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CONGRUENCES FOR THE FOURIER COEFFICIENTS OF CERTAIN MODULAR FORMS

GUNNAR DIRDAL

1.

Let

$$\varphi(x) = \prod_{n=1}^{\infty} (1-x^n),$$

$$\varphi(x)^k = \sum_{n=0}^{\infty} p_k(n)x^n .$$

Then $p_{-1}(n)=p(n)$ is the number of unrestricted partitions of n. In this paper we are concerned with congruences to prime moduli, involving $p(n)$ and the Fourier coefficients of certain modular forms of half-integral dimension.

In particular, application of Theorem 1 for the primes q, $13\leqq q\leqq 23$, gives congruences of the form

$$\alpha_1 p\left(qn-\frac{q^2-1}{24}\right)+\alpha_2 p\left(\frac{n}{q}\right) \equiv \alpha_3 p_{24k-1}\left(qn-\frac{q^2-1}{24}-k\right) \pmod q ,$$

where α_1 and α_2 are not congruent zero simultaneously.

Finally, we briefly mention the results obtained when $p(n)$ is replaced by $c(n)$, the Fourier coefficients of the modular invariant $j(\tau)$.

2.

Put

$$y = e^{\pi i\tau/12}, \quad x = y^{24},$$

$$\eta(\tau) = y\varphi(x), \quad \operatorname{Im}\tau>0 .$$

The congruence properties of $p(n)$ modulo q depend on the residue character of $24n-1$ modulo q. Therefore we define, as in Kløve [5],

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$$U(q;\varepsilon)\sum_n a(n)y^n = \sum_{\left(\frac{-n}{q}\right)=\varepsilon} a(n)y^n,$$

for any power series $\Sigma_n a(n)y^n$. Here and in the following q always denote a prime >3.

We shall examine the case $\varepsilon=0$. Results similar to Theorem 1 below do exist when $\varepsilon=\pm 1$, but this will not be considered here.

Let $\Gamma_0(m)$ denote the subgroup of the full modular group $\Gamma(1)$, defined by those matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,\ b,\ c,\ d \text{ integers, } ad-bc=1\ ,$$

of $\Gamma(1)$ that satisfy $c\equiv 0 \pmod m$. Further $C^+(\Gamma_0(m),\ -k,\ \chi)$ denotes the space of modular forms of dimension $-k$, regular in the fundamental domain, $\Delta(\Gamma_0(m))$, of $\Gamma_0(m)$, except possibly at $\tau=i\infty$, $\tau=0$ and with multiplier system χ. We denote by C_q the subspace of $C^+(\Gamma_0(q),0,1)$, consisting of all modular functions which are regular at $\tau=0$. $g=g(m)$ denotes the genus of the Rieman surface $H/\Gamma_0(m)$ (compactified) where H is the upper half plane. At a given point τ_0 of the Rieman surface $H/\Gamma_0(m)$, we say that k is a gap if no function exists with a pole of order k at τ_0 and regular elsewhere on $H/\Gamma_0(m)$. Weierstrass Gap Theorem asserts that there are just g gaps k at τ_0, and that these satisfy $1\leqq k\leqq 2g-1$. Moreover, except for finitely many τ_0, the gaps are just the integers 1 to g. Those exceptional τ_0 for which this is not so are called Weierstrass point of $H/\Gamma_0(m)$ (or, loosely, of $\Gamma_0(m)$).

No value of q is yet found for which $i\infty$ is a Weierstrass point of $\Gamma_0(q)$, hence it seems that the contrary is true. In any case this is so for $q<100$ (Atkin [1], [2]).

It is a conjecture tnat the elements, f_l, in a (polynomial) basis $\mathscr{B}$ for C_q may be taken as

$$f_l = \Omega_l(W_1,\ldots,W_{(q-1)/2}),$$

where Ω_l is a cyclic $(q,0)$-isobaric polynomial and

$$W_k = x^{k(6k-q)/q}\frac{C_{4k}(x)}{C_{2k}(x)}, \qquad k\not\equiv 0 \pmod q,$$

$$C_k(x) = \prod_{n=1}^{\infty}(1-x^{qn-k})(1-x^{qn-q+k}),$$

(see Fine [4]). Hence all elements of $\mathscr{B}$ have integral Fourier coefficients at the cusps $\tau=i\infty$ and $\tau=0$. The functions W_k were introduced by Atkin and Swinnerton-Dyer [3] and studied by Fine [4]. Since $W_k=W_{-k}=W_{k+q}$, only $(q-1)/2$ of the W_k are different.

