MATH. SCAND. 39 (1976), 91–101

ON APPROXIMATELY FINITE-DIMENSIONAL VON NEUMANN ALGEBRAS

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Abstract.

A classification is given of the biduals of separable approximately finite-dimensional C*-algebras. The basic result is that the properly infinite direct summand of the bidual is the same whenever the C*-algebra is not of type I.

1. Introduction.

We shall say that a von Neumann algebra is approximately finite-dimensional if any finite number of elements can be approximated arbitrarily closely in the $*$-ultrastrong topology by elements of a finite-dimensional sub von Neumann algebra. This property was studied by Murray and von Neumann in [9] in the case of a countably generated factor of type II_1, and shown to be equivalent to the existence of an increasing sequence of finite-dimensional sub von Neumann algebras with $*$-ultrastrongly dense union. One result of the present paper is that such a sequence of subalgebras exists in any countably generated approximately finite-dimensional von Neumann algebra. For a von Neumann algebra with separable predual, this was shown in [5].

The main result of this paper is that the maximal properly infinite approximately finite-dimensional direct summand of the bidual of a separable C*-algebra not of type I is independent of the C*-algebra. It is also shown that the maximal finite approximately finite-dimensional direct summand is determined by the number of its minimal direct summands of each type $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{II}_1$. The proof consists basically of two steps: analysis of the countably decomposable direct summands (which have separable preduals), and transfinite arithmetic.

The facts concerning approximately finite-dimensional von Neumann algebras with separable preduals which will be used are the result of [5] that a properly infinite such algebra is a direct summand of the bidual of the Glimm algebra M_{2^∞}, and hence by a construction of Glimm is

Received June 12, 1975.

a direct summand of the bidual of an arbitrary C*-algebra not of type I (see 2.1 below), and the result of Murray and von Neumann that a finite continuous factor in this class is unique. (The elaboration by Kaplansky of this result, in [7], expressing any finite continuous algebra in this class as the tensor product of a factor and a commutative von Neumann algebra, will also be used.)

The only facts concerning general approximately finite-dimensional von Neumann algebras which will be used are that the class is closed under direct sums and contains the discrete von Neumann algebras. For illustration, it may be remarked that the bidual of an inductive limit of C*-algebras of type I belongs to this class.

Naturally, a number of questions arise concerning approximately finite-dimensional von Neumann algebras. Let us comment briefly on two of them, although we shall not consider them further in this paper. First, in the definition it would seem almost as natural to use the ultraweak or the $*$-ultrastrong topology instead of the $*$-ultrastrong. A curious problem in this connection is which von Neumann algebras are approximately finite-dimensional in the norm topology. (An obvious conjecture presents itself, but does not seem easy to verify.[1]) Second, is the tensor product of von Neumann algebras A and B approximately finite-dimensional if and only if both A and B are? Sufficiency is easily verified. Necessity is an interesting question even in the case that B is commutative; if A and B have separable preduals this case can be settled by direct integral theory. In the case that B is discrete and A is a finite factor, the problem is well known.[2]

2. Properly infinite von Neumann algebras with separable predual.

2.1. THEOREM. *Let A be a properly infinite approximately finite-dimensional von Neumann algebra with separable predual. Let B be a separable C*-algebra not of type* I. *Then A is isomorphic to a direct summand of B^{**}.*

PROOF. In the case $B = M_{2^\infty}$, this is the main result of [5].

We shall deduce the general case from this special case by using a construction of Glimm. An examination of the proof of Theorem 1 of [6], where this construction is described (see also § 9 of [3]), reveals that in fact the following statement (among others) is proved. There exists

Footnotes added in proof (June 18, 1976).

[1] This conjecture has been verified independently by A. Connes and S. Wassermann, by an indirect argument using the ultraproduct technique of McDuff.

[2] In this case, for algebras with separable predual, the problem has been solved by A. Connes.

a sub-C*-algebra B_1 of B with a quotient B_0 isomorphic to M_{2^∞} and, moreover, such that if f is a state of B the restriction of which to B_1 is defined by a state of the quotient B_0, then there is a projection $e \in \pi_f(B)''$ (where π_f denotes the representation of B determined by f) with the following properties.

