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POLYNOMIAL APPROXIMATION IN CERTAIN WEIGHTED HILBERT SPACES OF ENTIRE FUNCTIONS

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Abstract.

Estimates for reproducing kernels are used to prove polynomial approximation results in weighted Hilbert spaces of entire functions. The idea is to show that the polynomials are dense in those spaces whose weight functions are nearly a function of $|z|$ alone. Theorem 4 shows that if φ is plurisubharmonic and

$$|\varphi - |z|^2| \leqq c(1+|z|^2)^{\frac{1}{2}} ,$$

where c is a constant, then the polynomials are dense in the Hilbert space of entire functions f with $\int |f|^2 \exp(-\varphi)\, d\lambda < \infty$ ($d\lambda$ is the Lebesgue measure on $\mathbf{C}^n$). Applications to a problem of D. J. Newman and H. S. Shapiro are also discussed.

1. Introduction.

Our aim is to apply estimates for reproducing kernels to prove polynomial approximation results in certain weighted Hilbert spaces of entire functions. The basic idea is that the linear span of the reproducing kernels is dense in the Hilbert space of functions. Thus, to prove that the polynomials are dense in the Hilbert space, it suffices to show that their closure contains the reproducing kernels.

In particular, suppose that φ is a real valued measurable function on $\mathbf{C}^n$ and let $A^2(\varphi)$ denote the set of entire functions f for which

$$\int |f|^2 \exp(-\varphi)\, d\lambda < \infty ,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbf{C}^n$. If φ is upper semicontinuous, then $A^2(\varphi)$ is a Hilbert space with norm

$$\|f\|_\varphi^2 = \int |f|^2 \exp(-\varphi)\, d\lambda .$$

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We will assume for the remainder of this paper that φ denotes an upper semicontinuous function. When φ is a function of radius alone (i.e. $\varphi(z) = \varphi(|z|)$), we say that φ is *radial.*

The motivation for our work derives from the fact that when φ is radial, the polynomials are easily seen to be dense in $A^2(\varphi)$. For example, the set of monomials, z^k, where k is a multi-index, in $A^2(\varphi)$ is a complete orthogonal system in $A^2(\varphi)$ and consequently the Taylor series of $f \in A^2(\varphi)$ converges to f. When φ is not radial, however, there does not seem to be an analogous complete orthogonal system of polynomials.

A particularly important example is $\varphi(z) = |z|^2$, the Fischer space, which has been studied because of its intimate connection with quantum mechanics. (See [1].) In [8], D. J. Newman and H. S. Shapiro studied the related question of polynomial approximation in $A^2(|z|^2 - 2\log|P|)$ when P is an entire function of order less than two. Newman and Shapiro were able to show that the polynomials are dense in $A^2(|z|^2 - 2\log|P|)$ when P is an exponential polynomial or a function without zeroes. No examples are known, however, in which the polynomials fail to be dense in $A^2(|z|^2 - 2\log|P|)$.

We will show, as a special case of Theorem 4, that if φ is plurisubharmonic and there is a constant c so that

$$|\varphi - |z|^2| \leqq c(1+|z|^2)^{\frac{1}{2}},$$

then the polynomials are dense in $A^2(\varphi)$. Informally, then, the polynomials are dense in $A^2(\varphi)$ whenever φ is plurisubharmonic and sufficiently near the weight function of the Fischer space. In Section 4 we indicate how to recover from Theorem 4 a suitable analogue of the Newman–Shapiro result for exponential polynomials. In Theorem 8 we obtain an analogous extension of the Newman–Shapiro result to functions of exponential type.

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2. Preliminaries.

We recall first an estimate for reproducing kernels in weighted Hilbert spaces of entire functions from [5]. To each $\zeta \in \mathbf{C}^n$ there corresponds an element K_ζ in $A^2(\varphi)$ with the property that $f(\zeta) = (f, K_\zeta)_\varphi$ for each $f \in A^2(\varphi)$. This entire function, K_ζ, is the *reproducing kernel at* ζ of $A^2(\varphi)$. (For the general theory of reproducing kernels, see [2].) For example, the reproducing kernel at ζ in the Fischer space ($\varphi(z) = |z|^2$) is

$$K_\zeta(z) = \pi^{-n} \exp(z \cdot \zeta).$$

While we know that K_ζ belongs to $A^2(\varphi)$ for each ζ in $\mathbf{C}^n$, the reproducing kernels of the Fischer space actually satisfy a much stronger condition. They belong to $A^2(\varphi-\varepsilon|z|^2)$ for each $\varepsilon<1$. Theorem 8 of [5] generalizes this result to a class of plurisubharmonic functions.

