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ON THE LINEAR PREDICTION PROBLEM OF CERTAIN NON-STATIONARY STOCHASTIC PROCESSES

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1. Introduction.

Tjöstheim and Thomas [12] have introduced the class of uniformly bounded linearly stationary stochastic processes. This class consists, in general, of non-stationary stochastic processes for which the shift operator group T_s, $s \in \mathsf{R}$, of the process is well-defined and uniformly bounded, that is, $\|T_s\| \leqq M$, $s \in \mathsf{R}$, for some constant $M>0$. In this paper the spectral properties and the linear prediction problem of these stochastic processes are studied.

It is shown here that every continuous uniformly bounded linearly stationary stochastic process $x(t)$, $t \in \mathsf{R}$, is V-bounded, that is, x has a spectral representation

$$x(t) = \int e^{it\lambda}\, d\mu(\lambda), \qquad t \in \mathsf{R}\,,$$

where μ is a bounded stochastic measure. Moreover, it is shown that the linear prediction problem of a uniformly bounded linearly stationary stochastic process is analogous to the linear prediction problem of a wide sense stationary stochastic process.

In section 4 the linear prediction problem of harmonizable uniformly bounded linearly stationary stochastic processes is studied. A necessary and sufficient condition for such a stochastic process to be purely non-deterministic (respectively deterministic) is derived. The conditions presented here have the same form as the corresponding conditions concerning a continuous wide sense stationary stochastic process. These results give a solution, in a special case, to the linear prediction problem of harmonizable stochastic processes considered e.g. by Cramér [2].

2. Uniformly bounded linearly stationary stochastic processes.

Let $(\Omega, \mathscr{A}, P)$ be a probability space. By $L_0^2(\Omega, \mathscr{A}, P)$ we denote the linear

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space of (equivalence classes of) complex-valued random variables ξ defined on $(\Omega, \mathscr{A}, P)$ for which

$$E\xi = 0 \quad \text{and} \quad E|\xi|^2 < \infty .$$

The space $L_0^2(\Omega, \mathscr{A}, P)$ is a Hilbert space if the norm and the inner product are defined in the well-known way.

In the following the underlying probability space $(\Omega, \mathscr{A}, P)$ is defined only through $L_0^2(\Omega, \mathscr{A}, P)$.

Following Tjöstheim and Thomas [12] we call a stochastic process $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ *uniformly bounded linearly stationary*, if there exists a constant $M > 0$ such that

$$\left\| \sum_{k=1}^{m} a_k x(t_k + h) \right\| \leqq M \left\| \sum_{k=1}^{m} a_k x(t_k) \right\| \tag{1}$$

for all $h, t_k \in \mathbf{R}$, $a_k \in \mathbf{C}$, $k = 1, \ldots, m$, $m \in \mathbf{N}$.

Let $\overline{\text{sp}}\,\{x\}$ be the closure in $L_0^2(\Omega, \mathscr{A}, P)$ of the linear span, $\text{sp}\,\{x\}$, of the set $\{x(t) \mid t \in \mathbf{R}\}$. If x is a uniformly bounded linearly stationary stochastic process, then for every $h \in \mathbf{R}$ there exists a bounded linear operator $T_h\colon \overline{\text{sp}}\,\{x\} \to \overline{\text{sp}}\,\{x\}$ such that

$$T_h\left(\sum_{k=1}^{m} a_k x(t_k) \right) = \sum_{k=1}^{m} a_k x(t_k + h)$$

for all elements $\sum_{k=1}^{m} a_k x(t_k) \in \text{sp}\,\{x\}$.

Moreover, if $M > 0$ is a constant for which the condition (1) is satisfied, then

$$\|T_h\| \leqq M \quad \text{for all } h \in \mathbf{R}$$

(Getoor [3], Tjöstheim and Thomas [12]). Clearly,

$$T_0 = I, \quad T_s T_t = T_{s+t} \quad \text{for all } s, t \in \mathbf{R}$$

(I stands for the identity operator). Thus the operators T_h, $h \in \mathbf{R}$, form a commutative group. Following Getoor [3] we call the group T_h, $h \in \mathbf{R}$, the *shift operator group* of the stochastic process x. (Note that in the present case $T_{-h} = T_h^{-1}$ for all $h \in \mathbf{R}$.)

