MATH. SCAND. 39 (1976), 125–130

UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTIONS

H. S. GOPALAKRISHNA and SUBHAS S. BHOOSNURMATH*

Abstract.

If f is a transcendental meromorphic function, a is an extended complex number and k is a positive integer or ∞, let

$$E(a,k,f) = \{z \in \mathbf{C} \mid z \text{ is a zero of } f-a \text{ of order } \leqq k\}$$

where C is the complex plane. If f_1, f_2 are distinct meromorphic functions and if there exist distinct extended complex numbers $a_1,\ldots,a_m$ such that $E(a_i,k_i,f_1) = E(a_i,k_i,f_2)$ for $i=1,2,\ldots,n$ where each k_i is a positive integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_m$, then it is shown that

$$\sum_{i=2}^{m} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} \leqq 2 .$$

Several consequences are deduced which include a theorem of Nevanlinna and the following result:

If the set of simple zeros of f_1-a coincides with the set of simple zeros of f_2-a for seven distinct values of a in the extended complex plane, the $f_1 \equiv f_2$.

1.

We denote by $\mathbf{C}$ the set of all finite complex numbers and by $\bar{\mathbf{C}}$ the extended complex plane consisting of all (finite) complex numbers and ∞. By a meromorphic function we mean a transcendental meromorphic function in the plane. We use the usual notations of the Nevanlinna theory of meromorphic functions as explained in [1] and [3].

If f is a meromorphic function, then as in [1], we denote by $S(r,f)$ any quantity satisfying

Received August 12, 1975.

* Research of the second author is supported by the Department of Atomic Energy, Bombay.

$$S(r,f) = o(T(r,f))$$

as $r \to \infty$, through all values if f is of finite order and outside a set of finite linear measure if f is of infinite order.

If f is a meromorphic function, $a \in \bar{\mathbf{C}}$ and k is a positive integer or ∞, we denote by $\bar{n}_k(r,a,f)$ the number of distinct zeros of order $\leqq k$ of $f-a$ in $|z| \leqq r$ (each zero of order $\leqq k$ is counted only once irrespective of its multiplicity). Thus, in particular. $\bar{n}_1(r,a,f)$ is the number of simple zeros and $\bar{n}_2(r,a,f)$ the number of distinct simple and double zeros of $f-a$ in $|z| \leqq r$. Also $\bar{n}_\infty(r,a,f) = \bar{n}(r,a,f)$. $\bar{N}_k(r,a,f)$ is defined in terms of $\bar{n}_k(r,a,f)$ in the obvious way. Clearly

$$\bar{n}(r,a,f) \leqq \frac{1}{k+1}\{k\bar{n}_k(r,a,f) + n(r,a,f)\}$$

so that

$$\bar{N}(r,a,f) \leqq \frac{1}{k+1}\{k\bar{N}_k(r,a,f) + N(r,a,f)\} \tag{1}$$

We also denote by $E(a,k,f)$ the subset of C consisting of all zeros of order $\leqq k$ of $f-a$. That is

$$E(a,k,f) = \{z \in \mathbf{C} \mid z \text{ is a zero of } f-a \text{ of order } \leqq k\}.$$

In particular, $E(a,\infty,f) = \{z \in \mathbf{C} \mid f(z) = a\}$ and we denote it simply by $E(a,f)$.

Nevanlinna proved the following theorem [2, page 48 and 1, Theorem 2.6]

THEOREM A. *If f_1, f_2 are meromorphic functions and if $E(a,f_1) = E(a,f_2)$ for five distinct values of a in $\bar{\mathbf{C}}$, then $f_1 \equiv f_2$.*

In this paper we obtain a general result of which Theorem A appears as a particular case.

Let f_1, f_2 be meromorphic functions. If $a \in \bar{\mathbf{C}}$ and k is a positive integer or ∞, then for $r > 0$, we denote by $n_0^{(k)}(r,a)$ the number of common zeros of order $\leqq k$ of $f_1 - a$ and $f_2 - a$ in $|z| \leqq r$, each zero of order $\leqq k$ being counted only once irrespective of its multiplicity. In particular $n_0^{(\infty)}(r,a)$ is the number of common zeros of $f_1 - a$ and $f_2 - a$ in $|z| \leqq r$ (all zeros are considered) and we also denote it simply by $n_0(r,a)$. As usual, we set

$$N_0^{(k)}(r,a) = \int_0^r \frac{n_0^{(k)}(r,a) - n_0^{(k)}(0,a)}{t}\,dt + n_0^{(k)}(0,a)\log r .$$

We also define

$$\bar{N}_{1,2}^{(k)}(r,a) = \bar{N}_k(r,a,f_1) + \bar{N}_k(r,a,f_2) - 2N_0^{(k)}(r,a)$$

and write $\bar{N}_{1,2}(r,a)$ for $\bar{N}_{1,2}^{(\infty)}(r,a)$.