PROPOSITION 1. *Suppose that φ is locally bounded, and that $\varphi-\varepsilon|z|^2$ is a plurisubharmonic function on $\mathbf{C}^n$ for some positive ε. Then the reproducing kernels of $A^2(\varphi)$ belong to $A^2(\varphi-\tau(\varepsilon)(1+|z|^2)^{\frac{1}{2}})$, where $\tau(\varepsilon)$ is a positive constant depending only on ε.*

As a final remark about reproducing kernels, notice that if $f\in A^2(\varphi)$ is orthogonal to the reproducing kernels of $A^2(\varphi)$, then f vanishes identically. Thus, the linear span of the reproducing kernels of $A^2(\varphi)$ is dense in $A^2(\varphi)$.

We will also require an estimate of L. Hörmander for the $\bar\partial$-operator. The Hilbert space of (*classes of*) functions u which are square integrable with respect to the measure $\exp(-\varphi)\,d\lambda$ is denoted by $L^2(\varphi)$ and

$$\|u\|_\varphi^2 = \int |u|^2 \exp(-\varphi)\,d\lambda\,.$$

Proposition 2 is a special case of Theorem 4.4.1 of [6].

PROPOSITION 2. *Suppose that $\varphi-\varepsilon|z|^2$ is a plurisubharmonic function on $\mathbf{C}^n$ for some positive ε. Then for every $(0,1)$ form $g=\Sigma_1^n g_i d\bar z_i$ with $g_i\in L^2(\varphi)$, and $\bar\partial g=0$, there exists $u\in L^2(\varphi)$ such that $\bar\partial u=g$ and*

$$\|u\|_\varphi^2 \leqq (2/\varepsilon)\int |g|^2 \exp(-\varphi)\,d\lambda\,.$$

REMARK. Theorem 4.4.1 of [6] requires that φ be of class $C^2(\mathbf{C}^n)$, but as noted in the text, this hypothesis can be removed.

3. Nearly radial plurisubharmonic weight functions.

In this section we prove the main results of this paper. We begin by proving a density result which is central to the polynomial approximation theorems which follow.

THEOREM 1. *Suppose that ψ is locally bounded, that $\varphi-\varepsilon|z|^2$ and $\psi-\varepsilon|z|^2$ are plurisubharmonic functions on $\mathbf{C}^n$ for some positive ε and that*

$$0 \leqq \varphi-\psi \leqq c(1+|z|^2)^{\frac{1}{2}}$$

where c is a constant. Then $A^2(\psi)$ is dense in $A^2(\varphi)$.

PROOF. Choose an integer N greater than $c/\tau(\varepsilon)$, and write $\omega_k = \psi + (k/N)(\varphi - \psi)$, for $k = 0, 1, \ldots N$. We proceed by induction to show that $A^2(\psi)$ is dense in ω_k for each k. The result then follows since $\omega_N = A^2(\varphi)$. We begin by observing that $A^2(\psi)$ is dense in $\omega_0 = A^2(\psi)$. Now suppose that $A^2(\psi)$ is dense in ω_k and $0 \leqq k < N$.