In this paper we call a stochastic process $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ *wide sense stationary*, if the covariance function $r(s,t) = E\,x(s)\overline{x(t)}$, $s, t \in \mathbf{R}$, of x has the form

$$E\,x(s)\overline{x(t)} = (x(s) \mid x(t)) = r(s-t) \quad \text{for all } s, t \in \mathbf{R} .$$

A stochastic process $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is wide sense stationary if and only if the shift operator group T_h, $h \in \mathbf{R}$, of x is a group of unitary operators.

Definition 1. *Let* $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ *be a stocnastic process. A couple* (y,B) *consisting of a wide sense stationary stochastic process* $y\colon \mathsf{R} \to L_0^2(\Omega',\mathscr{A}',P')$ *and a bounded linear operator* $B\colon \overline{\mathrm{sp}}\{y\} \to \overline{\mathrm{sp}}\{x\}$ *with a bounded inverse* B^{-1} *is called a stationary similarity of* x, *if*

$$x(t) = By(t) \quad \textit{for all } t \in \mathsf{R}\,.$$

Remark 2. A stochastic process $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ is uniformly bounded linearly stationary, if there exists at least one stationary similarity of x. Conversely, suppose $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ is a uniformly bounded linearly stationary stochastic process. Then there exists a stationary similarity (y,B) of x such that the random variables $y(t)$, $t \in \mathsf{R}$, are defined on $(\Omega,\mathscr{A},P)$, that is, $y(t) \in L_0^2(\Omega,\mathscr{A},P)$, $t \in \mathsf{R}$, and B is a bounded self-adjoint operator from $\overline{\mathrm{sp}}\{y\}$ onto $\overline{\mathrm{sp}}\{y\}$; in this case $\overline{\mathrm{sp}}\{y\} = \overline{\mathrm{sp}}\{x\}$ (Tjöstheim and Thomas [12], Theorems 1 and 2. See also Martin [5]). Moreover, it follows from Theorem 1 in [10] and from the method used in the proof of Theorem 2 in [12] that the self-adjoint operator $B\colon \overline{\mathrm{sp}}\{x\} \to \overline{\mathrm{sp}}\{x\}$ can be chosen such that $1/M \leqq \|B\| \leq M$ for any constant $M > 0$ satisfying the condition (1).

We recall that a stochastic process $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ is called (*q.m.*) *continuous*, if the mapping $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ is continuous. Let $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ be a uniformly bounded linearly stationary stochastic process and let T_h, $h \in \mathsf{R}$, be the shift operator group of x. Then x is continuous if and only if for all $z \in \overline{\mathrm{sp}}\{x\}$ the vector valued functions $T_h z$, $h \in \mathsf{R}$, are continuous.

The following lemma is obvious.

Lemma 3. *Suppose* $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ *is a continuous stochastic process. Then for any stationary similarity* (y,B) *of* x *the wide sense stationary stochastic process* y *is continuous. If there exists such a stationary similarity* (y,B) *of* x *that* y *is continuous, then* x *is continuous.*

Next we show that every continuous uniformly bounded linearly stationary stochastic process is V-bounded (for the definition of a V-bounded stochastic process see Bochner [1; p. 18] or [6; p. 34]).

We recall that a continuous stochastic process $x\colon \mathsf{R} \to L_0^2(\Omega,\mathscr{A},P)$ is V-bounded if and only if there exists a bounded (uniquely determined) vector measure $\mu\colon C_0(\mathsf{R}) \to L_0^2(\Omega,\mathscr{A},P)$ such that

$$x(t) = \int e^{it\lambda}\, d\mu(\lambda) \quad \text{for all } t \in \mathsf{R}; \tag{2}$$

here $C_0(\mathsf{R})$ is the linear space of all continuous functions $f\colon \mathsf{R} \to \mathsf{C}$ vanishing at infinity (see [6; Theorem 3.2.1] or Kluvánek [4; Theorem 2]. For a more general treatment see Ylinen [13]). The bounded vector measure μ on the right hand side of the representation (2) is called the *spectral measure* of the continuous V-bounded stochastic process x. (In this paper we use the integration technique of vector measures introduced by Thomas [11].)

THEOREM 4. *Every continuous uniformly bounded linearly stationary stochastic process $x: \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is V-bounded. Let (y, B) be any stationary similarity of x. Then the spectral measure μ_x of x can be represented in the form*

$$\mu_x = B \circ \mu_y, \tag{3}$$

where μ_y is the spectral measure of the continuous wide sense stationary stochastic process y.