(i) e has central support 1 in $\pi_f(B)''$.
(ii) $e \in \pi_f(B_1)'' \cap \pi_f(B_1)'$.
(iii) $\pi_f(B_1)e = e\pi_f(B)e$.
(iv) $\pi_f(B_1)''e$ is isomorphic to $\pi_{f|B_1}(B_1)''$.

From (iii) it follows that $\pi_f(B_1)''e = e\pi_f(B)''e$, and hence by (iv), that $e\pi_f(B)''e$ is isomorphic to $\pi_{f|B_1}(B_1)''$. If f is chosen so that $\pi_{f|B_1}(B_1)''$ is isomorphic to A (recall that the quotient B_0 of B_1 is isomorphic to M_{2^∞} and use the preceding paragraph), then $e\pi_f(B)''e$ is isomorphic to A. In particular, e is properly infinite in $\pi_f(B)''$; hence by (i) and the fact that $\pi_f(B)''$ is countably decomposable, e is equivalent to 1 in $\pi_f(B)''$. This implies that $\pi_f(B)''$ is isomorphic to $e\pi_f(B)''e$, and it follows that $\pi_f(B)''$ is isomorphic to A.

Since π_f is nondegenerate, $\pi_f(B)''$ is isomorphic to a direct summand of B^{**}.

2.2. Remarks. 2.1 still holds if the hypothesis that A has separable predual is replaced by the hypothesis that A is countably generated; see 4.2 below. The hypotheses are then minimal.

It would be of interest to prove 2.1 more directly in the case that B is approximately finite-dimensional. If $B = M_n$ for some generalized integer n then the method of [5] is applicable, but this seems to be inadequate even if B is a more general matroid C*-algebra (in the sense of [4]).

3. On the determination of the bidual of a separable C*-algebra by its countably decomposable direct summands.

3.1. Lemma. *Let A_1 and A_2 be von Neumann algebras. Suppose that each of A_1 and A_2 is isomorphic to a direct summand of the other. Then A_1 and A_2 are isomorphic.*

Proof. The proof is similar to the proof of the analogous Cantor–Bernstein theorem for equivalence of projections (or, for that matter, to the proof of the original theorem for sets).

3.2. Lemma. *Let A be the direct sum of a continuum of countably decomposable von Neumann algebras. Suppose that for every countably decom-*

posable direct summand A_0 of A there exists a direct summand of A isomorphic to the direct sum of a continuum of copies of A_0. Then A is isomorphic to the direct sum of a continuum of copies of itself.

PROOF. By Zorn's lemma there exists a maximal collection of mutually orthogonal direct summands of A each of which is isomorphic to the direct sum of a continuum of copies of itself. Denote by A_1 the smallest direct summand of A containing the members of such a collection; then A_1 also is isomorphic to the direct sum of a continuum of copies of itself. Moreover, if A_2 is a direct summand of A orthogonal to A_1 and isomorphic to the direct sum of a continuum of copies of itself, then $A_2=0$.

By Zorn's lemma there exists a maximal collection of mutually orthogonal direct summands of A each orthogonal to A_1 and isomorphic to a direct summand of A_1. By hypothesis, such a collection has at most a continuum of members. Hence the direct sum A_3 of the members of such a collection is isomorphic to a direct summand of A_1; in fact, A_1+A_3 is isomorphic to a direct summand of A_1. By 3.1, A_1+A_3 is isomorphic to A_1. Therefore, replacing A_1 by A_1+A_3, we may choose A_1 so that $A_3=0$, that is, so that no nonzero direct summand of A orthogonal to A_1 is isomorphic to a direct summand of A_1.