Since both $\varphi - \varepsilon|z|^2$ and $\psi - \varepsilon|z|^2$ are plurisubharmonic, it follows that

$$\omega_{k+1} - \varepsilon|z|^2 = (1 - (k+1)/N)(\psi - \varepsilon|z|^2) + ((k+1)/N)(\varphi - \varepsilon|z|^2)$$

is plurisubharmonic. We conclude from Proposition 1 that the reproducing kernels of $A^2(\omega_{k+1})$ belong to $A^2(\omega_{k+1} - \tau(\varepsilon)(1+|z|^2)^{\frac{1}{2}})$. Moreover, since

$$\{\omega_{k+1} - \tau(\varepsilon)(1+|z|^2)^{\frac{1}{2}}\} - \omega_k = (1/N)(\varphi - \psi) - \tau(\varepsilon)(1+|z|^2)^{\frac{1}{2}} < 0$$

it follows that the reproducing kernels of $A^2(\omega_{k+1})$ belong to $A^2(\omega_k)$. By the inductive hypothesis, the closure of $A^2(\psi)$ in $A^2(\omega_k)$ contains the linear span of the reproducing kernels of $A^2(\omega_{k+1})$. Since $\omega_k \leqq \omega_{k+1}$, convergence in $A^2(\omega_k)$ implies convergence in $A^2(\omega_{k+1})$ and thus, $A^2(\psi)$ is dense in $A^2(\omega_{k+1})$.

In our applications we actually need a slightly stronger version of Theorem 1.

COROLLARY 1. *Suppose that ψ is locally bounded, that $\varphi - \varepsilon|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and that $\psi - \varepsilon|z|^2$ is plurisubharmonic on the complement of a compact set K for some positive ε. Suppose also that $\psi + M|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and that*

$$0 \leqq \varphi - \psi \leqq c(1+|z|^2)^{\frac{1}{2}}$$

for constants c and M. Then $A^2(\psi)$ is dense in $A^2(\varphi)$.

PROOF. Let g be a negative, bounded function such that $g + (\varepsilon/2)|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and $g - (M + (\varepsilon/2))|z|^2$ is plurisubharmonic on a neighborhood of K. Then, applying Theorem 1 to φ and $\psi + g$, we conclude that $A^2(\psi + g)$ is dense in $A^2(\varphi)$. Since g is bounded, it follows that $A^2(\psi)$ is dense in $A^2(\varphi)$.

As an application of Corollary 1, we now prove a polynomial approximation theorem.

THEOREM 2. *Suppose that ψ is a locally bounded radial function, that $\varphi - \varepsilon|z|^2$ and $\psi - \varepsilon|z|^2$ are plurisubharmonic functions on $\mathbf{C}^n$ for some positive ε, and that $|\varphi - \psi| \leqq c(1+|z|^2)^{\frac{1}{2}}$ where c is a constant. Then the polynomials are dense in $A^2(\varphi)$.*

PROOF. Since ψ is radial, the polynomials are dense in $A^2(\psi)$. Our technique will be to adjust ψ so that we can apply Theorem 1, to conclude that the polynomials are dense in $A^2(\varphi)$. Let $\psi' = \psi - c(1+|z|^2)^{\frac{1}{2}}$ and notice that $\psi' + c|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and $\psi' - (\varepsilon/2)|z|^2$ is plurisubharmonic on the complement of a compact set. We conclude from Corollary 1 that $A^2(\psi')$ is dense in $A^2(\varphi)$. The result follows by observing that the polynomials are dense in $A^2(\psi')$ since ψ' is radial, and $\psi' \leqq \varphi$.

Theorem 2 is a result of the type in which we are interested. Our main application of theorems of this type will be to the Fischer space, in which we have $\psi(z) = |z|^2$. The major weakness of Theorem 2 is that it requires that $\varphi - \varepsilon|z|^2$ as well as $\psi - \varepsilon|z|^2$ be plurisubharmonic. In order to relax this restriction we require a density result which allows us to approximate certain plurisubharmonic functions by functions φ for which $\varphi - \varepsilon|z|^2$ is plurisubharmonic.

THEOREM 3. *Suppose that φ is plurisubharmonic on $\mathbf{C}^n$ and that $\varphi_k - (1/N_k^2)|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ for $k = 1, 2, \ldots$, where $N_k \to \infty$. Suppose also that $-c \leqq \varphi - \varphi_k$ on $\mathbf{C}^n$ and that $\varphi - \varphi_k \leqq c$ on the ball $0 \leqq |z| \leqq N_k$, where c is a constant independent of k. Then $\bigcup A^2(\varphi_k)$ is dense in $A^2(\varphi)$.*

PROOF. Choose $\alpha \in C^2(\mathbf{C}^n)$ with $0 \leqq \alpha \leqq 1$, $\alpha(z) = 1$ when $|z| \leqq \frac{1}{2}$ and $\alpha(z) = 0$ when $1 \leqq |z|$. Let $\alpha_k(z) = \alpha(z/N_k)$, and denote by $A(k)$ the annulus $\{z : \frac{1}{2}N_k \leqq |z| \leqq N_k\}$. Then notice that $\bar\partial\alpha_k$ is supported in $A(k)$ and $|\bar\partial\alpha_k|^2 \leqq a/N_k^2$, where a is a constant, depending on α.