Suppose that $i\infty$ is not a Weierstrass point of $\Gamma_0(q)$. Then a basis $\mathscr{B}$ for C_q,

$$\mathscr{B} = \{f_l \mid g+1 \leqq l \leqq 2g+1\},$$

is called perfect if all the elements of $\mathscr{B}$ has a zero at $\tau=0$ and

$$f_l(\tau) = q^{\psi_1(l)} \sum_{i=-l}^{\infty} a_{i,l} x^i, \tag{2.1}$$

$$f_l\left(-\frac{1}{q\tau}\right) = q^{\theta_1(l)} \sum_{i=1}^{\infty} a_{i,l}^* x^i, \tag{2.2}$$

$$\theta(l) = \theta_1(l) - \psi_1(l) > 0, \tag{2.3}$$

where $a_{i,l}$, $a_{i,l}^*$ and $\theta_1(l), \psi_1(l)$ are integers, and

$$0 = \pi_q(a_{-l,l}).$$

π_q is a valuation, defined by

$$q^{\pi_q(a)} \mid a, \qquad q^{\pi_q(a)+1} \nmid a,$$

integer a.

To each perfect basis $\mathscr{B}$ for C_q, we associate an integer $\xi, 0 \leqq \xi \leqq g+1$. If $\theta(2g+1) > 1$, ξ is given as the smallest integer such that $\theta(l) > 1$ when $l > \xi + g$. Otherwise we put $\xi = g+1$.

We also need the following definitions to formulate the results;

$$\sigma = \frac{12}{(12, q-1)},$$

$$\varrho = \frac{q-1}{(12, q-1)},$$

$$\nu = \left[\frac{q+11}{24}\right],$$

$$\lambda = \frac{(12, q-1)(q+1)}{24},$$

$$\gamma_k = \begin{cases} 0 & \text{if } -(\lambda-1) \leqq k \leqq \lambda-1 \\ p_{2q-2\sigma(\lambda+k)}(-\varrho(\lambda+k)) & \text{if } -(\lambda+(12, q-1)) \leqq k \leqq -\lambda, \end{cases}$$

$$\lambda^* = 3\lambda - \frac{(12, q-1)}{6} \quad \text{if } q \equiv 1, 7 \pmod{12},$$

$$\zeta_k = \begin{cases} (-1)^{(q-1)/2} & \text{if } 6(\lambda-k) = (12, q-1) \\ 0 & \text{otherwise}. \end{cases}$$

Further we put $\delta_q = 0, 0, 1, 0, 6, 2, 4, 1$ if $q \equiv 1, 5, 7, 11, 13, 17, 19, 23 \pmod{24}$ and

$\mu_q = 3, 1, 2, 1$ if $q \equiv 1, 5, 7, 11 \pmod{12}$ respectively. A rational number is called q-integral if the denominator is not divisible by q.

THEOREM 1. *Let q be a prime > 3 and $\mathscr{B}$ a perfect basis for C_q. Then there exist integers a_k, not all congruent zero, such that*

$$U(q;0)\left\{a_0\eta^{-1}(\tau)+\eta^{-1}(q^2\tau)\sum_{i=1}^{\xi} a_{k_i}\gamma_{k_i}\right\}$$

$$\equiv U(q;0)\sum_{i=1}^{\xi} a_{k_i}\{\eta^{24\varrho k_i-1}(\tau)+\zeta_{k_i}\eta^{24\varrho\lambda^*-1}(\tau)\} \pmod{q},$$

where $-(\lambda+(12,q-1)) \leqq k_i \leqq \lambda-1$.