We shall now show that $A_1=A$. Let A_0 be a countably decomposable direct summand of A orthogonal to A_1. By hypothesis there exists a direct summand A_2 of A isomorphic to the direct sum of a continuum of copies of A_0. If A_2 were not orthogonal to A_1, then by transitivity of isomorphism a nonzero direct summand of A_0 would be isomorphic to a direct summand of A_1. Therefore, the property of A_1 assumed in the preceding paragraph ensures that A_2 is orthogonal to A_1. On the other hand, A_2 is isomorphic to the direct sum of a continuum of copies of itself. Hence $A_2=0$, and therefore $A_0=0$. Since A is a direct sum of countably decomposable von Neumann algebras, it follows that $A_1=A$; in other words, the conclusion of the lemma holds.

3.3. LEMMA. *Let A be a von Neumann algebra with separable predual, let B be a separable* C**-algebra, and denote by C the diffuse commutative von Neumann algebra with separable predual. Suppose that there exists a direct summand of B^{**} isomorphic to $A\otimes C$. Then there exists a direct summand of B^{**} isomorphic to the direct sum of a continuum of copies of A.*

PROOF. We may represent C as $L^\infty(Z,\mu)$, where Z is a standard Borel space and μ is a diffuse finite positive measure on Z (Théorème 2, § 6, Chapitre II of [2]).

Representing A on a separable Hilbert space, we may write

$$A \otimes C = \int^{\oplus} A(\zeta) d\mu(\zeta) ,$$

where $A(\zeta) = A$ for all $\zeta \in Z$ (Corollaire to Proposition 3, § 3, Chapitre II of [2]). Then the algebra of diagonal operators is $1 \otimes C$.

Choose a nondegenerate representation π of B such that

$$\pi(B)'' = A \otimes C .$$

Then by 8.3.1 of [3] there exists for each $\zeta \in Z$ a representation $\pi(\zeta)$ of B on the Hilbert space of A such that

$$\pi = \int^{\oplus} \pi(\zeta) d\mu(\zeta) .$$

By 8.4.1 (i) of [3],

$$\pi(B)'' = \int^{\oplus} (\pi(\zeta)(B))'' d\mu(\zeta) .$$

Since $\pi(B)'' = A \otimes C$, by Proposition 1 (iii), § 3, Chapitre II of [2],

$$(\pi(\zeta)(B))'' = A(\zeta), \quad \mu\text{-almost all } \zeta \in Z .$$

Removing a μ-null Borel set from Z, we may suppose that this equation holds for all $\zeta \in Z$.

By 8.4.1 (iii) of [3], we may change Z by removing a μ-null Borel set so that the representations $\pi(\zeta)$ of B, $\zeta \in Z$, are mutually disjoint. In particular, at most one $\pi(\zeta)$ can be degenerate; since μ is diffuse we may suppose that all $\pi(\zeta)$ are nondegenerate. Then, if $B^{**}(\zeta)$ denotes the unique direct summand of B^{**} such that the corresponding representation of B is quasiequivalent to $\pi(\zeta)$, $B^{**}(\zeta)$ is isomorphic to $(\pi(\zeta)(B))''$ and hence to A for each $\zeta \in Z$, and the $B^{**}(\zeta)$, $\zeta \in Z$, are mutually orthogonal. Since μ is diffuse, Z is not countable. Therefore, by the continuum hypothesis Z is a continuum, and the conclusion of the lemma has been proved.

3.4. Lemma. *Let B be a separable* C*-*algebra. Then the continuous properly infinite approximately finite-dimensional direct summand of B^{**} is isomorphic to the direct sum of a continuum of copies of itself.*

Proof. Denote the continuous properly infinite approximately finite-dimensional direct summand of B^{**} by A. We shall show that the hypotheses of 3.2 are satisfied by A; the conclusion then follows by 3.2.

Since B is separable, B^{**} is the direct sum of a continuum of countably decomposable von Neumann algebras, and these in fact also have separable predual. In particular, the first hypothesis of 3.2 is verified.