Our main result is the following

THEOREM 1. *Let f_1, f_2 be distinct meromorphic functions (that is, $f_1 \not\equiv f_2$). If there exist distinct elements $a_1, \ldots, a_m$ in $\bar{\mathbf{C}}$ such that $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ for $i = 1, 2, \ldots, m$ for some $k_1, \ldots, k_m$ each of which is a positive integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_m$, then*

$$\sum_{i=2}^{m} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} \leqq 2 . \tag{2}$$

PROOF. Suppose, first, that $a_1, \ldots, a_m$ are all finite.

We have, by Nevanlinna's second fundamental theorem, for $j = 1, 2$,

$$\begin{aligned}(m-2)T(r, f_j) &\leqq \sum_{i=1}^{m} \bar{N}(r, a_i, f_j) + S(r, f_j)\\ &\leqq \sum_{i=1}^{m} \frac{1}{k_i+1} \{k_i \bar{N}_{k_i}(r, a_i, f_j) + N(r, a_i, f_j)\} + S(r, f_j) ,\end{aligned}$$

by (1)

$$\leqq \sum_{i=1}^{m} \frac{k_i}{k_i+1} \bar{N}_{k_i}(r, a_i, f_j) + \left\{\sum_{i=1}^{m} \frac{1}{k_i+1}\right\} T(r, f_j) + S(r, f_j) .$$

So,

$$\left\{m - 2 - \sum_{i=1}^{m} \frac{1}{k_i+1}\right\} T(r, f_j) \leqq \sum_{i=1}^{m} \frac{k_i}{k_i+1} \bar{N}_{k_i}(r, a_i, f_j) + S(r, f_j) .$$

Adding the two inequalities corresponding to $j = 1$ and $j = 2$, we obtain

$$\begin{aligned}&\left\{\sum_{i=1}^{m} \frac{k_i}{k_i+1} - 2\right\}\{T(r, f_1) + T(r, f_2)\}\\ &\qquad \leqq \sum_{i=1}^{m} \frac{k_i}{k_i+1} \{\bar{N}_{k_i}(r, a_i, f_1) + \bar{N}_{k_i}(r, a_i, f_2)\} + S(r, f_1) + S(r, f_2)\\ &\qquad = 2 \sum_{i=1}^{m} \frac{k_i}{k_i+1} N_0^{(k_i)}(r, a_i) + S(r, f_1) + S(r, f_2) ,\end{aligned}$$

since, by hypothesis, $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ so that $\bar{n}_{k_i}(r, a_i, f_1) = \bar{n}_{k_i}(r, a_i, f_2) = n_0^{(k_i)}(r, a_i)$ for $i = 1, 2, \ldots, m$.

The sequence $\langle k/(k+1) \rangle$ is increasing and so, since $k_1 \geqq k_2 \geqq \ldots \geqq k_m$, (3) yields

$$\begin{aligned}&\left\{\sum_{i=1}^{m} \frac{k_i}{k_i+1} - 2\right\} \{T(r, f_1) + T(r, f_2)\}\\ &\qquad \leqq \frac{2k_1}{k_1+1} \sum_{i=1}^{m} N_0^{(k_i)}(r, a_i) + S(r, f_1) + S(r, f_2)\end{aligned} \tag{4}$$

Now, since $f_1 \not\equiv f_2$, it follows that, for $a \in C$, each common zero of $f_1 - a$ and $f_2 - a$ is a pole of $1/(f_1 - f_2)$. Since $a_1, \ldots, a_m$ are distinct, we therefore have

$$\sum_{i=1}^{m} N_0^{(k_i)}(r, a_i) \leqq N\left(r, \frac{1}{f_1 - f_2}\right) \leqq T(r, f_1 - f_2) + O(1)$$

$$\leqq T(r, f_1) + T(r, f_2) + O(1)\,.$$

Hence, from (4), we obtain

$$\left\{\sum_{i=2}^{m} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} - 2\right\}\{T(r, f_1) + T(r, f_2)\} \leqq S(r, f_1) + S(r, f_2)\,, \tag{5}$$

which implies (2), as, otherwise, (5) would yield

$$T(r, f_1) + T(r, f_2) = o(T(r, f_1) + T(r, f_2))$$

as $r \to \infty$ outside a set of finite measure, which is impossible.