If, in particular, we choose all the k_i in Theorem 1 such that

$$-\delta_q \leqq k_i \leqq \lambda-\mu_q,$$

then there always exists a congruence similar to the one in Theorem 1. In fact

THEOREM 2. *Let q be a prime > 3. Then there exist integers a'_k, not all congruent zero, such that*

$$a'_0 U(q;0)\eta^{-1}(\tau) \equiv U(q;0)\sum_{i=1}^{\nu} a'_{k_i}\eta^{24\varrho k_i-1}(\tau) \pmod{q}$$

where $-\delta_q \leqq k_i \leqq \lambda-\mu_q$.

In [6], Kolberg proved Theorem 2 when $k_i = i$, using an identity of Ramanujan [12]. Our proof of Theorem 2 is just a slight modification of that of his.

3.

In this section we will apply Theorem 1 for the primes $q, 5 \leqq q \leqq 23$. The details in the construction of a perfect basis $\mathscr{B}$ for C_q where

(3.1) $$\xi = \begin{cases} 0 & \text{if } q=5,7,11 \\ 1 & \text{if } q=13,17,19,23 \end{cases}$$

will not be considered here, but the construction, may easily be carried out.

Hence, for $q = 5, 7, 11$ we obtain the well-known result

$$U(q;0)\eta^{-1}(\tau) \equiv 0 \pmod{q}.$$

This was first discovered and proved by Ramanujan [13], [14]. Further, when $q=13,17,19,23$ Theorem 1 and (3.1) gives

$$U(13;0)\eta^{-1}(\tau) \equiv U(13;0)\{\eta^{119}(\tau)+\eta^{455}(\tau)\} \pmod{13}$$

and

(3.2)
$$U(q;0)\{\alpha_{q,k,1}\eta^{-1}(\tau)+\alpha_{q,k,2}\eta^{-1}(q^2\tau)\}$$
$$\equiv \alpha_{q,k,3}U(q;0)\eta^{24\varrho k-1}(\tau) \pmod{q},$$

where the integers $\alpha_{q,k,i}$, $i=1,2,3$, are given by the tables 1–4.

Table 1.

k	$\alpha_{13,k,1}$	$\alpha_{13,k,2}$	$\alpha_{13,k,3}$
6	1	0	11
4	1	0	12
3	1	0	5
2	1	0	3
1	1	0	6
−1	1	0	11
−3	1	0	4
−4	1	0	9
−5	1	0	9
−6	1	0	3
−8	1	9	2
−9	0	1	7
−10	0	1	4
−11	0	1	5
−12	0	1	10
−13	0	1	6
−14	0	1	1
−16	0	1	11
−17	0	1	2
−18	0	1	2
−19	0	1	5

Table 2.

k	$\alpha_{17,k,1}$	$\alpha_{17,k,2}$	$\alpha_{17,k,3}$
1	1	0	1
−1	1	0	9
−2	1	0	8
−4	0	1	13
−5	0	1	16
−6	0	1	16
−7	0	1	12

Table 3.

k	$\alpha_{19,k,1}$	$\alpha_{19,k,2}$	$\alpha_{19,k,3}$
1	1	0	1
−2	1	0	8
−3	1	0	15
−4	1	0	11
−6	0	1	12
−7	0	1	18
−9	0	1	10
−10	0	1	18
−11	0	1	11

Table 4.

k	$\alpha_{23,k,1}$	$\alpha_{23,k,2}$	$\alpha_{23,k,3}$
−1	1	0	9
−3	0	1	16
−4	0	1	1

Some of these results are known. When $q=13$, $k=6$ (3.2) was proved by Zuckermann [15]. Kolberg [6], [7] proved (3.2) when $q=13$, $k=1$; $q=17$, $k=1,-1,-2$; $q=19$, $k=1,-2,-3,-4$; and $q=23$, $k=-1$. The other results seem to be new.