Let A_0 be a countably decomposable direct summand of A. Then A_0 has separable predual, so 3.3 is applicable. By 3.3, to verify the second hypothesis of 3.2 it is enough to show that $A_0 \otimes C$ is isomorphic to a direct summand of A, where C is the diffuse commutative von Neumann algebra with separable predual. If $A_0 = 0$ this is trivial. Suppose, then that $A_0 \neq 0$, so that B is not of type I. Then by 2.1 (with $A = A_0 \otimes C$), $A_0 \otimes C$ is isomorphic to a direct summand of B^{**}. By definition of A, this is also a direct summand of A.

3.5. Lemma. *Let B be a separable C^* algebra. Let A denote the continuous finite approximately finite-dimensional direct summand of B^{**}, or the homogeneous discrete direct summand of B^{**} of order n, $n = 1, 2, \ldots, \aleph_0$. Then either A is countably decomposable or A is isomorphic to the direct sum of a continuum of copies of itself.*

Proof. Let us first prove the conclusion for the direct summand of A with diffuse centre. We shall use 3.2, the first hypothesis of which is verified since B is separable. To see that the second hypothesis of 3.2 is verified, let A_0 be a countably decomposable direct summand of A with diffuse centre. Then A_0 has separable predual, and, moreover, A_0 is isomorphic to $A_0 \otimes C$ where C is the diffuse commutative von Neumann algebra with separable predual. (A_0 is isomorphic to $F \otimes C$ where F is an approximately finite-dimensional factor — see § 3, Proposition 2 and § 7, Exercice 16 of Chapitre III of [2] —, and C is isomorphic to $C \otimes C$.) Hence by 3.3 there is a direct summand of B^{**} isomorphic to the direct sum of a continuum of copies of A_0, and by definition of A this must be contained in A. This shows that the direct summand of A with diffuse centre verifies the hypotheses of 3.2, and hence also the conclusion of 3.2, that is, the conclusion of the present lemma.

Now consider the minimal direct summands of A. These are mutually isomorphic, so it is enough to show that if A is not countably decomposable then there is a continuum of them. If the direct summand of A with diffuse centre were zero, then since each minimal direct summand of A is countably decomposable, there would have to be uncountably many of these, and hence (by the continuum hypothesis) a continuum. If there is a nonzero direct summand A_0 of A with diffuse centre then by the decomposition $A_0 = F \otimes C$ described in the parenthetical remark in the preceding paragraph, and by 3.3 (with $A = F$), there is a continuum of minimal direct summands of A.

3.6. Lemma. *Let B be a separable C^*-algebra. Let A denote (as in 3.5) the continuous finite approximately finite-dimensional direct summand of*

B^{**}, or the homogeneous discrete direct summand of B^{**} of order n, $n=1,2,\ldots,\aleph_0$. Then A is countably decomposable if and only if it is a direct sum of factors.

PROOF. If A is not a direct sum of factors then there exists a nonzero countably decomposable direct summand A_0 of A with diffuse centre. Then as in the proof of 3.5, A_0 has separable predual and is isomorphic to $A_0 \otimes C$ where C is the diffuse commutative von Neumann algebra with separable predual. Hence by 3.3 there is a direct summand of B^{**} isomorphic to the direct sum of a continuum of copies of A_0. By definition of A, this is a direct summand of A. This shows that A is not countably decomposable.

Now suppose that A is not countably decomposable. We shall show that A is not a direct sum of factors. We may assume that A does have a minimal direct summand (otherwise there would be nothing to prove). Then by 3.5 A has a continuum of minimal direct summands. There are three cases, which we shall treat separately: (i) A is finite; (ii) B is of type I; (iii) A is properly infinite and B is not of type I.

