $Q_q(z_0)$ can be zero or infinity. Equating (2.6) and (2.7) we get

$$(p-q-1)\alpha = 1 \tag{2.8}$$

and since α is a positive integer we must have

$$\alpha = 1, \quad p = q+2 \tag{2.9}$$

as asserted.

We can also compute a, the residue of the pole, but we shall find it advantageous to impose the conditions

$$P_p(z) \equiv 1, \quad q_q(z) \equiv 1 . \tag{2.10}$$

The immediate effect is to make all residues (at points which are not fixed singularities of the DE) equal and equal to -1. Other and more important consequences will appear later.

At this stage we disregard the case of meromorphic coefficients, for which see Section 6, and restrict ourselves to the following two cases.

I. All coefficients are entire functions of z and at least one of the coefficients $P_q(z)$ and $P_{q+1}(z)$ is transcendental.

II. All coefficients are holomorphic in the open unit disk and at least one of the coefficients $P_q(z)$ and $P_{q+1}(z)$ has a singularity on the unit circle such that

$$(1-r)M(r;\, C_j) \tag{2.11}$$

has a positive lower bound for $0<r<1$.

Here $M(r;\, C_j)$ is the maximum modulus of $C_j(z)$.

Admissible solutions of (2.3) are in case I meromorphic transcendental functions in the finite plane with poles clustering at infinity while in case II $w(z)$ shall be meromorphic in the open unit disk with infinitely many poles which cluster at the fixed singularities on the unit circle.

We shall use a majorant method. Let

$$C_j(z) = \textstyle\sum_{n=0}^{\infty} c_{jn} z^n \tag{2.12}$$

and set

$$C_n = \max_j |c_{jn}|, \quad C(z) = \textstyle\sum_{n=0}^{\infty} C_n z^n . \tag{2.14}$$

For the discussion of the polar neighborhoods we need a restriction on $C(r)$, namely if

$$Q(r) = \frac{C[r+1/6C(r)]}{C(r)} \quad \text{then} \quad \sup Q(r) \leqq B < \infty . \tag{2.15}$$

In case I Borel's Lemma shows that $Q(r)<2$ outside a set of finite measure. In case II (2.15) requires that the lower bound in (2.11) exceeds $\frac{1}{6}$ in order that $Q(r)$ be definable. If $C(r)$ satisfies these various conditions it is rated as an acceptable majorant of the coefficients.

3. The pseudo-Riccati equation.

Consider equation (2.3) and divide the numerator by the denominator to obtain the pseudo-Riccati equation

$$w' = w^2 + A_1(z)w + A_0(z) + P_1(z,w)/\,Q(z,w) \; . \tag{3.1}$$

That the coefficient of w^2 is identically 1 follows from (2.10) and is the main reason for this assumption. The degree of $P_1(z,w)$ as a polynomial in w is at most $q-1$. Since $p=q+2$

$$A_1 = P_{q+1} - Q_{q-1}, \quad A_0 = P_q - Q_{q-1} - (P_{q+1} - Q_{q-1})Q_{q-1} , \tag{3.2}$$

$$(3.3) \qquad P_1(z,w) = \sum_{j=1}^{k} [P_{q-j} - Q_{q-j-2} - (P_{q+1} - Q_{q-1})Q_{q-j-1}]w^{q-j} .$$

Following the prescripts of Wittich we divide (3.1) by w^2 and aim to get an estimate of the form

$$(3.4) \qquad w'/w^2 = 1 + h(z) \quad \text{with } |h(z)| < \tfrac{1}{2}$$

in some neighborhood of the poles. It is supposed that $C(r)$ is an acceptable majorant for all the coefficients. Let B stand for the supremum of $Q(r)$ in (2.15). Then for $C(r) > 1$

$$(3.5) \qquad |A_0(z)| < 4[C(r)]^2, \quad |A_1(z)| < 2C(r), \quad z = re^{i\theta} ,$$

where $0 < r < 1$ in Case II but not in Case I. Now

$$(3.6) \qquad Q(z,w) = z^q[1 + \sum_{j=0}^{q-1} Q_j(z)w^{j-q}]$$

the absolute value of which exceeds

$$(3.7) \qquad |w|^q[1 - C(r)\sum_{n=1}^{\infty} |w|^{-n}] = |w|^q \frac{|w| - 1 - C(r)}{|w| - 1} > \frac{1}{2}\frac{|w|^{q+1}}{|w| - 1}$$

if $|w(z)| > 2C(r) + 2$. On the other hand, by (3.3)

$$(3.8) \qquad |P_1(z,w)| < 4[C(r)]^2 \, |w|^q/(|w| - 1)$$

so that

$$(3.9) \qquad |P_1(z,w)|/|Q(z,w)| < 8[C(r)]^2|w|^{-1}$$

provided $|w(z)| > 2C(r) + 2$.