PROOF. Since x is a uniformly bounded linearly stationary stochastic process there exists a stationary similarity (y, B) of x. Since x is continuous, the wide sense stationary stochastic process y is, by Lemma 3, continuous. Thus there exists a uniquely determined bounded vector measure $\mu_y: C_0(\mathbf{R}) \to \overline{\mathrm{sp}}\{y\}$ for which $(\mu_y(f) \mid \mu_y(g)) = 0$ for all $f, g \in C_0(\mathbf{R})$ with compact and disjoint supports and

$$y(t) = \int e^{it\lambda}\, d\mu_y(\lambda) \quad \text{for all } t \in \mathbf{R}$$

(see [7; Theorem 8]). Define $\mu_x: C_0(\mathbf{R}) \to \overline{\mathrm{sp}}\{x\}$ by setting

$$\mu_x = B \circ \mu_y .$$

Then μ_x is a bounded vector measure and by using a result of Thomas [11; pp 78–79] we get

$$x(t) = By(t) = B\left(\int e^{it\lambda}\, d\mu_y(\lambda)\right) = \int e^{it\lambda}\, dB \circ \mu_y(\lambda)$$

for all $t \in \mathbf{R}$. Thus x is V-bounded and $\mu_x = B \circ \mu_y$ is the spectral measure of x.

REMARK 5. There exist continuous V-bounded stochastic processes $x: \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ for which the shift operator group T_h, $h \in \mathbf{R}$, of x cannot be defined as a group of bounded linear operators (see Remark 15).

For the sake of completeness we briefly consider the discrete-time case, that is, the case where the parameter set of the process is the set of all integers. Suppose for a stochastic process $x: \mathbf{Z} \to L_0^2(\Omega, \mathscr{A}, P)$ a condition analogous to (1) is satisfied for some constant $M > 0$. In this case there exists a bounded linear operator $T\colon \overline{\mathrm{sp}}\{x\} \to \overline{\mathrm{sp}}\{x\}$ for which $\|T^h\| \leqq M$ for all $h \in \mathbf{Z}$ $(T^0 = I)$; and for any $h \in \mathbf{Z}$ we have

$$T^h\left(\sum_{j=1}^{m} a_j x(k_j)\right) = \sum_{j=1}^{m} a_j x(k_j + h)$$

for all elements $\Sigma_{j=1}^{m} a_j x(k_j) \in \mathrm{sp}\{x\}$.

It follows from Theorem 1 in Tjöstheim and Thomas [12] and Theorem 1 in Sz.-Nagy [10] that the results stated in Remark 2 are valid in the discrete-time case. Moreover, Theorem 4 is valid in the discrete-time case with an obvious modification.

3. On the linear prediction problem of uniformly bounded linearly stationary stochastic processes.

Let $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a stochastic process. For $t \in \mathbf{R}$ we denote by $\overline{\mathrm{sp}}\{x;\, t\}$ the closed linear subspace in $L_0^2(\Omega, \mathscr{A}, P)$ spanned by the set $\{x(s) \mid s \leqq t\}$. Moreover, we use the notation

$$\overline{\mathrm{sp}}\{x;\; -\infty\} = \bigcap_{t \in \mathbf{R}} \overline{\mathrm{sp}}\{x;\, t\}\,.$$

We recall that a stochastic process $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is called *purely non-deterministic*, if $\overline{\mathrm{sp}}\{x;\; -\infty\} = \{0\}$; and *deterministic*, if $\overline{\mathrm{sp}}\{x;\; -\infty\} = \overline{\mathrm{sp}}\{x\}$.

The proof of the following lemma is obvious and is therefore omitted.

LEMMA 6. *Let $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ and $y\colon \mathbf{R} \to L_0^2(\Omega', \mathscr{A}', P')$ be stochastic processes. Suppose there exists a bounded linear operator $A\colon \overline{\mathrm{sp}}\{y\} \to \overline{\mathrm{sp}}\{x\}$ with a bounded inverse $A^{-1}\colon \overline{\mathrm{sp}}\{x\} \to \overline{\mathrm{sp}}\{y\}$ such that*

$$x(t) = Ay(t) \quad \textit{for all } t \in \mathbf{R}\,.$$

Then for any $t \in \mathbf{R}$ one has

$$\overline{\mathrm{sp}}\{x;\, t\} = A\,\overline{\mathrm{sp}}\{y;t\}, \qquad \overline{\mathrm{sp}}\{y;\, t\} = A^{-1}\,\overline{\mathrm{sp}}\{x;\, t\};$$

and

$$\overline{\mathrm{sp}}\{x;\; -\infty\} = A\,\overline{\mathrm{sp}}\{y;\; -\infty\}, \qquad \overline{\mathrm{sp}}\{y;\; -\infty\} = A^{-1}\,\overline{\mathrm{sp}}\{x;\; -\infty\}\,.$$

The following theorem is a direct consequence of Lemma 6.

