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SOME ENUMERATIVE PROPERTIES OF SECANTS TO NON-SINGULAR PROJECTIVE SCHEMES

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1. Introduction.

In the following article we shall make some remarks on two recent articles which generalize, to non-singular algebraic varieties of arbitrary dimension, classical results about secants of algebraic curves. The two articles treat apparantly different questions. A. Holme (in [4]) is interested in giving numerical criteria for the possiblility of embedding non-singular projective varieties into projective spaces by projections from the ambient projective space. C. A. M. Peters and J. Simonis (in [8]), on the other hand, are interested in determining the number of secants of a non-singular projective variety passing through a general point of the ambient projective space. The key to the two articles is, however, the same, the determination, in terms of projective invariants, of the rational equivalence class of the bundle of secant lines to a given non-singular projective variety. The class considered in the Chow ring of the Grassmann variety of all lines of the ambient projective space. C. A. M. Peters and J. Simonis work directly on the Grassmann manifold and determine the class of the secant bundle using the well known description of the Chow ring of a Grassmann variety in terms of generators and relations given by the Schubert cycles of the Grassmann variety. A. Holme, on the other hand, uses a fibration of the Grassmann variety obtained by adding points to the lines of the ambient projective space. His approach has the advantage that the structures of the Chow rings of the varieties considered are extremely simple, all morphisms between the varieties being structure morphism of projective bundles. However, a major part of the work lies in the construction of the fibration map and in the ensuing cumbersome computations of the Chern classes involved. The main objective of the present article is to give a natural construction of the fibration map leading to a simple determination of the rational equivalence class of the secant bundle in question. In addition to the purely technical advantages, our approach brings forth the connection between the methods of A. Holme and Peters–Simonis and allows us to obtain their results from the same general formula. We show

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that the result of A. Holme is an immediate consequence of a slight generalization of the result of Peters–Simonis (Theorem 21 (ii)) together with the classical (and geometrically obvious) result that a projection between two projective spaces induces an embedding of varieties whose secants do not intersect the center of projection. On the other hand, the result of Peters–Simonis follows from the methods of Holme together with the result that a secant in general position on a projective variety (which is not a hypersurface in a linear subspace) intersect the variety in exactly two simple points. This result is classical when the ground field is of characteristic zero. Below, (Lemma 15) we offer a proof of the result valid for an arbitrary ground field.

A different approach to the construction of secant bundles in a far more general sense than that mentioned above was given by R. L. E. Schwarzenberger [10]. He also gave a beautiful method for solving problems of the kind mentioned above. Apart from the generality, Schwarzenberger's method has many advantages that may be worthwhile exploiting before trying new constructions of secant bundles. We shall return to Schwarzenberger's approach at a later occasion.

2. Monoidal transformations with center on a linear subspace.

Let X denote a scheme and Y a closed subscheme of X defind by an ideal I of the structure sheaf $\mathcal{O}_X$ of X. The monoidal transformation $B_I(X)$ of X with center on Y is the projective space $\operatorname{Proj}(\oplus_{n=0}^{\infty} I^n)$ over X associated to the graded $\mathcal{O}_X$-algebra $\oplus_{n=0}^{\infty} I^n$, and the exceptional divisor of the monoidal transformation is the projective space $\operatorname{Proj}(\oplus_{n=0}^{\infty} I^n/I^{n+1})$ over Y associated to the $\mathcal{O}_Y$-algebra $\oplus_{n=0}^{\infty} I^n/I^{n+1}$.

A surjective homomorphism $H \to I$ of quasi-coherent $\mathcal{O}_X$-modules induces a surjective homomorphism $\operatorname{Sym}_{\mathcal{O}_X}(H) \to \oplus_{n=0}^{\infty} I^n$ of $\mathcal{O}_X$-algebras and consequently a closed immersion $t: B_I(X) \to \boldsymbol{P}(H)$ of schemes. Moreover, the above surjection restricted to the open subscheme $(X-Y)$ of X clearly gives rise to a section $s: (X-Y) \to \boldsymbol{P}(H)$.

