MATH. SCAND. 39 (1976), 161–166

COMMUTATIVE SUBRINGS OF PERIODIC RINGS

THOMAS J. LAFFEY

A ring R is called *periodic* if for each $x \in R$ the set $\{x, x^2, x^3, \ldots\}$ is finite, or equivalently, for each $x \in R$, there exist natural numbers $m(x), n(x)$ such that $x^{m(x)} = x^{m(x)+n(x)}$. Examples of periodic rings are nil rings and direct sums of matrix rings over finite fields. In this note we prove the following result:

THEOREM 1. *Let R be an infinite periodic ring and assume:*

(a) *R has no infinite subring S with $S^2 = \{0\}$,*

(b) *R has no infinite set of mutually orthogonal idempotents.*

Then R has a commutative ideal I with R/I finite.

In proving Theorem 1 we introduce a new ideal $H(R)$ of a ring R and obtain some "radical-like" properties of it.

This paper is selfcontained modulo elementary group theory and the following well-known result:

Jacobson's $(x^n = x)$ THEOREM. *Let R be a ring such that for each $x \in R$, there exists $n(x) > 1$ such that $x^{n(x)} = x$. Then R is commutative.*

We also recall:

Poincare's THEOREM. *Let G be a group and $H_1, \ldots, H_n$ subgroups of G with $[G:H_i] < \infty$, $i = 1, 2, \ldots, n$.*

Then $[G: \bigcap_{i=1}^{n} H_i] < \infty$.

Let R be a ring and let R^+ be its additive group. If S is a subring of R, $[R:S]$ denotes the index $[R^+:S^+]$.

If X is a nonempty subset of R, write

$$A(X) = \{r \in R \mid rx = xr = 0 \text{ for all } x \in X\} .$$

If $X = \{r\}$, write $A(r)$ in place of $A(X)$. Define

$$H(R) = \{x \in R \mid [R:A(x)] < \infty\} .$$

Received February 7, 1976.

We note the following elementary properties of $H(R)$:

LEMMA 1. *Let* R *be a ring and let* $x \in R$. *Then*

(a) $x \in H(R)$ *if and only if* xR *and* Rx *are finite.*
(b) $H(R)$ *is an ideal of* R.
(c) $x \in H(R)$ *if and only if* RxR *is finite (where* RxR *is the set of all finite sums* $\Sigma r_i x s_i$ $(r_i, s_i \in R)$).
(d) $H(H(R)) = H(R)$.
(e) *if* I *is a subring of* R *with* $[R:I] < \infty$, *then* $H(I) = I \cap H(R)$.

The next result is important.

LEMMA 2. *Let* R *be a ring with no infinite subring* S *with* $S^2 = \{0\}$. *Then, if* $x \in R$ *is such that* $x^2 \in H(R)$, $x \in H(R)$.

PROOF. Let $B = \{xax \mid a \in A(x^2)\}$. Then $B^2 = \{0\}$, so B is finite. Thus

$$C = \{a \in A(x^2) \mid xax = 0\}$$

has finite index in $A(x^2)$ (since it is the kernel of the map: $y \to xyx$) and thus in R^+. Let $D = \{xa \mid a \in C\}$. Again $D^2 = \{0\}$, so D is finite and thus $E = \{a \in C \mid ax = 0\}$ has finite index in C and thus in R^+. The result follows using symmetry.

COROLLARY. *Let* R *be a ring with no infinite subring* S *with* $S^2 = \{0\}$. *Then* $H(R)$ *contains all nilpotent elements of* R.

LEMMA 3. *Let* R *be a periodic ring with no infinite subring* S *with* $S^2 = \{0\}$. *Then* $R/H(R)$ *is commutative.*

PROOF. Let $x \in R$. Then $x^m(x - x^n) = 0$ for some natural numbers m, n with $n > 1$. So $x - x^n$ is nilpotent. The result now follows from the Corollary to Lemma 2 and Jacobson's $(x^n = x)$ Theorem.

LEMMA 4. *Let* R *be a ring with* $H(R)$ *infinite. Then (at least) one of the following statements holds:*

(a) $H(R)$ *has an infinite subring* S *with* $S^2 = \{0\}$.
(b) $H(R)$ *has an infinite set of mutually orthogonal idempotents.*

PROOF (cf. Bell [1]). Suppose the result is false and let R be a counter example. Let $h \in H(R)$. Then the ring V generated by h is finite since otherwise $W = V \cap A(h)$ is an infinite subring of R with $W^2 = \{0\}$.