THEOREM 7. *Let $x\colon \mathbf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a uniformly bounded linearly stationary stochastic process. If there exists such a stationary similarity (y, B) of x that y is purely non-deterministic (respectively deterministic), then x is purely* *non-deterministic (respectively deterministic). If x is purely non-deterministic (respectively deterministic), then for all stationary similarities (y, B) of x the stochastic process y is purely non-deterministic (respectively deterministic).*

Let $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a stochastic process. By $P_{-\infty}$ we denote the orthogonal projection of $\overline{\mathrm{sp}}\{x\}$ to $\overline{\mathrm{sp}}\{x; -\infty\}$. Consider the decomposition

$$x(t) = x_1(t) + x_2(t), \qquad t \in \mathsf{R},$$

where $x_2(t) = P_{-\infty}x(t)$ and $x_1(t) = x(t) - x_2(t)$, $t \in \mathsf{R}$. Then

(i) $x_1: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is purely non-deterministic,
(ii) $x_2: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is deterministic and
(iii) $(x_1(t) \mid x_2(t)) = 0$ for all $t \in \mathsf{R}$.

If x is a (continuous) wide sense stationary stochastic process, then x_1 and x_2 are (continuous) wide sense stationary stochastic processes.

THEOREM 8. *Let $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a uniformly bounded linearly stationary stochastic process and let T_t, $t \in \mathsf{R}$, be the shift operator group of x. Let (y, B) be a stationary similarity of x. If*

$$y(t) = y_1(t) + y_2(t), \qquad t \in \mathsf{R};$$

where $y_2(t) = P_{-\infty}y(t)$, $y_1(t) = y(t) - y_2(t)$, $t \in \mathsf{R}$, then the stochastic processes

$$x_k(t) = By_k(t), \qquad t \in \mathsf{R},\ k = 1, 2,$$

are uniformly bounded linearly stationary; $x(t) = x_1(t) + x_2(t)$, $t \in \mathsf{R}$;

$$x_k(t+h) = T_h x_k(t), \qquad t, h \in \mathsf{R},\ k = 1, 2;$$

(y_k, B) is a stationary similarity of x_k, $k = 1, 2$; x_1 is purely non-deterministic and x_2 is deterministic. If x is continuous, then x_1 and x_2 are continuous.

PROOF. Since T_t, $t \in \mathsf{R}$, is the shift operator group of x the group $U_t = B^{-1}T_tB$, $t \in \mathsf{R}$, (of unitary operators) is the shift operator group of y. Since

$$y_2(t) = P_{-\infty}y(t) \quad \text{and} \quad y_1(t) = y(t) - y_2(t), \qquad t \in \mathsf{R},$$

the stochastic processes y_1 and y_2 are wide sense stationary and

$$y_k(t+h) = U_h y_k(t), \qquad t, h \in \mathsf{R},\ k = 1, 2$$

(Rozanov [9; pp. 54–55]). It follows that the stochastic processes

$$x_k(t) = By_k(t), \qquad t \in \mathsf{R},\ k = 1, 2,$$

are uniformly bounded and linearly stationary. Furthermore, T_t, $t \in \mathsf{R}$, is the shift operator group of x_1 and x_2. Clearly, (y_k, B) is a stationary similarity of x_k, $k=1,2$.

Since y_1 is purely non-deterministic and since y_1 is a stationary similarity of the uniformly bounded linearly stationary stochastic process x_1 we get, by applying Theorem 7, that x_1 is purely non-deterministic. In a similar way we see that x_2 is deterministic.

Clearly, the continuity of x implies the continuity of x_k, $k=1,2$.

The theorem is proved.

REMARK 9. Let $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a uniformly bounded linearly stationary stochastic process and let the purely non-deterministic (respectively deterministic) uniformly bounded stationary stochastic process x_1 (respectively x_2) be defined as in Theorem 8. In general, we then have

$$(x_1(t) \mid x_2(t)) \neq 0, \qquad t \in \mathsf{R}.$$

REMARK 10. Let $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a uniformly bounded linearly stationary stochastic process. For $t \in \mathsf{R}$ and $h>0$ we denote by $\hat{x}(t; h)$ the projection of $x(t+h)$ to $\overline{\mathrm{sp}}\{x; t\}$, that is, $\hat{x}(t; h)$ is the best linear least-square prediction of $x(t+h)$ in terms of the development of x up to and including the time t. Let (y, B) be a stationary similarity of x and let $\hat{y}(t; h)$ be defined in the similar way as $\hat{x}(t; h)$. Then