It follows that

$$h(z) = A_0(z)w^{-2} + A_1(z)w^{-1} + P_1(z,w)/w^2Q(z,w)$$

satisfies

$$(3.10) \qquad |h(z)| < 2C(r)|w|^{-1} + 4[C(r)]^2|w|^{-2}[1 + 2|w|^{-1}] .$$

Consider now the point set

$$(3.11) \qquad S = [z\,;\ |w(z)| > 6C(r)]$$

where z is restricted to the unit disk in Case II but not in Case I. Since by assumption the solution has infinitely many poles in the domain under consideration, S is not void. Now for $z \in S$

$$(3.12) \qquad |h(z)| < \tfrac{13}{27} < \tfrac{1}{2}$$

so that (3.4) holds in S. In general S has infinitely many maximal components. Each component contains a pole of $w(z)$ and if the component is convex, one and only one pole. This follows from (3.4) by integration from z_0 to z_1, two points in the same component of S. The integration gives

(3.13) $$w(z_0)^{-1} - w(z_1)^{-1} = \int_{z_0}^{z_1} [1+h(s)]ds = (z_1 - z_0)g(z_0, z_1)$$

where $\frac{1}{2} < |g(z_0, z_1)| < \frac{3}{2}$. In particular, this shows that no component can contain two poles, one visible from the other, for if z_0 and z_1 were the two poles then the first member of (3.13) would be zero while the third member cannot be zero.

4. Polar neighborhoods and polar statistics.

With Wittich we introduce for each pole $z = z_n$ a corresponding *polar neighborhood*

(4.1) $$U_n = \{z\ ;\ |z - z_n| < [9BC(|z_n|)]^{-1}\}$$

where in Case II z is restricted to the open unit disk and $B = \sup Q(r)$.

It should be shown that $U_n \subset S$. To this end consider the auxiliary neighborhood

(4.2) $$D_n = \{z\ ;\ |z - z_n| < [6C(|z_n|)]^{-1}\}\ .$$

If this set is already in S, we are though since $B > 1$. If D_n is not wholly in S, then we can shrink it to a concentric disk D_{n0} which lies entirely in S and its boundary passes through a point $z = t$ where

(4.3) $$|w(t)| = 6C(|t|)\ .$$

A lower bound for $|t - z_n|$ is obtainable from (3.13) with (4.2). Thus

(4.4) $$|t - z_n| > \tfrac{2}{3}|w(t)|^{-1} = [9C(|t|)]^{-1}$$

where

$$|t| \leqq |z_n| + |t - z_n| < |z_n| + [6C(|z_n|]^{-1}\ ,$$
$$C(|t|) < C\{|z_n| + [6C(|z_n|)]^{-1}\} < BC(|z_n|)\ .$$

This combined with (4.4) shows that $U_n \subset S$ and gives the desired polar neighborhood. Two such neighborhoods cannot overlap for if they did then the corresponding poles would be visible one from the other in S and this would contradict (3.13).

We can now proceed with the polar statistics as in the author's paper [7]. The number of poles of $w(z)$ in the disk $|z| < R$ equals the number of polar neighborhoods with centers in the disk. The radii of the polar disks are a decreasing function of the distance of the disk from the origin. It follows that an upper bound for the number of polar neighborhoods is given by the solution of the closest packing problem: how many disks of radius $[9BC(R)]^{-1}$ can be placed in the disk $|z|<R$ without overlapping? A comparison of the areas of the disks shows that the desired number is at most

$$81[BRC(R)]^2 \tag{4.5}$$

and this bound is quite generous. It shows that the Nevanlinna enumerative function satisfies

$$N(r,\infty; w) \ < \ 41[BrC(r)]^2 \,. \tag{4.6}$$

The estimate of the corresponding proximity function

$$m(r,\infty; w) \ = \ (2\pi)^{-1} \textstyle\int_0^{2\pi} \log^+|w(re^{i\theta})|\,d\theta \tag{4.7}$$

also leads to a packing problem. For a given r, the more polar neighborhoods are intersected by $|z|=r$, the larger is the integral. We get a generous upper bound by assuming that the circle $|z|=r$ is packed with polar neighborhoods with centers on the circle. The number of such disks is at most $9\pi BrC(r)$. If there is a pole at $z=z_n=re^{i\theta_n}$, then on the arc Γ_n of the circle $|z|=r$ which belongs to U_n we have by (3.13)

$$\log^+|w(re^{i\theta})| \ < \ \log^+(\pi/r)+\log^+|\theta-\theta_n|^{-1} \tag{4.8}$$

where θ goes from $\theta_n-\eta$ to $\theta_n+\eta$ and $\eta=[9BrC(r)]^{-1}$. It follows that Γ_n contributes

$$\frac{1}{\pi}\int_0^{\eta} \log\frac{1}{u}\,du+O(\eta) \ = \ \frac{\eta}{\pi}\log\frac{e}{\eta}+O(\eta)$$

to the proximily function. Since there are at most $9\pi BrC(r)$ such contributions it is seen that

$$m(r,\infty; w) \ < \ \log C(r)+O[\log r]\,. \tag{4.9}$$

A slightly smaller estimate is obtainable if the circle $|z|=r$ manages to avoid all polar neighborhoods.