Now suppose that $f \in A^2(\varphi)$. Then it is clear that $f\alpha_k$ converges to f in $L^2(\varphi)$. According to Proposition 2, we can find $u_k \in L^2(\varphi_k)$ with $\bar\partial u_k = f\bar\partial\alpha_k$ and

$$\|u_k\|_{\varphi_k}^2 \leqq 2N_k^2 \int |f\bar\partial\alpha_k|^2 \exp(-\varphi_k)\, d\lambda .$$

We write $f_k = f\alpha_k - u_k$ and observe that $f_k \in A^2(\varphi_k)$ for each k, since $\bar\partial f_k = 0$. We complete the proof by showing that u_k converges to zero in $L^2(\varphi)$, and thus that f_k converges to f in $A^2(\varphi)$.

Since $p_k \leqq \varphi + c$ we have

$$\begin{aligned} \|u_k\|_\varphi^2 &\leqq \exp(c)\|u_k\|_{\varphi_k}^2 \\ &\leqq 2\exp(c) N_k^2 \int |f\bar\partial\alpha_k|^2 \exp(-\varphi_k)\, d\lambda \\ &\leqq 2\exp(c) N_k^2 \int_{A(k)} |f|^2 (a/N_k^2) \exp(-\varphi) \exp(\varphi - \varphi_k)\, d\lambda \\ &\leqq 2a \exp(2c) \int_{A(k)} |f|^2 \exp(-\varphi)\, d\lambda . \end{aligned}$$

It follows, by the dominated convergence theorem, that u_k converges to zero in $L^2(\varphi)$.

THEOREM 4. *Suppose that ψ is a locally bounded radial function, that φ and $\psi - \varepsilon|z|^2$ are plurisubharmonic on $\mathbf{C}^n$ for some positive ε, and that*

$$|\varphi - \psi| \leqq c(1+|z|^2)^{\frac{1}{2}},$$

where c is a constant. Then the polynomials are dense in $A^2(\varphi)$.

PROOF. We shall construct φ_k which satisfy the conditions of Theorem 3, so that the polynomials are dense in $A^2(\varphi_k)$. Write

$$\varphi_k = \varphi + (1/k^2)(\psi - \varphi - (\varepsilon/2)|z|^2) \quad \text{and} \quad N_k = (2/\varepsilon)^{\frac{1}{2}}k\,.$$

Then

$$\varphi_k - (1/N_k^2)|z|^2 = (1 - 1/k^2)\varphi + (1/k^2)(\psi - \varepsilon|z|^2)$$

is plurisubharmonic on $\mathbf{C}^n$. Moreover, we have

$$\varphi - \varphi_k = (1/k^2)((\varepsilon/2)|z|^2 + \varphi - \psi) \geqq (1/k^2)((\varepsilon/2)|z|^2 - c(1+|z|^2)^{\frac{1}{2}})$$

so that $\varphi - \varphi_k$ is bounded below by a constant independent of k. Similarly, we have

$$\varphi - \varphi_k \leqq (1/k^2)((\varepsilon/2)|z|^2 + c(1+|z|^2)^{\frac{1}{2}}) \leqq 1 + c(1+2/\varepsilon)$$

on $0 \leqq |z| \leqq N_k$. Thus, by Theorem 3, $\bigcup A^2(\varphi_k)$ is dense in $A^2(\varphi)$. Now, since $\varphi - \varphi_k$ is bounded below by a constant, the proof is completed by showing that the polynomials are dense in $A^2(\varphi_k)$ for each k.