$$\begin{aligned} \|\hat{x}(t; h)-x(t+h)\| &\leqq \|B\hat{y}(t; h)-x(t+h)\| \\ &\leqq \|B\| \, \|\hat{y}(t; h)-y(t+h)\| \end{aligned}$$

and

$$\|\hat{x}(t; h)-x(t+h)\| \geqq \|B^{-1}\| \, \|\hat{y}(t; h)-y(t+h)\| .$$

Moreover, according to Remark 2, there exists a stationary similarity (y, B) of x such that

$$M^{-1}\|\hat{y}(t;h)-y(t+h)\| \leqq \|\hat{x}(t;h)-x(t+h)\| \leqq M\|\hat{y}(t;h)-y(t+h)\|$$

for any $M>0$ satisfying the condition (1).

4. Linear prediction of harmonizable uniformly bounded linearly stationary stochastic processes.

In this section we present a method to construct a stationary similarity for certain uniformly bounded linearly stationary stochastic processes. After that we show that by applying this method we get a necessary and sufficient condition for these stochastic processes and especially for harmonizable uniformly bounded linearly stationary stochastic processes to be purely non-deterministic (respectively deterministic).

Following Rozanov [8] we say that a stochastic process $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ has a *spectrum*, if the scalar function $\|x(t)\|^2$, $t \in \mathsf{R}$, is integrable on every finite interval, if the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (x(t+\tau) \mid x(t))\, dt = S(\tau) \tag{4}$$

exists for all $\tau \in \mathsf{R}$ and if the limit function $S: \mathsf{R} \to \mathsf{C}$ is continuous. We recall that the limit function $S: \mathsf{R} \to \mathsf{C}$ is, provided that it exists, positive definite, that is,

$$\sum_{j=1}^{m} \sum_{k=1}^{m} a_j \overline{a_k} S(t_j - t_k) \geqq 0$$

for all $t_j \in \mathsf{R}$, $a_j \in \mathsf{C}$, $j=1,\ldots,m$, $m \in \mathsf{N}$ (Rozanov [8]).

The proof of the following theorem is essentially based on the proof of Theorem 1 in Sz.-Nagy [10].

THEOREM 11. *Let* $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ *be a uniformly bounded linearly stationary stochastic process. If the limit* $S(\tau)$ *defined in* (4) *exists for all* $\tau \in \mathsf{R}$, *then there exists a stationary similarity* (y, B) *of* x *where* y *is such a wide sense stationary stochastic process that*

$$(y(s) \mid y(t)) = S(s-t) \quad \text{for all } s, t \in \mathsf{R};$$

and $\overline{\mathrm{sp}}\,\{y\} = \overline{\mathrm{sp}}\,\{x\}$. *The operator* $B: \overline{\mathrm{sp}}\,\{x\} \to \overline{\mathrm{sp}}\,\{x\}$ *is self-adjoint and*

$$\frac{1}{M} \leqq \|B\| \leqq M$$

for any constant $M > 0$ *satisfying the conditon* (1).

PROOF. Recall that $\mathrm{sp}\,\{x\}$ stands for the linear span of the set $\{x(t) \mid t \in \mathsf{R}\}$ We define a mapping $K: \mathrm{sp}\,\{x\} \times \mathrm{sp}\,\{x\} \to \mathsf{C}$ by

$$K(z_1, z_2) = \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} a_{1j} \overline{a_{2k}} S(t_{1j} - t_{2k}),$$

where

$$z_h = \sum_{j=1}^{m_h} a_{hj} x(t_{hj}) \in \mathrm{sp}\,\{x\}, \quad h=1,2.$$

Clearly,

$$
\begin{aligned}
K(z_1,z_2) &= \lim_{T\to\infty} \sum_{j=1}^{m_1}\sum_{k=1}^{m_2} a_{1j}\overline{a_{2k}}\,\frac{1}{T}\int_0^T (x(t+t_{1j}-t_{2k})\,|\,x(t))\,dt \\
&= \lim_{T\to\infty} \sum_{j=1}^{m_1}\sum_{k=1}^{m_2} a_{1j}\overline{a_{2k}}\,\frac{1}{T}\int_{-t_{2k}}^{T-t_{2k}} (x(t+t_{1j})\,|\,x(t+t_{2k}))\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T (T_t z_1\,|\,T_t z_2)\,dt\,.
\end{aligned}
$$

Thus K is bilinear and Hermitian.