Combining we get

$$T(r; w) \ < \ K[rC(r)]^2 \tag{4.10}$$

for a suitable constant K. This estimate, poor as it may be, leads to extensions of the Malmquist theorem.

Let us first note that the estimate agrees with a special case of Gol'dberg's Theorem 2 which deals with the DE

$$(w')^m+P_1(z,w)(w')^{m-1}+\ldots+P_m(z,w) \ = \ 0 \tag{4.11}$$

where the P's are polynomials in w with coefficients which are entire functions of z of finite order ϱ. Then for every single-valued solution $w(z)$ in $|z| < \infty$ we have

$$T(r; w) = O[\exp(r^{\varrho+\varepsilon})]$$

where $\varepsilon > 0$ is arbitrarily small. Here take $m = 1$ and consider case I with $C(z)$ an entire function of order ϱ. Then $C(r) < \exp(r^{\varrho+\varepsilon})$ if ϱ is finite and we have also

$$[rC(r)]^2 < \exp(r^{\varrho+\delta})$$

in agreement with Gol'dberg's result. Actually it is not necessary to assume ϱ to be finite: (4.10) is still valid.

The Riccati equation

$$w' = [F'(z)]^2 + \frac{F''(z)}{F'(z)} w + w^2 \tag{4.12}$$

has among its solutions $w(z) = F'(z) \tan[F(z)]$. Let $F'(z)$ be holomorphic in the unit disk, real positive together with all its derivatives on $(0, 1)$. We have then $C(r) = [F'(r)]^2$. Formula (4.10) gives

$$T(r; w) = O\{r^2[F'(r)]^2\} \tag{4.13}$$

which is normally vastly exaggerated though it is not too bad in case I.

5. Two Laine-Malmquist-Wittich-Yang-Yosida type theorems.

To complete the discussion of the pseudo-Riccati equation we use the methods of Wittich. Set

$$P_1(z, w)/Q(z, w) = F[z, w(z)] = F(z) \tag{5.1}$$

so that

$$F(z) = w'(z) - [w(z)]^2 - A_1(z)w(z) - A_0(z) . \tag{5.2}$$

We seek conditions which will ensure that $F(z)$ does not depend on w. Every pole of $w(z)$ is a zero of $F(z)$ since the degree of $P_1(z, w)$ as a polynomial in w is lower than that of $Q(z, w)$. It follows that $F(z)$ is holomorphic in the unit disk and in case I $F(z)$ is an entire function of z. By the classical relations of Nevanlinna for proximity functions, omitting the reference to infinity, we have

$$m(r; F) \leqq m(r; w') + 3m(r; w) + m(r; A_0) + m(r; A_1) . \tag{5.3}$$

Here

$$m(r; w') \leqq m(r; w) + m(r; w'/w) + \log 2$$

where the second term on the right is dominated by an expression

$$(5.4) \qquad O[\log r]+O[\log T(r;w)] \quad \text{in case I},$$

$$(5.5) \qquad O(\log(1-r)^{-1})+O[\log T(r;w)] \quad \text{in case II}.$$

These estimates hold for r outside of exceptional sets which in case I are of finite linear measure while in case II the total variation of $(1-r)^{-1}$ is finite. Since

$$|A_0(z)| \leqq 4[C(r)]^2, \quad |A_1(z)| \leqq 2C(r)$$

we have

$$(5.6) \qquad m(r,\infty;A_0) < 2\log C(r)+O(1),$$

$$(5.7) \qquad m(r,\infty;A_1) < \log C(r)+O(1).$$

Combining these estimates one obtains

$$(5.8) \qquad m(r,\infty;F) < 7\log C(r)+S(r)$$

where for r outside of the, possibly void, exceptional sets, $S(r)$ is dominated by (5.4) in case I and by (5.5) in case II. Since $N(r,\infty;F)=0$

$$(5.9) \qquad T(r;F) < 7\log C(r)+S(r).$$

By R. Nevanlinna's First Fundamental Theorem

$$T(r;1/F) = T(r;F)+O(1)$$

so that

$$(5.10) \qquad N(r,\infty,1/F) < 7\log C(r)+S(r)$$

for a sequence of r-values which tend to ∞ in case I and to $+1$ in case II.

But if $F(z,w)$ actually depends upon w then every pole of $w(z)$ is also a pole of $1/F$ so that we must have

$$(5.11) \qquad N(r,\infty;w) < 7\log C(r)+S(r).$$

Since we have $m(r,\infty;w)<\log C(r)+O(\log r)$ we get

$$(5.12) \qquad T(r;w) < 8\log C(r)+S(r).$$

But now in the expression for $S(r)$ we have $T(r;w)=O[\log C(r)]$ so that (5.11) may be sharpened to

$$(5.13) \qquad N(r,\infty;w) < [7+o(1)]\log C(r).$$

In the case considered by Malmquist and by Wittich $C(r)$ would be a polynomial in r and $w(z)$ would have to be a rational function against the hypothesis. This type of contradiction does not work in the cases here considered; it is not enough to assume infinitely many poles, some additional information about their frequency is called for if this method is to permit the conclusion that the equation is a Riccati DE. The obvious assumption is

$$N(r,\infty;w) > 7\log C(r) \tag{5.14}$$

for large values of r in case I and for r close to $+1$ in case II. This leads to the following two theorems of the L-M-W-Ya-Yo type.