LEMMA 1. *With the above notation, the monoidal transformation $B_I(X)$ of X with center on Y considered as a subscheme of $\boldsymbol{P}(H)$ via the immersion t, is the (scheme theoretic) closure of the subscheme $s(X-Y)$ of $\boldsymbol{P}(H)$.*

For a proof see [6, Proposition (3.2.1), p. 419].

We shall now give a different and more classical construction of the monoidal transformation of a projective bundle with center on a linear subspace. Let S denote an integral scheme and let

$$0 \to E \to F \to G \to 0$$

be an exact sequence of locally free $\mathcal{O}_S$-modules of finite rank. Put $W = \boldsymbol{P}(E)$, $X = \boldsymbol{P}(F)$ and $Y = \boldsymbol{P}(G)$ and denote by i the closed immersion of Y in X defined by the surjection $F \to G$ of $\mathcal{O}_S$-modules. Moreover, denote by K the kernel of the universal quotient map $E_W \to L_W$ from the pull back E_W of the sheaf E to W. The following exact diagram of locally free sheaves on W defines a quotient $\mathcal{O}_W$-module M of F_W of rank two,

$$(*)\qquad \begin{array}{ccccccccc} & & & & 0 & = & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & E_W & \to & L_W & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & F_W & \to & M & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G_W & = & G_W & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Denote by Z the projective bundle $\boldsymbol{P}(M)$ over W defined by M and by h and j the closed immersions $Z \to \boldsymbol{P}(F_W)$ and $\boldsymbol{P}(G_W) \to Z$ defined by the surjections $F_W \to M$ and $M \to G_W$ of the above diagram. Moreover, denote by f the morphism $Z \to X$ defined by the surjective map $(F_W)_Z \to L_Z$ of $\mathcal{O}_Z$-modules obtained by composing the pull back to Z of the surjection $F_W \to M$ with the universal quotient map $M_Z \to L_Z$ on Z. Note that we have obvious canonical isomorphisms $\boldsymbol{P}(F_W) \cong \boldsymbol{P}(E_X)$ and $\boldsymbol{P}(G_W) \cong \boldsymbol{P}(E_Y)$. From the morphisms defined above together with the structure morphisms of the schemes defined we obtain the following diagram,

$$\begin{tikzcd} \boldsymbol{P}(E_Y) \arrow[r, "j"] \arrow[d] & Z = \boldsymbol{P}(M) \arrow[r, "h"] \arrow[d, "f"] & \boldsymbol{P}(E_X) = \boldsymbol{P}(F_W) \arrow[d] \\ Y = \boldsymbol{P}(G) \arrow[r, "i"] & X = \boldsymbol{P}(F) \arrow[dr] & W = \boldsymbol{P}(E) \arrow[d] \\ & & S \end{tikzcd}$$

LEMMA 2. *The closed immersion i defined by the surjection $F \to G$ of $\mathcal{O}_X$-modules identifies Y with "the scheme of zeroes" of the homomorphism $E_X \to L_X$ of $\mathcal{O}_X$-modules obtained by composing the pull back of the inclusion $E \to F$ with the universal quotient $F_X \to L_X$ on X. Consequently, the image in $\mathcal{O}_X$ of the resulting homomorphism $E_X \otimes L_X^{-1} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules is the ideal defining the closed subscheme $i(Y)$ of X. Moreover, the restriction of the above homomorphism to Y (via i) gives an isomorphism*

$$E_Y \otimes L_Y^{-1} \to I/I^2 .$$

PROOF. Let T be an S-scheme and L an invertible $\mathcal{O}_T$-module. Then the composite map of a surjection $F_T \to \cdot L$ of $\mathcal{O}_T$-modules with the pull back $E_T \to F_T$ of the inclusion map $E \to F$ to T is zero if and only if the morphism $T \to X$ resulting from the map $F_T \to L$ factorizes via Y. The first assertion of the lemma follows.