We have seen that $\varphi_k - (1/N_k^2)|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and

$$|\varphi_k - (\psi - (\varepsilon/2k^2)|z|^2)| = (1 - 1/k^2)|\psi - \varphi| < c(1+|z|^2)^{\frac{1}{2}}\,.$$

Since $(\psi - (\varepsilon/2k^2)|z|^2) - (\varepsilon/2k^2)|z|^2$ is plurisubharmonic, and $\psi - (\varepsilon/2k^2)|z|^2$ is radial, it follows from Theorem 2 that the polynomials are dense in $A^2(\varphi_k)$ for each k.

In certain of our applications, we will not have a uniform estimate of the form $|\varphi - \psi| < c(1+|z|^2)^{\frac{1}{2}}$. Rather, we will have a uniform lower bound of the form $-c(1+|z|^2)^{\frac{1}{2}} < \varphi - \psi$ and an upper bound of the form $\varphi - \psi < c(1+|z|^2)^{\frac{1}{2}}$ on circles whose radii tend to infinity. We can generalize Theorem 2 to this situation.

THEOREM 5. *Suppose that ψ is continuous and radial, and that $\varphi - \varepsilon|z|^2$ and $\psi - \varepsilon|z|^2$ are plurisubharmonic functions on $\mathbf{C}^n$ for some positive ε. Suppose also that*

$$-c(1+|z|^2)^{\frac{1}{2}} < \varphi-\psi$$

and that there exists a sequence of circles $|z|=R_k$, *with* $R_k \to \infty$, *on which we have*

$$\varphi-\psi < c(1+|z|^2)^{\frac{1}{2}},$$

where c is a constant. Then the polynomials are dense in $A^2(\varphi)$.

PROOF. Let $\omega=\max(\varphi, \psi+c(1+|z|^2)^{\frac{1}{2}})$. Then $\omega-\varepsilon|z|^2$ is plurisubharmonic and

$$0 \leqq \omega-\varphi < 2c(1+|z|^2)^{\frac{1}{2}}.$$

Moreover, since $\varphi-(\psi+c(1+|z|^2)^{\frac{1}{2}})$ is upper semicontinuous and is negative on $|z|=R_k$, we conclude that $\omega=\psi+c(1+|z|^2)^{\frac{1}{2}}$ on a neighborhood of $|z|=R_k$.

Now let

$$\chi(z) = \omega(z)-2c(1+|z|^2)^{\frac{1}{2}}$$

and observe that $\chi+2c|z|^2$ is plurisubharmonic on $\mathbf{C}^n$, and $\chi-(\varepsilon/2)|z|^2$ is plurisubharmonic on the complement of a compact set. In addition, we have

$$0 \leqq \varphi-\chi < 2c(1+|z|^2)^{\frac{1}{2}}.$$

We conclude from Corollary 1 that $A^2(\chi)$ is dense in $A^2(\varphi)$. Thus, to complete the proof it suffices to prove that the polynomials are dense in $A^2(\chi)$.

Let g be a bounded function such that $g+(\varepsilon/4)|z|^2$ is plurisubharmonic on $\mathbf{C}^n$ and $g-c|z|^2$ is plurisubharmonic on $|z|<4c/\varepsilon$. Now, write

$$\varphi_k = \begin{cases} \chi+g & \text{when } |z|\leqq R_k \\ \psi-c(1+|z|^2)^{\frac{1}{2}}+g & \text{when } |z|>R_k\,. \end{cases}$$

Note that in a neighborhood of $|z|=R_k$, the function φ_k is equal to $\psi-c(1+|z|^2)^{\frac{1}{2}}+g$. Moreover, both $\varphi_k-(\varepsilon/2)|z|^2$ and $\chi+g-(\varepsilon/2)|z|^2$ are plurisubharmonic. We also have $\chi+g-\varphi_k=0$ on $|z|\leqq R_k$ and

$$\chi+g-\varphi_k = (\chi-\psi)+c(1+|z|^2)^{\frac{1}{2}} = (\omega-\psi)-c(1+|z|^2)^{\frac{1}{2}} \geqq 0$$

on $|z|>R_k$.

We conclude from Theorem 3 that $\bigcup A^2(\varphi_k)$ is dense in $A^2(\chi+g)$, and, since g is bounded, in $A^2(\chi)$.