THEOREM 1. *In equation* (2.3) *let* $p=q+2$, *let* (2.10) *hold and let the coefficients be entire functions of* z *such that at least one of* $P_q(z)$ *and* $P_{q+1}(z)$ *is transcendental. Let* $C(r)$ *be an acceptable majorant of the coefficients satisfying* (2.14) *and* (2.15). *If the equation has a solution which is meromorphic in the finite plane with infinitely many poles and if the enumerative function* $N(r,\infty;w)$ *satisfies* (5.14), *then the equation is a Riccati* DE.

THEOREM 2. *Suppose instead that the coefficients are holomorphic in the open unit disk and at least one of the functions* $P_q(z)$ *and* $P_{q+1}(z)$ *has a singularity on the unit circle such that* (2.11) *holds. Let the majorant* $C(r)$ *satisfy* (2.14) *and* (2.15) *for* $0<r<1$ *and let the infimum in* (2.11) *exceed* $\frac{1}{8}$. *If the equation has a solution which is meromorphic in the open unit disk with infinitely many poles and an enumerative function* $N(r,\infty;w)$ *satisfying* (5.14) *then the equation is a Riccati* DE.

Condition (5.14) is merely sufficient for the result. The DE (4.12) is a Riccati equation. If here we take $F(z)=\tan(\frac{1}{2}\pi z)$ and the solution

$$w(z) = \tfrac{1}{2}\pi\sec^2(\tfrac{1}{2}\pi z)\tan[\tan(\tfrac{1}{2}\pi z)]$$

we have $C(r)=\frac{1}{4}\pi^2\sec^4(\frac{1}{2}\pi r)$ and $N(r,\infty;w)\sim\pi^{-2}\log C(r)$ for $0<r<1$. Obviously $\pi^{-2}<7$. It would seem possible that $N(r,\infty;w)=o[\log C(r)]$ can hold for a Riccati equation.

There are Riccati equations which do not satisfy the Laine–Yang condition (1.4). The equation

$$w' = w^2+A(z)w+Q'(z)-[Q(z)]^2-A(z)Q(z) \tag{5.15}$$

is a case in point. If $A(z)$ and $Q(z)$ are single-valued analytic functions with a common domain of existence, then $w=Q(z)$ is a solution and (1.4) cannot hold.

6. Meromorphic coefficients.

The majorant method used above has to be modified if we wish to cope with coefficients and solutions which are transcendental meromorphic functions in the finite plane. The poles of the solution are now of two different types, *regular* and *singular*, according as the pole occurs at a regular or a singular point of the equation. All the poles of the solution may be singular. An example is furnished by equation (5.15) where we specify $Q(z)$ to be a meromorphic function with infinitely many poles clustering at infinity. Here every pole of $Q(z)$ is a fixed singular point of the equation and the solution $w(z)$ has only singular poles.

We exclude such cases from further consideration and suppose that the solution $w(z)$ has infinitely many regular poles. Then the discussion in Section 2 applies and we have (2.9), i.e. these poles are simple and $p=q+2$. To simplify matters we also assume (2.10). Then the corresponding pseudo-Riccati equation is

$$w' = w^2+A_1(z)w+A_0(z)+P_1(z,w)/Q(z,w) \tag{6.1}$$

where the coefficients are given by (3.2) and (3.3).

To each coefficient $C_j(z)$ corresponds a Nevanlinna characteristic $T(r;C_j)$. Suppose that $D(r)$ is a positive, increasing, logarithmically convex function and that

$$T(r;C_j) < D(r) \quad \forall j\,. \tag{6.2}$$

We set

$$S = \{z\,;\ |w(z)|>D(|z|)\} \tag{6.3}$$

and assume that the sets

$$U_n = \{z\,;\ |z-z_n|<[D(|z_n|)]^{-1}\} \tag{6.4}$$

are non-overlapping polar neighborhoods for the poles z_n.

As in Section 4 we see that

$$N(r,\infty;w) < \tfrac{1}{2}[rD(r)]^2\,. \tag{6.5}$$

The proximity function $m(r,\infty;w)$ presents more of a problem if the solution has singular poles. To handle this case and the discussion of the function $z\mapsto F[z,w(z)]=F(z)$ of (5.1) we find it desirable to introduce some restrictive hypotheses:

H_1. No function $Q_k(z)$ has a distant pole.

H_2. The functions $P_j(z)$ may have distant poles at most of the second order and the numerical coefficients of the corresponding principal parts are uniformly bounded.

H_3. At least one of the functions $P_q(z)$ and $P_{q+1}(z)$ has infinitely many poles which are simple in the case of P_{q+1}, at most double for P_q.