Note that not all values of k, $-(\lambda+(12,q-1))\leqq k\leqq\lambda-1$, are included in the tables above. $k=0$, $-\lambda$ are excluded since (3.2) is trivial in these cases. For the other excluded k-values we have

$$\alpha_{q,k,i} = 0, \quad \alpha_{q,k,3} = 1, \quad i=1,2\,,$$

except if $q=19$, $k=4$ where

$$U(19;0)\{\eta^{287}(\tau)-\eta^{1007}(\tau)\} \equiv 0 \pmod{19}\,.$$

When applying Theorem 1 to the primes $q>23$ further results are obtained, but ξ is probably >1 in these cases. In particular, there exist a perfect basis $\mathscr{B}$ for C_q, $q=29,31$, where $\xi=2$.

4.

If $h(\tau)$ is any function in the complex variable τ, the operator L_m, introduced by Lehner [9] for each positive integer m, is given by

$$L_m h(\tau) = m^{-1}\sum_{k=0}^{m-1} h\left(\frac{\tau+k}{m}\right). \tag{4.1}$$

It is immediately seen that L_m is linear and

$$L_m \sum_n a(n)x^n = \sum_n a(mn)x^n\,.$$

Hence we easily obtain

LEMMA 1. *If* $f(\tau)\in C^+(\Gamma_0(q^2),0,1)$ *then* $L_q f(\tau)\in C^+(\Gamma_0(q),0,1)$.

From Newman [10], [11] we have

LEMMA 2. *If*

$$\Phi_{q^i}(\tau) = \left(\frac{\eta(q^i\tau)}{\eta(\tau)}\right)^{24/(12i,q^i-1)}, \quad i=1,2\,,$$

then

$$\Phi_{q^i}(\tau)\in C^+(\Gamma_0(q^i),0,1)\,,$$

and

$$\Phi_{q^i}(-1/q^i\tau) = q^{-12i/(12i,q^i-1)}\Phi_{q^i}^{-1}(\tau)\,.$$

Put

(4.2) $$h_{u,v}(\tau) = \Phi_q^u(\tau)L_q\Phi_{q^2}^v(\tau), \qquad u \geqq 0,$$

then

LEMMA 3.
$$h_{u,v}(\tau) \in C^+(\Gamma_0(q), 0, 1),$$

and $h_{u,v}(\tau)$ has a zero of order $\varrho u - [-\varrho\lambda v/q]$ at $\tau = i\infty$ and a pole of order $\varrho\varkappa$ at $\tau = 0$ where

$$\varkappa = \begin{cases} u + \lambda v & \text{if } v \geqq 0 \\ u & \text{if } v < 0 . \end{cases}$$

PROOF. From the definition, $h_{u,v}(\tau)$ is seen to be regular for $\operatorname{Im}\tau > 0$, and from Lemma 1 and 2 we conclude that $h_{u,v}(\tau)$ is invariant on $\Gamma_0(q)$.

Now, $\Delta(\Gamma_0(q))$ has only two inequivalent cusps, viz. $\tau = i\infty$ and $\tau = 0$ with the uniformazing variables x and $e^{2\pi i(-1/q\tau)}$ respectively.

(4.2) gives
$$h_{u,v}(\tau) = \varphi(x)^{-2\sigma u}\varphi(x^q)^{2\sigma u + v}\sum_n p_{-v}(qn - q\varrho u - \varrho\lambda v)x^n ,$$

which shows that $h_{u,v}(\tau)$ has a zero of order $\varrho u - [-\varrho\lambda v/q]$ at $\tau = i\infty$.

Subjecting (4.2) to tne transformation $\tau \to -1/q\tau$, and applying Lemma 2 and (4.1) we obtain after some calculation

(4.3) $$h_{u,v}(-1/q\tau) = q^{-\sigma u}x^{-\varrho u}\varphi(x)^{2\sigma u + v}\varphi(x^q)^{-2\sigma u}\cdot$$
$$\cdot\{q^{-(v+1)}x^{-\varrho\lambda v}\varphi(x^{q^2})^{-v} + d(x)\}$$

where

$$d(x) = \begin{cases} (-1)^{(v+3)(q-1)/4 + [q/4]}q^{-(v+1)/2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{n+\varrho\lambda v}{q}\right)p_{-v}(n)x^n & \text{if } v \text{ is odd} \\ (-1)^{v(q-1)/4}q^{-v/2-1}\left(-\varphi(x)^{-v} + q\displaystyle\sum_{n=0}^{\infty}\left(1-\left(\frac{n+\varrho\lambda v}{q}\right)^2\right)p_{-v}(n)x^n\right) & \text{if } v \text{ is even ,} \end{cases}$$

