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ON VON NEUMANN REGULAR RINGS – II

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Introduction.

A well-known theorem of Kaplansky states that a commutative ring R is von Neumann regular iff every simple R-module is injective. Rings whose simple singular modules are injective are studied in [1] and [4]. In particular, it is proved in [4, Theorem 3.6] that a commutative ring R is regular iff every simple singular R-module is injective. In this note, the injectivity property is weakened to p-injectivity. Several other results in [2], [4] and [6] are generalised. Finally, arbitrary regular rings are characterised in terms of: (a) singular p-injective modules and annihilators; (b) semi-simple p-injective modules.

Throughout, A denotes an associative ring with identity and "module" means "left, unitary A-module". We recall that

(a) an A-module M is p-injective if, for any principal left ideal I of A and any left A-homomorphism $g\colon I \to M$, there exists $y \in M$ such that $g(b)=by$ for all b in I;

(b) the singular submodule of M is $Z(M)=\{y \in M/l(y) \text{ is essential in } A\}$ and M is called singular if $Z(M)=M$.

Write "A satisfies (*)" if every simple singular A-module is p-injective.

LEMMA 1. *If A satisfies (*), then for every element b of A, there exists a left ideal K such that $A=(AbA+l(b))\oplus K$.*

PROOF. For any $b \in A$, there exists a left complement ideal K such that $(AbA+l(b))\oplus K$ is essential in A. If $(AbA+l(b))\oplus K \neq A$, let L be a maximal left ideal containing $(AbA+l(b))\oplus K$. Define $g\colon Ab \to A/L$ by $g(ab)=a+L$ for all a in A. Then g is a well-defined left A-homomorphism and since A/L is p-injective, there exists c in A such that $g(ab)=ab(c+L)$ for all a in A. In particular, $1+L=g(b)=bc+L$ and since $bc \in AbA \subseteq L$, then $1 \in L$ which contradicts the maximality of L. Thus $(AbA+l(b))\oplus K=A$.

We now extend [4, Theorem 3.6] and [6, Proposition 3].

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Theorem 2. *The following conditions are equivalent*;

(i) *A is regular without non-zero nilpotent elements*;
(ii) *A satisfies* (*) *and every left ideal of A is two-sided.*

Proof. (i) implies (ii) by [6, Lemma 2].

Assume (ii). For any $b \in A$, $A=(AbA+l(b))\oplus K$ for some left ideal K of A (Lemma 1). Since every left ideal is a right ideal, $A=(Ab+l(b))\oplus K$ and $KAb \subseteq Ab \cap K=0$ implies $K=0$ and hence $A=Ab+l(b)$. Then $1=ab+d$, $a \in A$, $d \in l(b)$, yields $b=ab^2$. This proves that (ii) implies (i).

Throughout, N will denote the Jacobson radical of A.

Proposition 3. *If A satisfies* (*), *then*
(i) $Z(A)\cap N=0$;
(ii) $A=AcA$ *for every non-zero-divisor c of A*;
(iii) *Every essential left ideal of A is an idempotent.*

Proof. (i) Let $z \in Z(A)\cap N$. By Lemma 1, $A=(AzA+l(z))\oplus K$ which implies $K=0$ (since $l(z)$ is essential in A). Let $1=w+d$, $w \in AzA \subseteq N$ (two-sided), $d \in l(z)$. There exists $v \in A$ such that $vd=v(1-w)=1$. Then $z=vdz=0$ which proves that $Z(A)\cap N=0$.

(ii) If c is a non-zero-divisor of A, then $A=(AcA+l(c))\oplus K$ again by Lemma 1. Since $l(c)=0$ and $cK \subseteq AcA \cap K=0$, then $K=0$ and $A=AcA$.

(iii) Let I be an essential ideal of A. For any $b \in I$, $IA+l(b)$ is essential in A. If $IA+l(b)\neq A$, let L be a maximal left ideal containing $IA+l(b)$. Then A/L is p-injective and this leads to a contradiction as in Lemma 1. Therefore $A=IA+l(b)$ and $1=u+d$, $u \in IA$, $d \in l(b)$, which implies $b=ub \in I^2$ and hence $I=I^2$.

Definition. An A-module M is called semi-simple if the intersection of all maximal submodules of M is zero [3].

Corollary 4. *The following are equivalent*:

(i) *Every simple A-module is injective*;
(ii) *A satisfies* (*) *and has the following properties*:
  (a) *Every minimal left ideal is injective and*
  (b) *Every cyclic singular A-module is semi-simple.*

(Apply [4, Theorem 3.3] and Proposition 3 (i).)