These conditions are inspired by the requirement that they be satisfied by the Riccati equation

$$w' = w^2 + A_1(z)w + A_0(z) \tag{6.6}$$

in the case that it admits a transcendental meromorphic solution. For the solutions of (6.6) are of the form

$$w(z) = -v'(z)/v(z) \tag{6.7}$$

where $v(z)$ is a solution of the linear second order DE

$$v'' - A_1(z)v' + A_0(z)v = 0 . \tag{6.8}$$

If the logarithmic derivative of $v(z)$ is to be a meromorphic function equation (6.8) can only have regular singular points. By formulas (3.2) this requirement leads to H_3.

Since all the Q_k with $k<q$ are expected to be zero H_1 is a natural simplification. H_2 hails from similar considerations as H_3 and H_1.

Now a pole of one of the P_j's is not necessarily a pole of $w(z)$ but it is a possibility which must be taken into account in the discussion.

After these remarks let us return to the proximity function. To fix the idea suppose that the circle $|z|=r$ is packed with polar disks corresponding to singular poles of order 1 (no pole can have a higher order). Suppose that for z on the arc Γ_n of the circle $|z|=r$ in U_n we have

$$|z-z_n|\,|w(z)| \leqq M \tag{6.9}$$

for a fixed constant M, the same for all poles. Then on Γ_n

$$\log^+|w(re^{i\theta})| < \log^+(Mr/2\pi) + \log^+|\theta-\theta_n|^{-1} \tag{6.10}$$

so that this arc contributes at most

$$\frac{\eta}{\pi}\log\frac{e}{\eta} + O(\eta), \quad \eta = [rD(r)]^{-1} .$$

Since the number of contributing arcs is at most $\pi rD(r)$ we get

$$m(r,\infty; w) < \log D(r) + O[\log r] . \tag{6.11}$$

It follows that

$$T(r; w) < K[rD(r)]^2 \tag{6.12}$$

for a suitable constant K.

We have now to discuss

$$F(z) = F[z, w(z)] = w'(z) - [w(z)]^2 - A_1(z)W(z) - A_0(z) . \tag{6.13}$$

This function is holomorphic save for poles. Now the regular poles of w

make $F(z)=0$ and the same is true for singular poles. We do not have to pay any attention to poles of a $P_j(z)$ with $j<q$ which are not poles of w, P_q or P_{q+1}, they have to be regular points of F. On the other hand we must pay attention to those poles of P_q and P_{q+1} which are not singular poles of $w(z)$. It follows that

$$(6.14) \qquad N(r,\infty;F) < N(r,\infty;P_q)+N(r,\infty;P_{q+1}) < 2D(r).$$

Further we have

$$m(r,\infty;F) < 4m(r,\infty;w)+S(r)+m(r;\infty,A_0)+m(r,\infty;A_1)$$
$$< 7\log D(r)+S(r) < 9\log D(r)+O[\log r]$$

so that

$$(6.15) \qquad m(r,\infty;w) < 9\log D(r)+O[\log r]$$

outside an exceptional set of finite linear measure. Thus

$$T(r;w) < [2+o(1)]D(r).$$

The First Fundamental Theorem of R. Nevanlinna gives

$$(6.16) \qquad T(r;1/F) < [2+o(1)]D(r), N(r,\infty;1/F) < [2+o(1)]D(r).$$

But if F actually depends upon w, every pole of w is pole of F so that

$$N(r,\infty;w) \leqq N(r,\infty;1/F) < [2+o(1)]D(r).$$

If this relation is false, that is if

$$(6.17) \qquad N(r,\infty;w) > (2+\varepsilon)D(r)$$

for large values of r, then the DE is a Riccati equation. This gives

THEOREM 3. *In the equation* (2.3) *let* $p=q+2$ *and assume* (2.10). *Let the coefficients be meromorphic functions satisfying* H_1, H_2 *and* H_3. *Let* $w(z)$ *be a transcendental meromorphic solution of* (2.3) *with infinitely many poles* $\{z_n\}$. *Let* $r \mapsto D(r)$ *satisfy*

(1) $D(r)$ *is positive, increasing and logarithmically convex.*
(2) $T[r,C_j(z)]<D(r)$ *for all* j.
(3) *The sets* (6.4) *are non-overlapping polar neighborhoods.*

Then if (6.17) *holds* (2.3) *is a Riccati equation.*

The constant "2" in (6.17) may not be the best possible but at least there exist equations for which we have equality. The Lamé–Riccati equation (2.1) with $m=1$, $B=2$ has the solution

$$w(z) = -\tfrac{1}{2}\frac{\wp'(z)}{\wp(z)-e_1}. \tag{6.18}$$

Here the poles are given by the two nets $2j\omega_1+2k\omega_3$ and $(2j+1)\omega_1+2k\omega_3$. Here the first set is the singular poles, the latter the regular ones and for $D(r)$ we can take Ar^2 where A exceeds π divided by the area of the period parallelogram of $\wp(z)$. We have $N(r,\infty;w) \sim 2D(r)$.