Suppose, now, that some a_i is ∞. Then, let a be a (finite) complex number different from $a_1, \ldots, a_m$. Then $1/(a_1 - a), \ldots, 1/(a_m - a)$ are all distinct and finite. If $g_j = 1/(f_j - a)$ for $j = 1, 2$, then g_1, g_2 are distinct meromorphic functions and

$$E\left(\frac{1}{a_i - a}, k_i, g_1\right) = E\left(\frac{1}{a_i - a}, k_i, g_2\right)$$

for $i = 1, 2, \ldots, m$. Hence, by what we have proved above, (2) holds.

This completes the proof of Theorem 1.

Consequences of Theorem 1. Let f_1, f_2 be meromorphic functions.

(i) Suppose that there exist seven distinct elements $a_1, \ldots, a_7$ in $\bar{C}$ such that $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ for $i = 1, \ldots, 7$, where each k_i is either a positve integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_7$ and $k_2 \geqq 2$ if $k_1 = 0$. Then $k_1/(k_1+1) \leqq 1$ with equality holding only when $k_1 = \infty$ and $k_i/(k_i+1) \geqq \frac{1}{2}$ for $i = 2, \ldots, 7$ with $k_2/(k_2+1) \geqq \frac{2}{3}$ if $k_1 = \infty$.

Hence

$$\sum_{i=2}^{7} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} > 2\,.$$

Hence by Theorem 1, $f_1 \equiv f_2$.

In particular, with $k_1 = \ldots = k_7 = 1$, it follows that if the set of simple zeros of $f_1 - a$ coincides with the set of simple zeros of $f_2 - a$ for seven distinct values of a in $\bar{C}$ then $f_1 \equiv f_2$.

(ii) If there exist six distinct elements $a_1,\ldots,a_6$ in $\bar{C}$ such that $E(a_i,k_i,f_1) = E(a_i,k_i,f_2)$ for $i=1,\ldots,6$ where each k_i is a positive integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_6$, $k_3 \geqq 2$ and

$$\frac{k_1}{k_1+1} < \frac{k_2}{k_2+1} + \frac{1}{6}$$

(which holds, in particular, if $k_1 = k_2$) then

$$\sum_{i=2}^{6} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} > 2 .$$

Hence, by Theorem 1, $f_1 \equiv f_2$.

(iii) If there exist five distinct elements $a_1,\ldots,a_5$ in $\bar{C}$ such that $E(a_i,k_i,f_1) = E(a_i,k_i,f_2)$ for $i=1,\ldots,5$ where each k_i is a positive integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_5 \geqq 2$, $k_3 \geqq 3$ and

$$\frac{k_1}{k_1+1} < \frac{k_2}{k_2+1} + \frac{1}{12}$$

(which holds if $k_1 = k_2$), then

$$\sum_{i=2}^{5} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} > 2$$

and so $f_1 \equiv f_2$ by Theorem 1.

This includes Theorem A of Nevanlinna mentioned earlier.

(iv) If there exist five distinct elements $a_1,\ldots,a_5$ in $\bar{C}$ such that $E(a_i,k_i,f_1) = E(a_i,k_i,f_2)$ for $i=1,\ldots,5$ where each k_i is a positive integer or ∞ with $k_1 \geqq k_2 \geqq \ldots \geqq k_5$, $k_4 \geqq 4$ and

$$\frac{k_1}{k_1+1} < \frac{k_2}{k_2+1} + \frac{1}{10},$$

then, again

$$\sum_{i=2}^{5} \frac{k_i}{k_i+1} - \frac{k_1}{k_1+1} > 2$$

and so $f_1 \equiv f_2$.

REFERENCES

1. W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.
2. R. Nevanlinna, *Le théorème de Picard–Borel et la théorie des fonctions méromorphes*, Gauthier–Villars, Paris, 1929.
3. S. K. Singh and H. S. Gopalakrishna, *Exceptional values of meromorphic functions*, Math. Ann. 191 (1971), 121–142.

DEPARTMENT OF MATHEMATICS,
KARNATAK UNIVERSITY,
DHARWAR, INDIA