in observing that

$$\Phi_{q^2}(-1/q^2\tau + k/q) = \Phi_{q^2}\left(\frac{k(\tau + k'/q) - (1 + k'k)/q}{q(\tau + k'/q) - k'}\right)$$
$$= q^{-\frac{1}{2}}\left(\frac{-k'}{q}\right)i^{(q-1)/2}\omega^{k'\varrho\lambda}\varphi(x)\varphi(\omega^{k'}x)^{-1} ,$$

where $k'k \equiv -1 \pmod q$, $1 \leqq k' \leqq q-1$ and $\omega = e^{2\pi i/q}$. Hence (4.3) completes the proof of Lemma 3.

Now, suppose that

$$\mathscr{B} = \{f_l(\tau) \mid g+1 \leqq l \leqq 2g+1\},$$

is a perfect basis for C_q where $f_l(\tau) = \Phi_q^{-1}(\tau)$ if $l = \varrho$. Let $t = k(g+1)+l$, $k \geqq 0$, and

$$F_t(\tau) = f_{g+1}^k(\tau) f_l(\tau) . \tag{4.4}$$

Hence from this and (2.2)

$$F_t(-1/q\tau) = q^{\theta_2(t)} F_t^*(\tau) ,$$

where

$$\theta_2(t) = k\theta_1(g+1) + \theta_1(l) .$$

Further, if $j = k\varrho + r \geqq g+1$, $r = 0, \ldots, \varrho - 1$, put

$$S_j(\tau) = \begin{cases} \Phi_q^{-k}(\tau) & \text{if } r=0,\ k \geqq 1 \\ F_{r+\varrho}(\tau)\Phi_q^{-k+1}(\tau) & \text{if } 0<r \leqq g,\ k \geqq 1 \\ F_r(\tau)\Phi_q^{-k}(\tau) & \text{if } g<r<\varrho,\ k \geqq 0 \end{cases} \tag{4.5}$$

and

$$S_j(-1/q\tau) = q^{\theta_3(j)} T_j(\tau) , \tag{4.6}$$

where

$$\theta_3(j) = \begin{cases} k\sigma & \text{if } r=0,\ k \geqq 1 \\ \theta_2(r+\varrho) + (k-1)\sigma & \text{if } 0<r \leqq g,\ k \geqq 1 \\ \theta_2(r) + k\sigma & \text{if } g<r<\varrho,\ k \geqq 0 . \end{cases}$$

In particular we notice that θ_3 satisfies the recursion formula

$$\theta_3(j+\varrho) = \theta_3(j) + \sigma .$$

From Lemma 2, (2.1), (2.2), (4.4)–(4.6) we obtain

LEMMA 4.

$$T_j(\tau) \in C^+(\Gamma_0(q), 0, 1)$$

and $T_j(\tau)$ *has a pole of order* j *at* $\tau = 0$ *and is otherwise regular.*

Now, Lemma 3, 4 and Weierstrass Gap Theorem assert that there exist constant b_j such that

(4.7) $$h_{0,1}(\tau) = \sum_{j=g+1}^{\varrho\lambda} b_j T_j(\tau) .$$

Subjecting this to the transformation $\tau \to -1/q\tau$, we obtain

(4.8) $$q^2 h_{0,1}(-1/q\tau) = \sum_{j=g+1}^{\varrho\lambda} b_j q^{2-\theta_3(j)} S_j(\tau) .$$

