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ON THE SPECTRUM OF A ONE-PARAMETER STRONGLY CONTINUOUS REPRESENTATION

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Let α_t be a bounded strongly continuous one-parameter group on a Banach space X, with infinitesimal generator iZ. If $\mathrm{sp}\,\alpha$ denotes the spectrum introduced by Arveson [1] of the representation α, we show that $\mathrm{sp}\,\alpha = \sigma Z$. This generalises Olesen's result [3, 2.5] for norm continuous one-parameter groups of isometries.

We recall [1] that if $f \in L^1(\mathsf{R})$,

$$Ax = \int \alpha_t(x) f(t)\, dt \qquad x \in X ,$$

defines a bounded linear operator A on X. Thus we can lift α to a representation π_α of $L^1(\mathsf{R})$. The spectrum of α, written $\mathrm{sp}\,\alpha$, is defined as the hull of the ideal kernel (π_α).

The following proposition and the usual characterisation of $\mathrm{sp}\,\alpha$ (e.g. [4, 2.4.1]) prove our claim that $\mathrm{sp}\,\alpha = \sigma Z$. If I is a subset of R, we will denote by L_I^1, the set of f in $L^1(\mathsf{R})$ such that $\hat{f}$ has support in I.

Proposition. *Let I be a compact interval in* R. *Then*

i) $I \subseteq \varrho(Z)$ *implies that* $\pi_\alpha(f) = 0$ *for all* $f \in L_I^1$.
ii) $\pi_\alpha(f) = 0$ *for all* $f \in L_I^1$ *implies that* $I^0 \subseteq \varrho(Z)$.

Proof. The proof is inspired by that of [2, Lemma 5.5].

i) Take $f \in L_I^1$. Then

$$f(t) = \int_{-\infty}^{+\infty} e^{-ita} \hat{f}(a)\, da \qquad \text{a.e. in } \mathsf{R} .$$

For $\delta > 0$, define

$$f_\delta^+(t) = \int_{-\infty}^{+\infty} \hat{f}(a) \exp(-iat - \delta t)\, da \qquad \text{for } t > 0$$
$$f_\delta^-(t) = \int_{-\infty}^{+\infty} \hat{f}(a) \exp(-iat + \delta t)\, da \qquad \text{for } t < 0 .$$

Then

$$f_\delta^+(t) - f(t) = f(t)(e^{-\delta t} - 1) \qquad \text{for } t > 0 ,$$

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and

$$f_\delta^-(t)-f(t) = f(t)(e^{+\delta t}-1) \quad \text{for } t<0 .$$

Hence by dominated convergence, there exists $\delta_0 > 0$ such that

$$\int_0^\infty |f_\delta^+(t)-f(t)|\,dt,\ \int_{-\infty}^0 |f_\delta^-(t)-f(t)|\,dt < \varepsilon/2 \quad \text{for } 0<\delta<\delta_0 .$$

Since $\|\alpha_t x\| \leqq M\|x\|$ for some $M<\infty$, all $t \in \mathsf{R}$, and

$$\pi_\alpha(f)x = \int_0^\infty f(t)\alpha_t(x)\,dt + \int_{-\infty}^0 f(t)\alpha_t(x)\,dt \quad \forall x \in X$$

we get for $0<\delta<\delta_0$ that

$$\begin{aligned}
&\|\pi_\alpha(f)x\| \\
&\quad \leqq M\|x\|\varepsilon + \|\textstyle\int_0^\infty \int_{-\infty}^\infty \alpha_t(x)\hat{f}(a)\exp(-iat-\delta t)\,da\,dt + \\
&\qquad + \textstyle\int_{-\infty}^0 \int_{-\infty}^0 \alpha_t(x)\hat{f}(a)\exp(-iat+\delta t)\,da\,dt\| \\
&\quad = M\|x\|\varepsilon + \|\textstyle\int_{-\infty}^\infty \hat{f}(a)\int_0^\infty \exp(-iat-\delta t)\alpha_t(x)\,dt\,da + \\
&\qquad + \textstyle\int_{-\infty}^\infty \hat{f}(a)\int_{-\infty}^0 \exp(-iat+\delta t)\alpha_t(x)\,dt\,da\| \\
&\quad = M\|x\|\varepsilon + \|\textstyle\int_{-\infty}^\infty \hat{f}(a)[R(a-i\delta,Z)x - R(a+i\delta,Z)x]\,da\| .
\end{aligned}$$

The integral tends to zero as $\delta \downarrow 0$, since $\hat{f}$ has compact support in I and $I \subseteqq \varrho(Z)$. Hence $\pi_\alpha(f)=0$.

(ii) Suppose $\pi_\alpha(f)=0\ \forall f \in L_I^1$.

We restrict ourselves to $f \in \mathscr{S}(\mathsf{R})$, such that $\hat{f}$ has support in I. Then with the same notation, for $\delta>0$

$$\begin{aligned}
\int_0^\infty |f_\delta^+(t)-f(t)|\,dt &= \int_0^\infty |f(t)|\,|e^{-\delta t}-1|\,dt \\
&= \int_0^\infty (1+t^2)|f(t)|\,|e^{-\delta t}-1|/(t^2+1)\,dt \\
&\leqq \varepsilon \int_0^\infty |f(t)|^2(1+t^2)^2\,dt .
\end{aligned}$$

for suitably small δ, independently of f.

Then looking at the previous manipulations we see

$$\|\int_I [R(a-i\delta,Z)-R(a+i\delta,Z)] \times \hat{f}(a)\,da\| \leqq |\hat{f}|\varepsilon\|x\|M ,$$

where $|\cdot|$ is a continuous seminorm on $\mathscr{S}(\mathsf{R})^\wedge$.

We now apply an edge of the wedge theorem [5, Theorem 2.16] and deduce that $R(\lambda,Z)$ is analytic for $\lambda \in I^0$. Hence $I^0 \subseteqq \varrho Z$.

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