In case (i), consider the set T of all finite positive traces on B of norm less than one. T is a convex subset of B^*, compact in the weak (B^*,B) topology, and since B is separable, T is metrizable. Hence (as in 7.4.2 of [3]) the set $\partial_e T$ of extreme points of T is a G_δ in T, and in particular is Polish. The proof of 7.4.3 of [3] shows that there is a Borel isomorphism ϱ from $\partial_e T$ onto a Borel subset of $\mathrm{Rep}(B)$ such that for each $g \in \partial_e T$, $\varrho(g)$ is a finite factor representation of B with character g. Since the map $\mathrm{Rep}(B) \ni \pi \mapsto \pi(B)''$ is Borel, the proof of Theorem 1 of [12] shows that the set of $g \in \partial_e T$ such that $(\varrho(g)(B))''$ is approximately finite-dimensional is a Souslin set. Moreover, as shown also in the proof of 7.4.3 of [3], for each $n=1,2,\ldots$ the set of $g \in \partial_e T$ such that $(\varrho(g)(B))''$ is of type I_n is a Borel set. It follows that the set $E_A \subset \partial_e T$ corresponding to the set of minimal direct summands of A is a Souslin set. By the Alexandroff–Hausdorff theorem (or, more precisely, its proof — see Corollaire 4, § 33.II of [8]), E_A contains a Borel set with the same cardinality as E_A. Since A has a continuum of minimal direct summands, E_A contains a Borel set isomorphic to the unit interval; in particular, there exists a diffuse normalized positive measure μ on $\partial_e T$ supported in E_A (which is a universally measurable set). Denote by f the barycentre of μ, that is, $f=\int g\,d\mu(g)$, and denote by π_f the representation of B determined by f. We shall show that the centre of $\pi_f(B)''$ is diffuse and that $\pi_f(B)''$ is isomorphic to a direct summand of A. By Lemma 2 of [11], μ is the central measure of f; in other words, $\int^{\oplus} \pi_g\,d\mu(g)$ is the central decomposition of π_f. Since μ is concentrated in E_A, μ-almost all π_g, and hence also π_f, are of the same type as A. Moreover, by Theorem 2 of [12], $\pi_f(B)''$ (which is equal to $\int^{\oplus}\pi_g(B)''d\mu(g)$) is approximately finite-dimensional. This shows that $\pi_f(B)''$ is isomorphic to a direct summand of A. Since the centre of $\pi_f(B)''$ is the algebra of diagonal operators, it is isomorphic to $L^\infty(\mu)$ and is therefore diffuse.

In case (ii), we may suppose that A is of type $\mathrm{I}_{\aleph_0}$. By Theorem 2 of [6], the Mackey Borel structure on $B^\wedge$ is standard. Since A has a continuum of minimal direct summands, the Borel subset $B^\wedge{}_{\aleph_0}$ of $B^\wedge$ has the cardinality of the continuum and is therefore Borel isomorphic to the unit interval; in particular, $B^\wedge{}_{\aleph_0}$ has a diffuse finite positive measure μ. By 8.6.5(i) of [3] there is a nondegenerate representation π of B such that $\pi(B)''$ is of type $\mathrm{I}_{\aleph_0}$ and the centre of $\pi(B)''$ is isomorphic to $L^\infty(\mu)$. Since $\pi(B)''$ is isomorphic to a direct summand of A, this shows that A is not a direct sum of factors.

In case (iii), choose a nonzero countably decomposable direct summand A_0 of A; A_0 has in fact separable predual. Denote by C the diffuse commutative von Neumann algebra with separable predual. Then by 2.1, $A_0 \otimes C$ is isomorphic to a direct summand of B^{**}; by definition of A this must be a direct summand of A. Since $A_0 \otimes C$ has diffuse centre, this shows that A is not a direct sum of factors.

3.7. **Theorem.** *Let B_1 and B_2 be separable* C*-*algebras. Suppose that B_1^{**} and B_2^{**} have, up to isomorphism, the same countably decomposable approximately finite-dimensional direct summands. Then the maximal approximately finite-dimensional direct summands of B_1^{**} and B_2^{**} are isomorphic.*

Proof. We shall show that the theorem follows from 3.4, 3.5, 3.6 and 3.1.

Denote by A_1 (respectively A_2) the maximal approximately finite-dimensional direct summand of B_1^{**} (respectively B_2^{**}) of type $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_{\aleph_0}, \mathrm{II}_1, \mathrm{II}_\infty$, or III. We shall show that A_1 is isomorphic to A_2.