By the first assertion of the lemma, the map $E_Y \otimes L_Y^{-1} \to I/I^2$ is surjective. The second assertion then follows from the fact that I/I^2 is locally free of rank equal to the rank of $E_Y \otimes L_Y^{-1}$ (see [1, VII Theorem (5.8)]).

By Lemma 2 we have a surjection $E_X \otimes L_X^{-1} \to I$ where I is the ideal in $\mathcal{O}_X$ defining the subscheme $i(Y)$ of X and consequently we have a closed immersion $t: B_I(X) \to \boldsymbol{P}(E_X \otimes L_X^{-1})$. Moreover, by the second assertion of Lemma 2, we conclude that under this immersion the exceptional locus $\boldsymbol{P}(I/I^2)$ of the monoidal transformation of X with the center on Y is mapped isomorphically onto $\boldsymbol{P}(E_Y \otimes L_Y^{-1})$.

The following proposition is a particular case of a result about monoidal transformations of special Schubert schemes along their singular locus (see [6, Theorem (5.2), p. 426]).

PROPOSITION 3. *With the above notation, the canonocal isomorphism* $\boldsymbol{P}(E_X) \to \boldsymbol{P}(E_X \otimes L_X^{-1})$ *of projective bundles over V defined by "twisting by L_X^{-1}" sends Z, considered as a subscheme of $\boldsymbol{P}(E_X)$ via h, isomorphically onto $B_I(X)$ considered as a subscheme of $\boldsymbol{P}(E_X \otimes L_X^{-1})$ via t. Under this isomorphism the projective bundle $\boldsymbol{P}(E_Y) \cong \boldsymbol{P}(G_W)$ over Y, considered as a subscheme of Z via j, is sent isomorphically onto the exceptional locus $\boldsymbol{P}(E_Y \otimes L_Y^{-1})$ of the monoidal transformation $B_I(X)$ of X with center on Y.*

PROOF. By Lemma 1, the morphsim t sends $B_I(X)$ isomorphically onto the closure of the section s: $(X - Y) \to \boldsymbol{P}(E_X \otimes L_X^{-1})$ defined by the restriction of the surjection $E_X \otimes L_X^{-1} \to I$ of Lemma 2 to $(X - Y)$. On the other hand, the same surjection $E_{X-Y} \to (L_X \mid X - Y)$ defines a morphism $(X - Y) \to \boldsymbol{P}(M) = Z$ which is a section to the morphism f (restricted to $f^{-1}(X - Y)$). Indeed, the surjection defines a morphism w: $(X - Y) \to W$ such that the surjection is the pull back by f of the universal quotient $E_W \to L_W$ on W. Considering the pull back of the diagram (*) by w we see that the composite map

$$w^*K \to w^*F_W = F_{X-Y} \to (L_X \mid X - Y)$$

is zero and consequently that the universal quotient $F_{X-Y} \to (L_X \mid X - Y)$ on $(X - Y)$ factorizes via a quotient map $w^*M \to (L_X \mid X - Y)$. The latter defines a morphism $(X - Y) \to Z$ which clearly is the inverse to the morphism $f \mid f^{-1}(X - Y)$, as asserted.

From the above we conclude that the image of Z in $\boldsymbol{P}(E_X \otimes L_X^{-1})$ contains the image of $B_I(X)$ and that the two images have a non empty open subset in common. Since $\boldsymbol{P}(M)$ is an integral scheme, (S being integral) the first part of the proposition follows. Moreover, using Lemma 2, we easily see that the open set common to the images in $\boldsymbol{P}(E_X \otimes L_X^{-1})$ of Z and $B_I(X)$ is the image in $\boldsymbol{P}(E_X \otimes L_X^{-1})$ of the complement in Z of $j(\boldsymbol{P}(E_Y))$ and of the complement in $B_I(X)$ of the exceptional locus. The second part of the proposition consequently follows by the first part of the proposition.