7. The case $n > 1$. Preliminaries.

Take equation (1.5), that is,

$$(w')^n = P(z,w)/Q(z,w) \tag{7.1}$$

where $n>1$ and P and Q are given by (2.4). Here the substitution (2.5) with $z=z_0$ non-singular gives

$$(p-q-n)\alpha = n\,. \tag{7.2}$$

Here again α is a positive integer, obviously a divisor of n. If $n=k\alpha$ we see that

$$p = q+n+k, \quad 1 \leqq k \leqq n, \quad n+k = N\,. \tag{7.3}$$

All regular poles of the equation are of the same order α. We assume (2.10).

The equation (6.1) is now brought to the form

$$\begin{aligned}(w')^n = {} & w^N+A_{N-1}(z)w^{N-1}+A_{N-2}(z)w^{N-2}+\ldots+\\ & +A_0(z)+P_1(z,w)/Q(z,w)\end{aligned} \tag{7.4}$$

where $P_1(z,w)$ is a polynomial in w of degree $<q$. Here

$$\begin{aligned}A_{N-1} &= P_{q+N-1}-Q_{q-1}\,,\\ A_{N-2} &= P_{q+N-2}-Q_{q-2}-(P_{q+N-1}-Q_{q-1})Q_{q-1}\end{aligned} \tag{7.5}$$

and A_{N-j} is a multinomial in the P's and the Q's of total degree j and numerical coefficients ± 1, involving 2^j power products. Similarly for the coefficients of $P_1(z,w)$ which look like A_0.

For the coefficients we consider two cases.

Case IV. All coefficients are entire functions of z and at least one P_j with $j \geqq q$ shall be transcendental.

Case V. All coefficients are holomorphic in the open unit disk and at least one of the P_j's with $j \geqq q$ shall have a singular point on the unit circle such that

$$(7.6) \qquad (1-r)^{1/\alpha}M(r; P_j)$$

has a lower bound >1 on $0<r<1$.

In case IV the solution to be considered shall be a transcendental meromprhic function in the finite plane with infinitely many poles. In case V $w(z)$ shall be meromorphic in the open unit disk with infinitely many poles that cluster at one of the fixed singularities on the unit circle.

As in case I we define a common majorant $C(r)$ of the maximal moduli of the coefficients by formulas (2.12) and (2.13). We replace (2.14) by the condition that if

$$(7.7) \qquad Q(r) = C\{r+[C(r)]^{-\alpha}\}/C(r) \quad \text{then} \quad \sup Q(r) \equiv B < \infty .$$

In case V it is required that the lower bound in (7.6) exceeds 1 in order that $Q(r)$ be definable.

We imitate the procedure for the case $n=1$. Equation (7.4) is divided by w^N. The result is

$$(7.8) \qquad w^{-N}(w')^n = 1+h(z)$$

and we want to determine an open set in the z-plane where $|h(z)|<1$. We have

$$(7.9) \qquad |A_{N-j}(z)| < [2C(r)]^j \quad \text{if } 1<C(r) .$$

Further

$$(7.10) \qquad |P_1(z,w)| < [2C(r)]^N|w|^q/(|w|-1) ,$$

while $|Q(z,w)|$ is still dominated by $\frac{1}{2}|w|^{q+1}/(|w|-1)$ so that

$$(7.11) \qquad |P_1(z,w)/|Q(z,w)| < 2[2C(r)]^N|w|^{-1} .$$

This gives

$$|h(z)| < \textstyle\sum_{j=0}^{N-1} (2C(r)/|w|)^j + 2[2C(r)]^N|w|^{-N-1} .$$

Consider now the set

$$(7.12) \qquad S = \{z ;\ |w(z)| > 4C(|z|)\} .$$

Then for $z \in S$

$$(7.13) \qquad |h(z)| < \textstyle\sum_{j=0}^{N} (\frac{1}{2})^j < 1 .$$

Thus for $z \in S$

$$(7.14) \qquad w^{-N}(w')^n = 1+h(z) \text{ with } |h(z)| < 1 .$$

Here we extract the nth root with $(1)^{1/n}=+1$ and note that $N/n = 1+k/n = 1+1/\alpha = 1+\beta$ so that

$$(7.15) \qquad w^{-1-\beta}w' = 1+h_0(z) \quad \text{with } |h_0(z)|<1/n \leqq \tfrac{1}{2} .$$