Since A_1 and A_2 are each direct sums of countably decomposable von Neumann algebras (B_1 and B_2 are separable) it follows by the hypothesis that A_1 is a direct sum of factors if and only if A_2 is. Hence by 3.6 and 3.4, A_1 is countably decomposable if and only if A_2 is.

If both A_1 and A_2 are countably decomposable, by the hypothesis each of A_1 and A_2 is isomorphic to a direct summand of the other. Hence by 3.1, A_1 and A_2 are isomorphic.

If neither A_1 nor A_2 is countably decomposable, then by 3.4 and 3.5, each is isomorphic to the direct sum of a continuum of copies of itself. Since each of A_1 and A_2 is also the direct sum of a continuum of countably decomposable von Neumann algebras (B_1 and B_2 are separable) it follows that each of A_1 and A_2 is isomorphic to a direct summand of the other. By 3.1 again, A_1 and A_2 are isomorphic.

4. Classification of the maximal approximately finite-dimensional direct summands of the biduals of separable C*-algebras.

4.1. THEOREM. *The properly infinite approximately finite-dimensional direct summand of the bidual of a separable* C*-*algebra* B *is the same whenever* B *is not of type* I, *and when* B *is of type* I *is determined by the number of its minimal direct summands.*

PROOF. By 2.1, the countably decomposable properly infinite approximately finite-dimensional direct summands of B^{**} are, up to isomorphism, independent of B if B is separable and not of type I. Therefore by 3.7 this is true also for the maximal properly infinite approximately finite-dimensional direct summand of B^{**}.

If B is a separable C*-algebra of type I, the conclusion can be deduced immediately from the description by Davies in [1] of all the possibilities for the bidual of B. (Actually, Davies described only the sequential weak* closure of B in its bidual, but it is clear that in this case the possibilities for the bidual are in one-to-one correspondence with the possibilities for the sequential closure of B.) The conclusion may also be shown by the proof of 4.3 below, with A denoting the properly infinite direct summand of B^{**}.

4.2. COROLLARY. 2.1 *holds if* A *is assumed just to be countably generated* (*instead of to have separable predual*).

PROOF. A countably generated von Neumann algebra is a direct summand of the bidual of a separable C*-algebra. Clearly we may choose such a C*-algebra which is not of type I (e.g., by adding a direct summand not of type I to any such C*-algebra). The conclusion is then immediate from 4.1.

4.3. THEOREM. *The finite approximately finite-dimensional direct summand of type* $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{II}_1$ *of the bidual of a separable* C*-*algebra is determined by the number of its minimal direct summands.*

PROOF. Let A be one of these direct summands. By 3.7, A is determined by its countably decomposable direct summands.

Since all minimal direct summands of A are isomorphic, the direct sum of these is clearly determined by their number.

By § 7, Exercice 16 of Chapitre III of [2], all nonzero countably decomposable direct summands of A with diffuse centre are isomorphic. By 3.5 and 3.6 such a direct summand occurs if and only if there is a continuum of minimal direct summands of A.

4.4. Remark. In 4.1 and 4.3 we have not described explicitly the approximately finite-dimensional direct summand of the bidual of a separable C*-algebra. Suffice it to say that examples are easily given to show that the numbers mentioned in 4.1 and 4.3 may take, independently, any value between 0 and the cardinality of the continuum (we are assuming the continuum hypothesis). Of course this paper yields very little information about the structure of the continuous properly infinite part, except that it is independent of the C*-algebra.

4.5. Remark. The results 4.1 and 4.3 can be restated in a special case as follows: the bidual of an approximately finite-dimensional separable C*-algebra A not of type I is determined by the number of extreme tracial states of A of each type $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{II}_1$. In particular, the biduals of any two Glimm algebras (see [4]) are isomorphic. Pedersen, in [10], raised the question whether this is the case for the enveloping Borel algebras (in place of the biduals, or enveloping von Neumann algebras). An affirmative answer to this would in particular yield a new proof of the result for biduals.

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