EXAMPLE 4. Let V be a vector space over a field k. On the projective space $P = \boldsymbol{P}(V)$, there is a canonical exact sequence

$$0 \to \Omega_P^1 \to V_P \otimes L_P^{-1} \to \mathcal{O}_P \to 0$$

of locally free $\mathcal{O}_P$-modules, where Ω_P^1 is the sheaf of Kähler differentials on P (see [1, I, Theorem (3.1)]). We can apply the constructions above to this sequence and conclude from Proposition 3 that the monoidal transformation of the scheme $\boldsymbol{P}(V_P \otimes L_P^{-1})$ with center on the closed subscheme $P = \boldsymbol{P}(\mathcal{O}_P)$ defined by the surjection $V_P \otimes L_P^{-1} \to \mathcal{O}_P$ is canonically isomorphic to the projective bundle $\boldsymbol{P}(M)$ over the scheme $T = \boldsymbol{P}(\Omega_P^1)$, where M is the locally free $\mathcal{O}_T$-module of rank two defined by the following diagram corresponding to the diagram (*) above,

$$(***)\qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & (\Omega_P^1)_T & \to & L_T & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & (V_P \otimes L_P^{-1}) & \to & M & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_T & = & \mathcal{O}_T & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Moreover, we conclude from Proposition 3 that by the above isomorphism the exceptional locus $\boldsymbol{P}(\Omega_P^1 \otimes L_P^{-1})$ of the monoidal transformation of $\boldsymbol{P}(V_P \otimes L_P^{-1})$ with center on P is mapped isomorphically onto the subscheme T of $\boldsymbol{P}(M)$ defined by the surjection $M \to \mathcal{O}_T$ of $\mathcal{O}_T$-modules defined above.

Note that the latter isomorphism $\boldsymbol{P}(\Omega_P^1 \otimes L_P^{-1}) \to T = \boldsymbol{P}(\Omega_P^1)$ is defined by the quotient map $\Omega_P \to L_P \otimes L_{T'}$ on $T' = \boldsymbol{P}(\Omega_P^1 \otimes L_P^{-1})$. Indeed, it is, by Proposition 3, the restriction of the morphism

$$\boldsymbol{P}((\Omega_P^1 \otimes L_P^{-1})_{P'}) \to \boldsymbol{P}((\Omega_P^1)_{P'})$$

to $\boldsymbol{P}(\Omega_P^1 \otimes L_P^{-1})$, where $P' = \boldsymbol{P}(V_P \otimes L_P^{-1})$. On the other hand, the monoidal transformation of $\boldsymbol{P}(V_P \otimes L_P^{-1})$ with center on P is isomorphic to the monoidal transformation $B_\Delta(P\times P)$ of $P\times P$ along the diagonal δ: $P \to P\times P$. Indeed, "twisting by L_P" the projective bundle $\boldsymbol{P}(V_P\otimes L_P^{-1})$ over P maps isomorphically onto the projective bundle $\boldsymbol{P}(V_P)=P\times P$ and the composite of this isomorphism with the inclusion $P \to \boldsymbol{P}(V_P\otimes L_P^{-1})$ defined by the surjection $V_P\otimes L_P^{-1} \to \mathcal{O}_P$ is easily checked to be the diagonal morphism d. Consequently, we have that the scheme $B_\Delta(P\times P)$ is canonically isomorphic to the bundle $\boldsymbol{P}(M)$ over T and that this isomorphism induces the identity map between the exceptional locus $T=\boldsymbol{P}(\Omega_P^1)$ of $B_\Delta(P\times P)$ and the subscheme T of $\boldsymbol{P}(M)$ defined by the surjection $M \to \mathcal{O}_T$ of $\mathcal{O}_T$-modules. The latter isomorphism is, in fact, the identity morphism. Indeed, as noted above the isomorphism is defined by the quotient map $(\Omega_P^1)_T \to L_T$.