Let $M>0$ be a constant for which the condition (1) is satisfied and let $z\in \mathrm{sp}\{x\}$. Then for all $s\in \mathbf{R}$ we get

$$
\|T_s z\| \leqq M\|z\|; \qquad \text{and} \qquad \frac{1}{M}\|z\| \leqq \|T_s z\|\,,
$$

since $\|z\|=\|T_{-s}T_s z\|\leqq M\|T_s z\|$. Therefore for all $z_1,z_2\in \mathrm{sp}\{x\}$ we have

$$
\text{(5)}\qquad |K(z_1,z_2)| \leqq \lim_{T\to\infty}\frac{1}{T}\int_0^T |(T_t z_1\,|\,T_t z_2)|\,dt \leqq M^2\|z_1\|\,\|z_2\|\,.
$$

Moreover, for any $z\in \mathrm{sp}\{x\}$

$$
\text{(6)}\qquad K(z,z) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \|T_t z\|^2\,dt \geqq \frac{1}{M^2}\|z\|^2\,.
$$

From the inequality (5) it follows that the bilinear mapping $K\colon \mathrm{sp}\{x\}\times \mathrm{sp}\{x\}\to \mathbf{C}$ is continuous (when $\mathrm{sp}\{x\}$ carries the norm topology). Thus K can be extended by continuity to a continuous bilinear form $\hat{K}\colon \overline{\mathrm{sp}}\{x\}\times\overline{\mathrm{sp}}\{x\}\to \mathbf{C}$. Since K is Hermitian the extended bilinear form is Hermitian. Thus there exists a bounded self-adjoint operator $A\colon \overline{\mathrm{sp}}\{x\}\to\overline{\mathrm{sp}}\{x\}$ such that

$$
\hat{K}(z_1,z_2) = (Az_1\,|\,z_2) \qquad \text{for all } z_1,z_2\in \overline{\mathrm{sp}}\{x\}\,.
$$

Moreover, by applying (5) and (6) we get

$$
\frac{1}{M^2}I \leqq A \leqq M^2 I\,.
$$

Therefore the self-adjoint operator $Q=A^{\frac{1}{2}}$ exists and we have

$$
\frac{1}{M}I \leqq Q \leqq MI\,.
$$

Let the stochastic process $y\colon \mathbf{R}\to\overline{\mathrm{sp}}\{x\}$ be defined by

$$
y(t) = Qx(t), \qquad t\in \mathbf{R}\,.
$$

Then for all $s, t \in \mathsf{R}$ we get

$$(y(s) \mid y(t)) = (Qx(s) \mid Qx(t)) = (Ax(s) \mid x(t))$$
$$= K(x(s), x(t)) = S(s-t),$$

which proves that y is a wide sense stationary stochastic process. Denote $B = Q^{-1}$; then the pair (y, B) is a stationary similarity of x satisfying the required conditions.

REMARK 12. Suppose $x\colon \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is a continuous uniformly bounded linearly stationary stochastic process and suppose the limit function S, defined in (4) exists. Then, by Theorem 11, there exists a stationary similarity (y, B) of x such that S is the covariance function of y. Thus S is continuous, since y is by Lemma 2 continuous. Moreover, since S is positive definite (Rozanov [8]), there exists a uniquely determined bounded positive Radon measure, that is, a regular Borel measure ν_0 on R such that

$$S(t) = \int e^{it\lambda}\, d\nu_0(\lambda) \quad \text{for all } t \in \mathsf{R}. \tag{7}$$

THEOREM 13. *Let $x\colon \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a continuous uniformly bounded linearly stationary stochastic process such that the spectrum of x exists. Let S be the (continuous) limit function defined in* (4) *and let ν_0 be the bounded positive Radon measure on the right hand side of the representation* (7) *of S. Let*

$$\nu_0 = f.d\lambda + \Sigma$$

be the decomposition of ν_0 to its absolutely continuous and singular parts. If the density f vanishes on some set of positive Lebesgue measure or, if

$$\int \frac{\log f(\lambda)}{1+\lambda^2}\, d\lambda = -\infty, \tag{8a}$$

then x is deterministic; if

$$\int \frac{\log f(\lambda)}{1+\lambda^2}\, d\lambda > -\infty \tag{8b}$$

and if

$$x_1(t) = \int_{\bar{\Delta}_0} e^{it\lambda}\, d\mu(\lambda), \quad t \in \mathsf{R},$$

and

$$x_2(t) = \int_{\Delta_0} e^{it\lambda}\, d\mu(\lambda), \quad t \in \mathsf{R};$$

where μ is the spectral measure of (the continuous V-bounded stochastic process) x and Δ_0 is the set of Lebesgue measure zero on which the measure Σ is concentrated and $\bar{\Delta}_0$ is its complement. Then $x(t)=x_1(t)+x_2(t)$, $t \in \mathsf{R}$; x_1 and x_2 are uniformly bounded linearly stationary; x_1 is purely non-deterministic and x_2 is deterministic.