The proof is completed by observing that, for each k, φ_k is plurisubharmonic and $|\varphi_k-\psi|$ is bounded above by a constant multiple of $(1+|z|^2)^{\frac{1}{2}}$. Hence, it follows from Theorem 4 that the polynomials are dense in $A^2(\varphi_k)$ for each k.

We conclude this section by using Theorem 3 to extend Theorem 5 to a more general class of plurisubharmonic weight functions.

THEOREM 6. *Suppose that ψ is continuous and radial and that φ and $\psi-\varepsilon|z|^2$ are plurisubharmonic for some positive ε. Suppose also that*

$$-c(1+|z|^2)^{\frac{1}{2}} < \varphi-\psi < c(1+|z|^2)$$

and that there exists a sequence of circles, $|z|=R_k$, with $R_k \to \infty$, on which

$$\varphi-\psi < c(1+|z|^2)^{\frac{1}{2}},$$

where c is a constant. Then the polynomials are dense in $A^2(\varphi)$.

PROOF. As in the proof of Theorem 4, we let

$$\varphi_k = \varphi+(1/k^2)(\psi-\varphi-(\varepsilon/2)|z|^2).$$

Then $(1/k^2)((\varepsilon/2)|z|^2-c(1+|z|^2)^{\frac{1}{2}})<\varphi-\varphi_k$ so that $\varphi-\varphi_k$ is bounded below by a constant independent of k. We also have

$$\varphi-\varphi_k < (1/k^2)((\varepsilon/2)+c)|z|^2$$

which insures that $\varphi-\varphi_k$ is bounded from above by a constant on $0 \leqq |z| \leqq (2/\varepsilon)^{\frac{1}{2}}k$. Since

$$\varphi_k-(\varepsilon/2k^2)|z|^2 = (1/k^2)(\psi-\varepsilon|z|^2)+(1-1/k^2)\varphi$$

is plurisubharmonic, it follows from Theorem 3, with $N_k=(2/\varepsilon)^{\frac{1}{2}}k$, that $\bigcup A^2(\varphi_k)$ is dense in $A^2(\varphi)$.

Now, we have

$$\varphi_k-(\psi-(\varepsilon/2)|z|^2) = (1-1/k^2)(\varphi-\psi) > -c(1+|z|^2)^{\frac{1}{2}}$$

and on $|z|=R_k$, we have

$$\varphi_k-(\psi-(\varepsilon/2)|z|^2) \leqq 2c(1+|z|^2)^{\frac{1}{2}}.$$

In addition, we have $(\psi-(\varepsilon/2)|z|^2)-(\varepsilon/2)|z|^2$ is plurisubharmonic. It follows from Theorem 5 that the polynomials are dense in $A^2(\varphi_k)$ and consequently in $A^2(\varphi)$.

4. A problem of Newman and Shapiro.

In this section we restrict our attention to the case $n=1$. In [8], D. J. Newman and H. S. Shapiro proved that the polynomials are dense in $A^2(|z|^2-2\log|P|)$ when P is an exponential polynomial. Inasmuch as our interest is in polynomial approximation is spaces whose weight functions are plurisubharmonic, (i.e. subharmonic since $n=1$) we shall study this problem in $A^2(\varphi)$ when φ is subharmonic and $A^2(\varphi)$ is very nearly equal to $A^2(|z|^2-2\log|P|)$.

In order to motivate our approach to this problem we begin by studying the situation when P is an exponential polynomial. That is, we assume that

$$P(z) = \sum_1^m p_k(z) \exp(a_k z) ,$$

where p_k is a polynomial and $a_k \in \mathbf{C}$ for $k=1,2,\ldots m$. Let gsm $(|z|^2-2\log|P|)$ denote the greatest subharmonic minorant of $|z|^2-2\log|P|$. (See [7] for details.) We begin by showing that the polynomials are dense in $A^2(\text{gsm}\,(|z|^2-2\log|P|))$, when P is an exponential polynomial.