From (4.3) we see that $q^2 h_{0,1}(-1/q\tau)$ has integral coefficients in the Fourier expansion at $\tau = i\infty$. In particular

$$b_{\varrho\lambda} q^{2-\theta_3(\varrho\lambda)} = 1 ,$$

so that

(4.9) $$b_{\varrho\lambda} = q^{(q-3)/2} .$$

(4.3) also gives

$$q^2 h_{0,1}(-1/q\tau) \equiv \Phi_{q^2}^{-1}(\tau) \pmod q ,$$

and since

$$\varphi(x^q) \equiv \varphi(x)^q \pmod q ,$$

we obtain

$$\Phi_{q^2}^{-1}(\tau) \equiv \Phi_q^{-\lambda}(\tau) \pmod q .$$

Together with (4.5) this gives

$$q^2 h_{0,1}(-1/q\tau) \equiv S_{\varrho\lambda}(\tau) \pmod q .$$

Hence there exists a function $h(\tau)$ with integral Fourier coefficients at $\tau = i\infty$, such that

$$q^2 h_{0,1}(-1/q\tau) = S_{\varrho\lambda}(\tau) + qh(\tau) .$$

Thus by (4.8) and (4.9)

(4.10) $$h(\tau) = \sum_{j=g+1}^{\varrho\lambda-1} b_j q^{1-\theta_3(j)} S_j(\tau) .$$

Put

$$\psi_2(t) = k\psi_1(g+1) + \psi_1(l) ,$$

$$\psi_3(j) = \begin{cases} 0 & \text{if } r=0,\ k \geqq 1 \\ \psi_2(r+\varrho) & \text{if } 0<r\leqq g,\ k\geqq 1 \\ \psi_2(r) & \text{if } g<r<\varrho,\ k\geqq 0 , \end{cases}$$

and

$$\theta(j) = \theta_3(j) - \psi_3(j),$$

where $t = k(g+1)+l$ and $j = k\varrho + r$. Note that the last identity agrees with (2.3) when $g+1 \leqq j \leqq 2g+1$.

If

$$S_j(\tau) = \sum_{i=-j}^{\infty} a_{i,j} x^i, \qquad a_{i,j} \text{ integers},$$

and

$$a_{-j,j} = q^{\pi_q(a_{-j,j})} a'_{-j,j},$$

then (4.10) asserts that all the

$$b_j q^{1-\theta_3(j)} a_{-j,j} = b_j q^{1-\theta(j)} a'_{-j,j}, \qquad j = g+1, \ldots, \varrho\lambda - 1,$$

are q-integral.

Now, $\theta(j) > 0$ when $g+1 \leqq j \leqq 2g+1$, thus $\theta(j) > 0$ for all $j \geqq g+1$. Hence we conclude that

(4.11) $$b_j = q^{\theta(j)-1} b'_j, \qquad j = g+1, \ldots, \varrho\lambda - 1,$$

where b'_j are q-integral.

Together with (4.7), (4.9), and (4.11) this gives

(4.12) $$h_{0,1}(\tau) = \sum_{j=g+1}^{\varrho\lambda-1} b'_j q^{\theta(j)-1} T_j(\tau) + q^{(q-3)/2} T_{\varrho\lambda}(\tau).$$

Let

$$\zeta_u^* = \begin{cases} 0 & \text{if } q \equiv 1 \pmod{12} \text{ and } u = 1 \\ (-1)^{(q-1)/2} q^{\sigma u - 2} & \text{otherwise}, \end{cases}$$

(4.13) $$H_u(\tau) = h_{u,-2\sigma u+2}(\tau) + \zeta_u^* \Phi_q^u(\tau),$$

and

$$A_q = \{k \mid 1 \leqq k \leqq 2\lambda + (12, q-1)\}.$$

If $u \in A_q$, Lemma 3, 4, (4.3) and Weierstrass Gap Theorem assert that there exist constants $b_{j,u}$ such that

$$H_u(\tau) = \sum_{j=g+1}^{\varrho u} b_{j,u} T_j(\tau) + \gamma_{\lambda-u}.$$

When subjecting this to the transformation $\tau \to -1/q\tau$ and observing from (4.3) that the coefficients in the Fourier expansion of $qH_u(-1/q\tau)$ at $\tau = i\infty$ are integers, we conclude that

$$b_{j,u} = q^{\theta(j)-1} b'_{j,u} ,$$

where $b'_{j,u}$ are q-integral. Hence

(4.14) $$H_u(\tau) - \gamma_{\lambda-u} = \sum_{j=g+1}^{\varrho u} b'_{j,u} q^{\theta(j)-1} T_j(\tau), \quad \text{if } u \in A_q .$$