We now pick a sequence of elements of R as follows: Let

$$y_1 \in H(R),\ y_2 \in H(R) \cap A(y_1) - X_1, \dots ,$$
$$y_{n+1} \in H(R) \cap A(y_1) \cap \dots \cap A(y_n) - (X_1 + \dots + X_n), \dots ,$$

where X_n denotes the ring generated by y_n. Since X_n is finite for each n and $H(R) \cap A(y_1) \cap \dots \cap A(y_n)$ has finite index in $H(R)$, by Poincare's Theorem, and is therefore infinite, this process can be continued indefinitely.

Since the elements of a finite ring with no nonzero idempotent are nilpotent and (b) does not hold, X_n is nilpotent for all but finitely many (and thus, without loss of generality, for all) n. Let S_n be a subring of X_n maximal with respect to $S_n^2 = \{0\}$. Since $X_n X_m = \{0\}$ $(m \neq n)$, the ring T generated by $\bigcup_{n=1}^{\infty} S_n$ satisfies $T^2 = \{0\}$ and is thus finite.

Hence there exists $N \geqq 1$ such that

$$S_m \subseteqq S_1 \cup \dots \cup S_N$$

for all m. In particular, for $m > N$, $S_m X_m = X_m S_m = \{0\}$. By the maximality of S_m we thus find that for $m > N, S_m$ contains all elements $y \in X_m$ with $y^2 = 0$.

Let $x \in X_m$ $(m > N)$ and let q be the least integer such that $x^q = 0$. Then $x^{[(q+1)/2]} \in S_m$ where $[\cdot]$ as usual denotes the greatest-integer function, so $x^{[(q+1)/2]+1} = 0$ and thus $q \leqq 3$. Hence $y_m^3 = 0$ $(m > N)$ and $X_m^3 = \{0\}$ $(m > N)$.

Now $y_m^2 \in T$ $(m > N)$ so there exists $z \in T$ such that $y_m^2 = z$ holds for infinitely many (and thus, without loss of generality, for all) $m > N$. Now $sz = 0$ for some $s \geqq 1$. Hence the elements

$$\begin{aligned} w_1 &= y_{n+1} + \dots + y_{N+s} \\ w_2 &= y_{N+s+1} + \dots + y_{N+2s} \\ &\vdots \\ w_{n+1} &= y_{N+ns+1} + \dots + y_{N+(n+1)s} \end{aligned}$$

satisfy $w_i w_j = 0$ $(i,j = 1, 2, \dots)$ and are distinct (by our choice of $\{y_k\}$). So the ring generated by $\{w_i\}$ is an infinite subring of R with square zero, giving a final contradiction.

COROLLARY (cf. Bell [1]). *Let R be an infinite nil ring. Then R has an infinite subring S with $S^2 = \{0\}$.*

PROOF. If the result fails, the Corollary to Lemma 2 implies that $H(R) = R$ and Lemma 4 gives a contradiction.

We now prove Theorem 1.

Let R be an infinite periodic ring and suppose that R has no infinite subring S with $S^2=\{0\}$ and no infinite set of mutually orthogonal idempotents.

By Lemma 4, $H(R)$ is finite and, by Lemma 3, $R/H(R)$ is commutative. Now $A(H(R))$ is an ideal of R and $R/A(H(R))$ is finite, by the finiteness of $H(R)$ and Poincare's Theorem. Let $x, y \in A(H(R))$. There exists $n>1$ such that $x-x^n \in H(R)$ (cf. proof of Lemma 3) and $xy-yx \in H(R)$ (since $R/H(R)$ is commutative). Now since

$$H(R)A(H(R)) = A(H(R))H(R) = \{0\}$$

it follows that

$$\begin{aligned} xy-yx &= x^n y - yx^n \\ &= x^{n-1}(xy-yx)+x^{n-2}(xy-yx)x+\ldots+(xy-yx)x^{n-1} \\ &= 0\,. \end{aligned}$$

Thus $I=A(H(R))$ satisfies the conclusions of Theorem 1.

COROLLARY TO THEOREM 1 (cf. [2]). *Let R be an infinite ring. Then R has an infinite commutative subring.*

We now obtain a general result on $H(R)$.