Theorem 7. *If P is an exponential polynomial then the polynomials are dense in $A^2(\text{gsm}\,(|z|^2-2\log|P|))$.*

Proof. Inasmuch as P is of order 1, it is clear that there is a constant, c_1, such that $2\log|P(z)|<c_1(1+|z|^2)^{\frac{1}{2}}$, and thus that

$$|z|^2-2\log|P| > |z|^2-c_1(1+|z|^2)^{\frac{1}{2}} .$$

Hence, it follows that there is a constant, c_2, so that

$$\text{(1)} \qquad \text{gsm}\,(|z|^2-2\log|P|) > |z|^2-c_2(1+|z|^2)^{\frac{1}{2}} .$$

In addition, by Theorem 1 in [10], there is a constant M so that each disk of radius 2 contains fewer than M zeroes of P. We denote by Δ the set of points in $\mathbf{C}$ whose distance to at least one zero of P does not exceed $1/3M$. Then there is a circle of radius less than 1 about each point $\mathbf{C}$ which does not meet Δ.

By Theorem 3 in [4] there is a constant, c_3, so that if z is not in Δ, then $-2\log|P|<c_3(1+|z|^2)^{\frac{1}{2}}$. Hence, it follows that

$$|z|^2-2\log|P| < |z|^2+c_3(1+|z|^2)^{\frac{1}{2}} ,$$

for such z, and by the subaveraging principle for subharmonic functions that

$$\text{(2)} \qquad \text{gsm}\,(|z|^2-2\log|P|) < |z|^2+c_4(1+|z|^2)^{\frac{1}{2}} ,$$

for all $z \in \mathbf{C}$. The result now follows by Theorem 5, in view of (1) and (2).

Theorem 7 is the polynomial approximation result which corresponds to the result of Newman and Shapiro. We can now extend this theorem to a more general class of weight functions.

Theorem 8. *Suppose that ψ is continuous and radial, that $\psi-\varepsilon|z|^2$ is plurisubharmonic, and that*

$$\psi(w) \leqq \psi(z)+c|z|^2 \quad \textit{when } |w-z|<|z| ,$$

where c is a constant. If P is an entire function of exponential type, then the polynomials are dense in $A^2(\text{gsm}\,(\psi-2\log|P|))$.

PROOF. As in the proof of Theorem 7, it is clear that there is a constant, c_1, so that

$$\text{gsm}\,(\psi-2\log|P|) > \psi-c_1(1+|z|^2)^{\frac{1}{2}}\,. \tag{3}$$

Furthermore, by (3.7.2) of [3], there are constants c_2 and R such that whenever $|z|>R$, there is a circle of radius $\varrho(z)\leqq 2|z|$ about the origin such that $-2\log|P(w)|\leqq c_2(1+|w|^2)$ if $|w|=\varrho(z)$. It follows that there is a constant c_3 so that

$$\text{gsm}\,(\psi-2\log|P|) \leqq \psi+c_3(1+|z|^2) \tag{4}$$

for all $z\in \mathbf{C}$.

Moreover, by (3.3.1) of [3], there exist circles of radii R_k and a constant c_4 so that $R_k\to\infty$ and

$$-2\log|P(z)| < c_4(1+|z|^2)^{\frac{1}{2}}\,,$$

if $|z|=R_k$. We conclude that, when $|z|=R_k$,

$$\text{gsm}\,(\psi-2\log|P|) < \psi+c_4(1+|z|^2)^{\frac{1}{2}}\,. \tag{5}$$

The result now follows from Theorem 6, in view of (3), (4) and (5).

We emphasize, in the following Corollary, that the most important special case of Theorem 8 is $\psi(z)=|z|^2$.

COROLLARY 2. *If P is an entire function of exponential type then the polynomials are dense in* $A^2(\text{gsm}\,(|z|^2-2\log|P|))$.

Corollary 2 provides information about the closure of the polynomials in $A^2(|z|^2-2\log|P|)$ when P is of exponential type.