From the definition of $\theta(j)$ and ξ, it is clear that

$$\theta(j) > 1 \quad \text{for all } j > \xi + g .$$

By (4.12) and (4.14), we thus have

$$h_{0,1}(\tau) \equiv \sum_{j=g+1}^{\xi+g} b'_j T_j(\tau) \pmod q ,$$

$$H_u(\tau) - \gamma_{\lambda-u} \equiv \sum_{j=g+1}^{\xi+g} b'_{j,u} T_j(\tau) \pmod q, \quad u \in A_q ,$$

where b'_j and $b'_{j,u}$ are q-integral. Hence, by suitable choice of the integers a_k we obtain

$$a_0 h_{0,1}(\tau) \equiv \sum_{i=1}^{\xi} a_{\lambda-k'_i}\{H_{k'_i}(\tau) - \gamma_{\lambda-k'_i}\} \pmod q ,$$

where $k'_i \in A_q$. This together with (4.2) and (4.13) gives

$$a_0 L_q \Phi_{q^2}(\tau)$$

$$\equiv \sum_{i=1}^{\xi} a_{\lambda-k'_i}\{\Phi_q^{k'_i}(\tau) L_q \Phi_{q^2}^{-2\sigma k'_i+2}(\tau) + \zeta_{\lambda-k'_i} \Phi_q^{k'_i}(\tau) - \gamma_{\lambda-k'_i}\} \pmod q ,$$

so that

$$a_0 \varphi(x)^q \sum_n p(qn - \varrho\lambda) x^n$$

$$\equiv \sum_{i=1}^{\xi} a_{\lambda-k'_i}\{\varphi(x)^{2q-2\sigma k'_i} \sum_n p_{2\sigma k'_i-2}(qn + \varrho k'_i - 2\varrho\lambda) x^n +$$

$$+ \zeta_{\lambda-k'_i} x^{\varrho k'_i} \varphi(x)^{24\varrho k'_i} - \gamma_{\lambda-k'_i}\} \pmod q .$$

Theorem 1 follows immediately if we put $k_i = \lambda - k'_i$ and observe that

$$\varphi(x)^s \sum_n p_k(qn+m) x^n \equiv \sum_n p_{k+qs}(qn+m) x^n \pmod q ,$$

and

(4.15) $$U(q;0)\eta^k(\tau) = x^{k/24} \sum_n p_k(qn + k\varrho\lambda) x^{qn+k\varrho\lambda} .$$

Now, we turn to the proof of Theorem 2. Let

$$E_{a,b}(x) = \sum_{u,v=1}^{\infty} u^a v^b x^{uv},$$

$$Q = 1+240E_{0,3}, \quad R = 1-504E_{0,5},$$

$$D = 12^3F = Q^3-R^2, \quad j = 12^3J = Q^3F^{-1}.$$

Kolberg [6] has shown

$$D^{-[q/12]} \equiv \begin{array}{lll} f(J) & q\equiv 1 & (\mathrm{mod}\ 12) \\ Qf(J) & q\equiv 5 & (\mathrm{mod} 12) \\ Rf(J) & q\equiv 7 & (\mathrm{mod}\ 12) \\ QRf(J) & q\equiv 11 & (\mathrm{mod}\ 12), \end{array}$$

where $f(J)$ is a polynomial in J of
degree $[q/12]$ and with integral coefficients. Further

$$D^{kq-s} \equiv V^q G_k(J)\delta J \quad (\mathrm{mod}\ q), \tag{4.16}$$

where $24s\equiv 1 \pmod q$, $0<s<q$,

$$\delta = x\frac{d}{dx},$$

and for $q\equiv 1,5,7,11,13,17,19,23 \pmod{24}$ respectively

$$-V = \begin{cases} Q^{-2}R^{-1}D^{\nu} \\ Q^{-1}R^{-1}D^{\nu} \\ Q^{-2}R^{-1}D^{3\nu+1}(J-1)f(J) \\ Q^{-1}R^{-1}D^{7\nu+3}J(J-1)^2f(J)^3 \\ Q^{-2}R^{-1}D^{\nu} \\ Q^{-1}R^{-1}D^{\nu} \\ Q^{-2}R^{-1}D^{3\nu}(J-1)f(J) \\ Q^{-1}R^{-1}D^{7\nu}J(J-1)^2f(J) \end{cases}$$