For any two points z_1 and z_2 in the same component of S

$$[w(z_1)]^{-\beta}-[w(z_2)]^{-\beta} = \beta\int_{z_1}^{z_2}[1+h_0(s)]ds = (z_2-z_1)g(z_1,z_2) \tag{7.16}$$

where

$$\beta(1-1/n) < |g(z_1,z_2)| < \beta(1+1/n)\,. \tag{7.17}$$

If in particular $z_1=z_0$ is a pole of $w(z)$ and $z_2=z$ lies in the same component of S as z_0, then

$$\beta(1-1/n)|z-z_0| < |w(z)|^{-\beta} < \beta(1+1/n)|z-z_0|\,. \tag{7.18}$$

8. Polar neighborhoods and polar statistics.

We can take

$$U_m = \left\{z\,;\ |z-z_m| < \frac{n\alpha}{B(n+1)}[C(|z_m|)]^{-1}\right\} \tag{8.1}$$

as the specific polar neighborhood of the pole z_m. In this case the auxiliary neighborhood may be

$$D_m = \{z\,;\ |z-z_m|<[C(|z_m|)]^{-\alpha}\}\,. \tag{8.2}$$

If D_m lies entirely in S we are through. If D_m is not entirely in S we shrink D_m to a concentric neighborhood in S the perimeter of which passes through a point $z=t$ where

$$|w(t)| = [C(|t|)]^{\alpha}\,.$$

We have then

$$|t-z_m| > \frac{\alpha n}{n+1}|w(t)|^{-\beta} = \frac{\alpha n}{n+1}[C(|t|)]^{-1} > \frac{\alpha n}{n+1}C\{|z_m|+[C(|z_m|)]^{-\alpha}\}^{-1} > \frac{n\alpha}{(n+1)B}[C(|z_m|)]^{-1} \tag{8.3}$$

and $U_m\subset S$ as desired.

The number of polar neighborhoods in the disk $|z|<r$ is at most $O\{[rC(r)]^2\}$ as in the case $n=1$. Here, however, each polar neighborhood contributes α units to $n(r,\infty;w)$. For the proximity function we also get

$$m(r,\infty;w) < \log C(r)+O[\log r]\,. \tag{8.4}$$

Thus as for $n=1$

$$T(r;w) = O\{[rC(r)]^2\}\,. \tag{8.5}$$

In case IV this is in agreement with the results of Gol'dberg for equation (4.11). Here $m=n$ and the P_j's are identically zero for $1 \leqq j \leqq h-1$. If the degree with respect to w of $P_n(z,w)$ is N, $n<N \leqq 2n$ and if the coefficients of the powers of w are entire functions of finite order ϱ and if $w(z)$ is a meromorphic solution then according to Gol'dberg

$$T(r; w) < \exp(r^{\varrho+\varepsilon}) \quad \text{for any positive } \varepsilon .$$

Here we have

$$C(r) < \exp(r^{\varrho+\varepsilon}) \quad \text{and} \quad [rC(r)]^2 < \exp(r^{\varrho+\delta}) .$$

The discussion of the remainder in (7.4) follows the same pattern and we shall find that

$$N(r,\infty; w) < [n+N(N+1)+o(1)] \log C(r) . \tag{8.6}$$

For we have

$$F[z,w(z)] = (w')^n - \textstyle\sum_{j=0}^{N} A_j(z)w^j, A_N(z) \equiv 1 ,$$

whence

$$m(r,\infty; F) \leqq [n+\tfrac{1}{2}N(N+1)]m(r,\infty; w)+\tfrac{1}{2}N(N+1) \log C(r)+S(r)$$

since by (7.5)

$$m(r,\infty; A_{N-j}) < j \log C(r)+O(1) .$$

Combining this inequality with (8.4) we get the stated inequality for w replaced by F. As in Section 5 we see that

$$N(r,\infty; 1/F) < [n+N(N+1)] \log C(r)+S(r) .$$

If F depends on w, then every pole of w is also a pole of $1/F$ of the same multiplicity and this means that (8.6) holds for w as stated. Hence in order to conclude that (7.1) reduces to polynomial form (becomes a hyper-Riccati equation) it is enough to assume that

$$N(r,\infty; w) > [n+N(N+1)+\varepsilon] \log C(r) . \tag{8.7}$$

This gives two more L-M-W-Ya-Yo type theorems.

THEOREM 4. *With assumptions and notations as above suppose that the coefficients of* (7.1) *are entire functions of z and at least one of the P_j's with $j \geqq q$ is transcendental. Let $C(r)$ be a majorant of the coefficients with properties as stated. Let the equation have a single-valued solution, meromorphic in the finite plane, whose enumerative function $N(r,\infty; w)$ satisfies* (8.7). *Then the equation is a hyper-Riccati* DE

$$(w')^n = w^N+\textstyle\sum_{j=0}^{N-1} A_j(z)w^j . \tag{8.8}$$

THEOREM 5. *The same conclusion holds in case* V *if the equation has a solution meromorphic in the unit disk where it has infinitely many poles and an enumerative function satisfying* (8.7).

A few words should be added about the case of meromorphic coefficients. Here we assume H_1 as is but H_2 and H_3 have to be replaced by

H^*_2. The functions $P_j(z)$ may have distant poles of order at most $n(\alpha+1)$ and the numerical coefficients of the principal parts are uniformly bounded.

H^*_3. At least one of the functions P_j with $j \geqq q$ has infinitely many poles which in the case of P_{q+N-m} are at most of order $m\alpha$.