PROOF. Since the spectrum of x exists, there exists, by Theorem 11, a stationary similarity (y, B) of x such that the (continuous) limit function S defined in (4) is the covariance function of y.

If the density f vanishes on some set of positive Lebesgue measure or, if (8a) holds, then y is deterministic (Rozanov [9; pp. 115–116]). Thus, in this case, it follows from Theorem 7 that x is deterministic.

Suppose (8b) holds and suppose μ_y is the spectral measure of y. Define

$$y_1(t) = \int_{\bar{\Delta}_0} e^{it\lambda}\, d\mu_y(\lambda), \qquad t \in \mathsf{R},$$

and

$$y_2(t) = \int_{\Delta_0} e^{it\lambda}\, d\mu_y(\lambda), \qquad t \in \mathsf{R}.$$

Then $y(t)=y_1(t)+y_2(t)$, $t \in \mathsf{R}$; and y_1, y_2 are wide sense stationary stochastic processes satisfying the conditions stated in Theorem 8, that is,

$$y_2(t) = P_{-\infty}y(t), \qquad y_1(t) = y(t)-y_2(t), \qquad t \in \mathsf{R}$$

(Rozanov [9; pp. 115–116]). It then follows from Theorem 8 that the stochastic processes

$$x_k(t) = By_k(t), \qquad t \in \mathsf{R},\ k=1,2,$$

are uniformly bounded linearly stationary. Moreover, x_1 is purely non-deterministic and x_2 is deterministic.

As in the proof of Theorem 4 we get

$$x_1(t) = By_1(t) = \int_{\bar{\Delta}_0} e^{it\lambda}\, dB\circ\mu_y(\lambda), \qquad t \in \mathsf{R},$$

and

$$x_2(t) = By_2(t) = \int_{\Delta_0} e^{it\lambda}\, dB\circ\mu_y(\lambda), \qquad t \in \mathsf{R},$$

which proves the theorem, since $\mu = B\circ\mu_y$ is the spectral measure of x.

Next we consider harmonizable uniformly bounded linearly stationary stochastic processes. We show that in this case the results stated in Theorem 13 can be improved. Our result is essentially based on a result of Rozanov [8] concerning the existence of the spectrum and the representation of the limit function S defined in (4) in case of a harmonizable stochastic process.

First we recall that a stochastic process $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ is called *harmonizable*, if the covariance function $r(s,t)$, $s,t \in \mathsf{R}$, of x can be represented as

$$r(s,t) = \int e^{is\lambda} e^{-it\theta} \, d\nu(\lambda,\theta), \qquad s,t \in \mathsf{R} \,,$$

where ν is a bounded Radon measure on $\mathsf{R} \times \mathsf{R}$ for which $\nu(f \otimes \bar{f}) \geqq 0$ for all $f \in C_0(\mathsf{R})$. Here

$$(f \otimes \bar{f})(s,t) = f(s)\overline{f(t)}, \qquad s,t \in \mathsf{R} \,.$$

Every harmonizable stochastic process is continuous. Moreover, every harmonizable stochastic process is V-bounded (Bochner [1]).

THEOREM 14. *Let $x: \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ be a harmonizable uniformly bounded linearly stationary stochastic process and let*

$$r(s,t) = \int e^{is\lambda} e^{-it\theta} \, d\nu(\lambda,\theta), \qquad s,t \in \mathsf{R} \,,$$

be its covariance function. Then (the spectrum of x exists and) the results stated in Theorem 13 are valid for x with

$$\nu_0(f) = \int f(\lambda) \chi_\Delta(\lambda,\theta) \, d\nu(\lambda,\theta), \qquad f \in C_0(\mathsf{R}); \tag{9}$$

here χ_Δ is the characteristic function of the main diagonal $\Delta = \{(\lambda,\lambda) \in \mathsf{R} \times \mathsf{R} \mid \lambda \in \mathsf{R}\}$.