$$G_k(J) = \begin{cases} J^{16\nu}(J-1)^{12\nu}f(J)^{12\nu-2-k} \\ J^{12\nu+1-k}(J-1)^{12\nu+2}f(J)^{12\nu-3k} \\ J^{16\nu+4}(J-1)^{6\nu-k}f(J)^{12\nu+1-2k} \\ J^{12\nu+4-2k}(J-1)^{6\nu+1-3k}f(J)^{12\nu+3-6k} \\ J^{16\nu-8}(J-1)^{12\nu-6}f(J)^{12\nu-8-k} \\ J^{12\nu-5-k}(J-1)^{12\nu-4}f(J)^{12\nu-6-3k} \\ J^{16\nu-4}(J-1)^{6\nu-3-k}f(J)^{12\nu-5-2k} \\ J^{12\nu-2-2k}(J-1)^{6\nu-2-3k}f(J)^{12\nu-3-6k} \end{cases}$$

If

$$B_q = \{k \mid -\delta_q \leqq k \leqq \lambda - \mu_q\}$$

and $k \in B_q$, it is easily seen that the degree of $G_k(J)$ is less than $(\nu+1)q-1$. Hence

$$G_k(J)\delta J \equiv \delta P_k(J) + \sum_{i=1}^{\nu} c_{i,k} J^{iq-1}\delta J \pmod q,$$

where $k \in B_q$ and $P_k(J)$ is a polynomial with integral coefficients. Thus, by suitable choice of a''_{k_i} we obtain

$$\sum_{i=0}^{\nu} a''_{k_i} G_{k_i}(J)\delta J \equiv \delta P(J), \qquad k_i \in B_q,$$

and by (4.16)

$$\sum_{i=0}^{\nu} a''_{k_i} D^{k_i\varrho - s} \equiv V^q \delta P(J) \pmod q.$$

Hence with $a'_k = 12^{3k\varrho - 3s} a''_k$ and $k_0 = 0$ we get

$$a'_0 x^{-s} \varphi(x)^{-1}$$

$$\equiv \sum_{i=1}^{\nu} a'_{k_i} x^{k_i\varrho - s} \varphi(x)^{24k_i\varrho - 1} + \varphi(x)^{24s-1} V^q \delta P(J) \pmod q,$$

and Theorem 2 follows from this and (4.15) in observing that

$$U(q;0)\varphi(x)^{24s-1} V^q \delta P(J) \equiv 0 \pmod q.$$

5.

In [8] Kolberg proved a result akin to Theorem 2 when $k_i = i$ involving $U(q;0)j(\tau)$ instead of $U(q;0)\eta^{-1}(\tau)$. Here we mention the results obtained with this change. Let

$$j(\tau) = \sum_{i=-1}^{\infty} c_i x^i,$$

$$M_q = \{k \mid -(12, q-1) \leqq k \leqq 2\lambda - 1\},$$

$$N_q = \{k \mid -(12, q-1) \leqq k \leqq 2\lambda - \mu_q\}.$$

If $\mathscr{B}$ is a perfect basis for C_q we may prove quite similar as for Theorem 1 that there exist integers e_k, not all congruent zero, such that

$$(5.1) \qquad e_0 U(q;0)(j(\tau)-c_0)$$
$$\equiv U(q;0) \sum_{i=1}^{\xi} e_{k_i}(\eta^{24\varrho k_i}(\tau)+\zeta_{k_i-\lambda}\eta^{4q(q-1)}(\tau)-\gamma_{k_i-\lambda}) \pmod q ,$$

where $k_i \in M_q$.

In particular, if we choose all $k_i \in N_q$ and $\xi = [q/12]$ then (5.1) is always satisfied.

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