We assume that $D(r)$ is an increasing logarithmically convex majorant of the characteristics $T(r; C_j)$ and that the sets U_n defined by (6,3) form a set of non-overlapping polar neighborhoods of the poles $\{z_m\}$ of the solution $w(z)$. We have then

THEOREM 6. *If the equation* (7.1) *under the stated conditions has a meromorphic solution such that*

$$N(r, \infty; w) > (N+\varepsilon)D(r) \, , \tag{8.9}$$

then the equation is hyper-Riccati.

9. Concluding remarks.

If the condition (2.10) is not assumed, the pseudo-Riccati equation takes the form

$$w' = A_2(z)w^2 + A_1(z)w + A_0(z) + F[z, w] \, , \tag{9.1}$$

This may be reduced to the normal form

$$v' = v^2 + B(z) + F^*[z, v] \tag{9.2}$$

by the transformation

$$w = f(z)v + g(z), \quad f = \frac{1}{A_2}, \quad g = -\tfrac{1}{2}\frac{A_1}{A_2} - \tfrac{1}{2}\frac{A_2'}{A_2^2}. \tag{9.3}$$

The expression for B is quite complicated and is omitted. It involves A_0, A_1, A_1', A_2, A_2' and A_2'' and if the A's are meromorphic so is B. The zeros of $A_2(z)$ are fixed singularities of (9.2). This may serve as a justification for the simplifying assumption (2.10).

Similar transformations may be used to reduce the pseudo-hyper-Riccati equation to the form used above. In this case an nth root of

$A_N(z)$ is introduced so the new coefficients are normally not single-valued unless $A_N(z)$ is the nth power of a single-valued function. The zeros of $A_N(z)$ are fixed singularities of the equation.

The following simple example is instructive. Take

$$(9.4) \qquad (w')^2 = z^m C_m (w^3 - g_2 w - g_3)$$

where m is an integer and C_m is to be chosen so that a solution is

$$(9.5) \qquad w = \wp\left\{\frac{2z^{\frac{1}{2}m+1}}{m+2} + K \,;\; g_2, g_3\right\}, \quad m \neq -2\,,$$

$$(9.6) \qquad w = \wp\{\log z + K \,;\; g_2, g_3\}, \quad m = -2\,,$$

where K is an arbitrary constant. If m is zero or a positive even integer the solution is a meromorphic function in the finite plane and $z=0$ is not singular. It is a singular point of all non-constant solutions, if m is an odd integer $m > -1$, in which case the origin is an algebraic branch point. The singularity is severe if m is a negative integer < -1 for then the origin is a cluster point of poles and if $m = -2$ or is odd the origin is also a branch point. Note, however, that if $m = -2$, $K=0$ and $2\pi i$ is a period of $\wp(u \mid 2\omega_1, 2\omega_3)$ then the solution is meromorphic in the plane punctured at $z=0$ and $z=\infty$. In a forthcoming paper S. Bank and R. P. Kaufman have proved a result which implies that

$$\wp\{\log[z + (z^2+1)^{\frac{1}{2}}] \mid 1, 2\pi i\}$$

is meromorphic in the finite plane.

Thus the presence of a multiplier $A_N(z) \neq 1$ may introduce fixed singularities and may prevent the equation from having solutions meromorphic in the finite plane or in the unit disk. This is why condition (2.10) was imposed.

The methods used in this paper are tailored to the needs of first order equations. But they may also be used, for instance, to estimate the order of the meromorphic functions which are solutions of the Painlevé second order equations. Take the first Painlevé equation in the normal form of Boutroux [5]

$$(9.7) \qquad w'' = 6w^2 - 6z\,.$$

It is easy to show that the movable singularities are double poles with the expansion

$$(9.8) \qquad w(z) = (z-z_0)^{-2} + O[(z-z_0)^2]\,.$$

Painlevé showed at the beginning of the century that the solutions are meromorphic functions in the finite plane and Boutroux showed

(1913–14) that the solutions are asymptotic to elliptic functions $\wp(Z-Z_0;\,12,g_3)$ with $Z=\frac{4}{5}z^{\frac{5}{4}}$ with a multiplier $z^{-\frac{1}{2}}$. Boutroux's results imply that the distance between nearby poles is $O(|z|^{-\frac{1}{4}})$. From the last result one may show that the sets

$$U_n = \{z\,;\ |z-z_n| < \tfrac{1}{4}|z_n|^{-\frac{1}{4}}\} \tag{9.9}$$

form non-overlapping neighborhoods of the poles. The polar statistics now gives

$$N(r,\infty;w) < \tfrac{16}{3}r^3, \qquad m(r,w;w) < \log r, \qquad T(r;w) < 6r^3\,. \tag{9.10}$$

This estimate is too high; the results of Boutroux show that the true order is $\frac{5}{2}$.

We have better luck with the second Painlevé equation

$$w'' = 2w^3-2zw+a\,. \tag{9.11}$$

Here we can use the same polar neighborhoods and hence find that $T(r;w)<6r^3$ and here 3 is the correct order.

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