Corollary 5. *If A is a left continuous ring satisfying* (*), *then A is regular.*

PROOF. If A is left continuous, $Z(A)=N$ and A/N is regular [5, Lemma 4.1]. By Proposition 3 (i), $N=0$.

The next result generalises [4, Prop. 4.3 and Prop. 4.5].

PROPOSITION 6. *Let A be a semi-prime ring satisfying* (*). *Then every left ideal of A is an idempotent.*

PROOF. For any left ideal I of A, suppose there exists $b \in I$, $b \notin I^2$. Then $Ab \neq (Ab)^2$ but $(Ab)^2$ is essential in Ab since A is semi-prime. By Zorn's Lemma, the set of left ideals J such that $(Ab)^2 \subseteqq J \subset Ab$ has a maximal member L. Then Ab/L is simple singular and therefore p-injective by hypothesis. If $g: Ab \to Ab/L$ is the canonical homomorphism, there exists $c \in A$ such that $g(ab)=ab(cb+L)$ for all a in A. Then $b-bcb \in L$ which implies $b \in L$. This contradicts $L \subset Ab$. Thus $I=I^2$.

COROLLARY 7. *If A is a P.I. ring, then A is regular iff A is a semi-prime ring satisfying* (*).

(Apply Proposition 6 to [2, Theorem 1]).

COROLLARY 8. *If A is a prime left Goldie ring satisfying* (*), *then A is simple,* (see [4, Corollary 4.6].)

So far, only certain classes of regular rings have been characterised in terms of singular p-injective modules. We now turn our attention to arbitrary regular rings.

THEOREM 9. *The following are equivalent:*

(i) *A is regular:*
(ii) *Every principal left ideal of A is the left annihilator of an element of A and every left cyclic singular A-module is p-injective;*
(iii) *Every principal right ideal of A is a right annihilator and every left cyclic singular A-module is p-injective;*
(iv) *Every left cyclic semi-simple A-module is p-injective.*

PROOF. (i) implies (ii), (iii) and (iv) by [6, Lemma 2].

Assume (ii). For any $b \in A$, let $Ab=l(t)$, $t \in A$. Let K be a left complement ideal such that $l(t)\oplus K$ is essential in A. Then $A/(l(t)\oplus K)$ is p-injective by hypothesis and if $g: At \to A/(l(t)\oplus K)$ is defined by $g(at)=a+(l(t)\oplus K)$ for all a in A, there exists c in A such that $1+(l(t)\oplus K)=g(t)=tc+(l(t)\oplus K)$. If $1-tc$

$=db+k$, $d \in A$, $k \in K$, then $b=bdb+bk$ and $bt=0$ implies $bk \in l(t) \cap K=0$. Thus $b=bdb$ which proves that (ii) implies (i).

Assume (iii). For any $b \in A$, let $bA=r(S)$ for some subset S of A. Since $l(b) \oplus K$ is essential in A for some left complement K, if $g: Ab \to A/(l(b) \oplus K)$ is defined by $g(ab)=a+(l(b) \oplus K)$, there exists c in A such that $1+(l(b) \oplus K)=g(b)$ $=bc+(l(b) \oplus K)$. Then $1-bc=t+k$, $t \in l(b)$, $k \in K$. For any $s \in S$, $s=st+sk$ and $sk=s-st \in l(b) \cap K=0$. Therefore $Sk=0$ implies $k \in r(S)=bA$. If $k=bd$, $d \in A$, then $b-bcb=bdb$ which implies $b=b(c+d)b$. Thus (iii) implies (i).

Assume (iv). Then every simple A-module is p-injective. We prove that every principal left ideal I of A is semi-simple. Then (iv) will imply (i) as in [6, Lemma 2]. For any $0 \neq b \in I$, by Zorn's Lemma, the set of all left subideals K of Ab such that $b \notin K$ has a maximal member J. Then Ab/J is simple, p-injective and the canonical homomorphism $f: Ab \to Ab/J$ may be extended to $g: A \to Ab/J$. Restrict g to $h: I \to Ab/J$. Then $I/\ker h \approx Ab/J$ which proves that $\ker h$ is a maximal subideal of I. Since $\ker h \cap Ab = \ker f$, $b \in Ab$, $b \notin \ker f$, then $b \notin \ker h$. Thus I is semi-simple which completes the proof.

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