PROOF. It follows from a result of Rozanov [8] that the spectrum of x exists and the limit function S, defined in (4), can be represented as

$$S(t) = \int e^{it\lambda} \, d\nu_0(\lambda), \qquad t \in \mathsf{R} \,,$$

where ν_0 is defined as in (9).

To complete the proof we show that ν_0 is a bounded positive Radon measure.

Clearly,

$$|\nu_0(f)| \leqq \sup|f| \int d|\nu| \quad \text{for all } f \in C_0(\mathsf{R});$$

thus ν_0 is bounded.

To show that ν_0 is positive, it is enough to show that $\nu_0(f) \geqq 0$ for all $f \in C_0(\mathsf{R})$, $f \geqq 0$.

Let $f \in C_0(\mathsf{R})$, $f \geqq 0$, be given. Let $\varepsilon > 0$. Then there exists a compact set $K \subset \mathsf{R} \times \mathsf{R}$ and open set $G \subset \mathsf{R} \times \mathsf{R}$ such that $K \subset \Delta \subset G$ and

$$\int \chi_{G\setminus K}\, d|\nu| < \varepsilon , \tag{10}$$

where $\chi_{G\setminus K}$ is the characteristic function of $G\setminus K$.

Denote

$$K' = \{s \in \mathsf{R} \mid (s,s) \in K\} .$$

Since $K' \subset \mathsf{R}$ is compact we can find a finite collection G_j, $j = 1,\ldots,L$, of open subsets of R such that

$$K' \subset \bigcup_{j=1}^{L} G_j \quad \text{and} \quad K \subset \bigcup_{j=1}^{L} G_j \times G_j \subset G .$$

Moreover, there exist continuous functions $g_j \colon \mathsf{R} \to [0,1]$ such that the support of each g_j is contained in G_j, $j = 1,\ldots,L$, and

$$\sum_{j=1}^{L} g_j(t) \leqq 1, \quad t \in \mathsf{R}; \qquad \sum_{j=1}^{L} g_j(t) = 1, \quad t \in K' .$$

Denote

$$h = \sum_{j=1}^{L} (f^{\frac{1}{2}} g_j^{\frac{1}{2}} \otimes f^{\frac{1}{2}} g_j^{\frac{1}{2}}) \quad \text{and} \quad V = \nu(h) .$$

It then follows from the properties of ν that

$$V = \sum_{j=1}^{L} \nu(f^{\frac{1}{2}} g_j^{\frac{1}{2}} \otimes f^{\frac{1}{2}} g_j^{\frac{1}{2}}) \geqq 0 .$$

Moreover, a straightforward calculation shows that for all $s, t \in \mathsf{R}$

$$h(s,s) = f(s)\chi_\Delta(s,t)$$

and

$$h(s,t) = \sum_{j=1}^{L} f(s)^{\frac{1}{2}} g_j(s)^{\frac{1}{2}} f(t)^{\frac{1}{2}} g_j(t)^{\frac{1}{2}}$$

$$\leqq \sum_{j=1}^{L} (f(s)g_j(s)+f(t)g_j(t)) \leqq f(s)+f(t)\,.$$

Thus, by using (10), we get

$$\begin{aligned} |v_0(f)-V| &= \left|\int (\chi_\Delta(s,t)f(s)-h(s,t))\,dv(s,t)\right| \\ &\leqq \int |\chi_\Delta(s,t)f(s)-h(s,t)|\,d|v|(s,t) \\ &\leqq \int \chi_{G\setminus K}(s,t)h(s,t)\,d|v|(s,t) \leqq 2\varepsilon \sup f\,, \end{aligned}$$

which proves that $v_0(f) \geqq 0$.

The theorem is proved.

REMARK 15. There exist harmonizable stochastic processes which are not uniformly bounded linearly stationary. Consider for example the stochastic process $x\colon \mathsf{R} \to L_0^2(\Omega, \mathscr{A}, P)$ defined by

$$x(t) = (e^{it}-e^{-it})\xi, \qquad t \in \mathsf{R}\,,$$

where $\xi \in L_0^2(\Omega, \mathscr{A}, P)$, $\xi \neq 0$. Moreover, there exist continuous stochastic processes for which the spectrum exists and which are not V-bounded. Consider for example the stochastic process $x(t)=f(t)\xi$, $t \in \mathsf{R}$, where

$$f(t) = \sum_{n=2}^{\infty} \frac{\sin nt}{n \log n}, \qquad t \in \mathsf{R}\,,$$

and $\xi \in L_0^2(\Omega, \mathscr{A}, P)$, $\xi \neq 0$.

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