THEOREM 2. *Let R be an infinite ring with no infinite subring S with $S^2=\{0\}$. Then $H(R/H(R))$ is commutative.*

PROOF. Let $x+H(R) \in H(R/H(R))$. By definition

$$C = \{y \in R \mid xy \text{ and } yx \in H(R)\}$$

has finite index in R^+. So $x^m+C=x^{m+n}+C$ for some natural numbers m,n. Thus $x^{m+1}-x^{m+n+1} \in H(R)$ and thus $(x-x^n)^{m+1} \in H(R)$. So, by repeated application of Lemma 2, we get $x-x^n \in H(R)$. The result now follows from Jacobson's $(x^n=x)$ Theorem.

We now give an example.

EXAMPLE. Let S be a finite ring with $A(S)=\{0\}$. Let T be the set of infinite sequences $\{s_n\}$ of elements of S made into a ring by defining $\{s_n\}+\{s_n'\}=\{s_n+s_n'\}$, $\{s_n\}\{s_n'\}=\{s_n s_n'\}$. For $t=\{s_n\} \in T$ and $s \in S$, define $st=\{ss_n\}$ and $ts=\{s_n s\}$. Let $R=S\times T$ made into a ring by defining

$$(s_1, t_1)+(s_2, t_2) = (s_1+s_2, t_1+t_2)$$

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, s_1 t_2+t_1 s_2+t_1 t_2)$$

($s_1, s_2 \in S$, $t_1, t_2 \in T$). Then $H(R)=\{(0,t) \mid t \in T\}$ and $R/H(R)=H(R/H(R)) \cong S$. Thus $H(R/H(R))$ is not necessarily commutative. In this example, if S has no nonzero nilpotent elements, then R has no infinite subring U with $U^2=\{0\}$, so S is commutative by Theorem 2. In particular taking S to be a finite division ring, we get Wedderburn's Theorem, namely: every finite division ring is commutative. However Jacobson's ($x^n=x$) Theorem which is used in the proof of Theorem 2 is, of course, a generalization of Wedderburn's Theorem.

The well-known Kurosh problem on rings is: "Is a finitely generated ring in which each element generates a finite subring necessarily finite." While the answer is in general negative we get an affirmative answer for rings not having an infinite subring S with $S^2=\{0\}$. We prove:

THEOREM 3. *Let R be a finitely generated ring in which each element generates a finite subring. If R is infinite, then R has an infinite subring S with $S^2=\{0\}$.*

PROOF. By Lemma 3, $R/H(R)$ is commutative. Also for each $x \in R$, there exists $n(x)>1$ such that $x-x^{n(x)} \in H(R)$. Also $q(x)x \in H(R)$ for some natural number $q(x)$. So $R/H(R)$ is finite.

Let $y_1+H(R),\dots,y_m+H(R)$ be a basis for the finite abelian group $[R/H(R)]^+$. Let $x_1,\dots,x_n$ be a set of generators for R. Now

$$x_i = \sum_{k=1}^{m} a_k y_k+h_i \qquad (i=1,2,\dots,n)$$

$$y_i y_j = \sum_{k=1}^{m} b_{ijk} y_k+h_{ij} \qquad (i,j=1,2,\dots,m)$$

for some integers a_k, b_{ijk} and elements $h_i, h_{ij} \in H(R)$. Let e_i be the order of $y_i+H(R)$, $i=1,2,\dots,m$. Let L be the ideal of R generated by

$$\{h_i \mid i=1,2,\dots,n\} \cup \{h_{ij} \mid i,j=1,2,\dots,m\} \cup \cup \{e_i y_i \mid i=1,2,\dots,m\}.$$

Since R has no infinite subring S with $S^2=\{0\}$, Lemma 1 (c) implies that L is finite. Also if $r \in R$, then

$$r = \sum_{k=1}^{m} c_k y_k+w$$

for some integers c_i with $0 \leqq c_i < e_i$ and some $w \in L$. If $r \in H(R)$, then $r - w \in H(R)$ and thus $\Sigma_{k=1}^{m} c_k(y_k + H(R)) = 0$, forcing $c_1 = c_2 = \ldots = c_m = 0$ (since $0 \leqq c_i < e_i$). This implies that $L = H(R)$. So R is finite.

REFERENCES

1. H. E. Bell, *Infinite subrings of infinite rings and near rings*, Pacific J. Math. 59 (1975), 345–348.
2. T. J. Laffey, *On commutative subrings of infinite rings*, Bull. London Math. Soc. 4 (1972), 3–5.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE
DUBLIN 4
IRELAND