THEOREM 9. *If P is an entire function of exponential type, then the closure of the polynomials in $A^2(|z|^2-2\log|P|)$ contains $A^2(|z|^2-2\log|P|-\log(1+|z|^2))$.*

PROOF. Suppose that $f\in A^2(|z|^2-2\log|P|-\log(1+|z|^2))$ and write $\varphi=\text{gsm}\,(|z|^2-2\log|P|)$. By (3) and (4) there exists a constant c so that

$$-c(1+|z|^2)^{\frac{1}{2}} < \varphi-|z|^2 < c(1+|z|^2)\,.$$

Let g be a bounded function such that $|z|^2-c(1+|z|^2)^{\frac{1}{2}}+g$ is plurisubharmonic and let

$$\varphi_k = \varphi+(1/k^2)(|z|^2-c(1+|z|^2)^{\frac{1}{2}}+g-\varphi)\,.$$

Then, with $N_k = k$ and α_k as in the proof of Theorem 3, we can find $u_k \in A^2(\varphi_k)$ so the $\bar{\partial} u_k = f \bar{\partial} \alpha_k$ and

$$\int |u_k|^2 \exp(-\varphi_k)\, d\lambda \leqq 2k^2 \int |f \bar{\partial} \alpha_k|^2 \exp(-\varphi_k)\, d\lambda .$$

Proceeding as in the proof of Theorem 3, we find that $\int |u_k|^2 \exp(-\varphi)\, d\lambda$ is bounded above by a constant multiple of $\int_{A(k)} |f|^2 \exp(-\varphi)\, d\lambda$.

By the defining property of the reproducing kernels in the Fischer space and the fact that $zf \in A^2(|z|^2 - 2\log|P|)$ it follows that $|zf(z)|^2$ is bounded above by a constant times $\exp(|z|^2 - 2\log|P|)$ for each $z \in \mathbf{C}$. Moreover, inasmuch as $\log|zf|$ is subharmonic it also follows that $|zf(z)|^2$ is bounded above by a constant times $\exp(\varphi)$. Thus, the sequence $\|u_k\|_\varphi$ is bounded and hence u_k has a weakly convergent subsequence in $L^2(\varphi)$, which we continue to denote by u_k. Writing the weak limit of u_k as u, we observe that $f\alpha_k - u_k + u$ is a sequence in $A^2(\varphi)$ which converges weakly to f in $A^2(|z|^2 - 2\log|P|)$. It follows that f is in the closure of $A^2(\varphi)$ in the Hilbert space $A^2(|z|^2 - 2\log|P|)$. The result follows by Corollary 2, since $\varphi \leqq |z|^2 - 2\log|P|$.

5. Conclusion.

We have shown that the polynomials are dense in $A^2(\varphi)$ when φ is nearly equal to a radial function ψ, with $\psi - \varepsilon|z|^2$ plurisubharmonic. We remark that the polynomials are not always dense in $A^2(\varphi)$. In order to illustrate this point, we consider some weight functións of one variable.

If $\varphi(z) = x^2\ (z = x + iy)$, then $A^2(\varphi)$ is not empty, but contains no polynomials. In this case, $\varphi - \frac{1}{2}|z|^2$ is plurisubharmonic, but φ is not sufficiently near a radial function for the polynomials to be dense in $A^2(\varphi)$.

In [9], B. A. Taylor showed that the polynomials are not dense in $A^2(\varphi)$ when $\varphi = (1 + x^2)^{\frac{1}{4}}$. In this case, φ is nearly radial, since

$$|\varphi - (1 + |z|^2)^{\frac{1}{4}}| < (1 + |z|^2)^{\frac{1}{4}} ,$$

but there is no positive constant ε for which $(1 + |z|^2)^{\frac{1}{4}} - \varepsilon|z|^2$ is plurisubharmonic.

We know of no example in which the polynomials fail to be dense in $A^2(\varphi)$ when φ is plurisubharmonic and

$$|\varphi - |z|^2| < c(1 + |z|^2)^{\frac{1}{4} + d} \quad \text{when } 0 < d < \tfrac{1}{2} .$$

However, we have been unable to prove this result, except under the additional conditions of Theorem 6. We remark that the estimates we have used for the reproducing kernels are not known to be sharp. Suitable improvements in those estimates would lead to corresponding improvements in our results.

Finally, we considered an appropriate analogue of a problem of D. J. Newman and H. S. Shapiro. We were able to prove more general analogous results, but our methods seem to give slightly less precise answers to the original question of Newman and Shapiro.

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