Corresponding to the diagram (**) above, we have a commutative diagram

(****)
$$\begin{tikzcd}
T = \boldsymbol{P}(\Omega_P^1) \arrow[rr, "j"] \arrow[dr] \arrow[dd, "g"'] & & B \arrow[r, phantom, "="] \arrow[dd, "f"] & \boldsymbol{P}(M) \arrow[dd, "\lambda"] \\
& B_\Delta(P\times P) \arrow[ur, "\cong"] \arrow[dr] & & \\
P \arrow[rr, "\delta"] & & P\times P \arrow[dr, "p_2"] & T \arrow[d, "g"] \\
& & & P
\end{tikzcd}$$

where g and λ are the structure morphisms of the projective bundle $T=\boldsymbol{P}(\Omega_P^1)$ over P and of the projective bundle $B=\boldsymbol{P}(M)$ over T, and p_2 is the projection onto the second factor.

Note. The morphism λ defined above is the map λ defined by A. Holme ([4, III, § 8, Proposition 8.7 and Lemma 8.73, pp. 26–31 and IV, § 13, pp. 60–64]) by a 'glueing together" method and is essential in his as well as in our treatment of secant schemes. In the approach of Peters–Simonis on the other hand the key morphism is the composite map of λ with the morphism α: $T \to G_2(V)$ from T to the Grassmann scheme parametrizing lines in P, defined by the surjection $(V_P)_T \to (L_P)_T\otimes M$. However, it is easy to check that T (via α) is canonically isomorphic to the projective bundle $\boldsymbol{P}(Q)$ over $G_2(V)$ where Q is the universal sheaf of rank two on $G_2(V)$. This explains the close connection between the approaches of Holme and Peters–Simonis.

We collect the observations about the morphisms of the diagram (****) that we shall need below in the following lemma.

LEMMA 5. (i) *The image of T by the morphism j is the scheme of zeroes of the inclusion map $(L_T)_B \to L_B$ obtained by composing the pull back of the inclusion map $L_T \to M$ of the diagram (***) to B (by λ), with the universal quotient map $M_B \to L_B$ on B.*

(ii) *The morphism $f: B \to P \times P$ is defined by the quotient map*

$$(((V_P)_T)_B) \to ((L_P)_T)_B \otimes L_B$$

obtained by tensoring with $((L_P)_T)_B$ the map obtained by composing the pull back to B, $((V_P \otimes L_P^{-1})_T)_B \to M_B$, of the corresponding map of the diagram (***), with the universal quotient map $M_B \to L_B$ on B.

(iii) *The image of P by d is the scheme of zeroes of the composite map*

$$(\Omega_P^1 \otimes L_P)_{P \times P} \to L_{P \times P}$$

of the pull back $(\Omega_P^1 \otimes L_P)_{P \times P} \to (V_P)_{P \times P}$ to $P \times P$ of the corresponding inclusion on P, with the universal quotient map $(V_P)_{P \times P} \to L_{P \times P}$ on $P \times P$.

PROOF. (i) The morphism j is defined by the surjective map $M_T \to \mathcal{O}_T$ of the diagram (***). Consequently, the assertion (i) follows from Lemma 2 applied to this surjection.

(ii) Part (ii) is simply the definition of the morphism f.

(iii) The morphism $\delta: P \to P \times P = \boldsymbol{P}(V_P)$ is defined by the universal quotient map $V_P \to L_P$ on P. Consequently, the assertion (iii) follows from Lemma 2 applied to this surjection.

3. Computation of the Chow ring of monoidal transformations with center on a linear subspace.

Let S denote a non singular quasi-projective scheme over an algebraically closed field. Moreover, let $F \to G$ be a surjective homomorphism between two locally free $\mathcal{O}_S$-modules and consider $\boldsymbol{P}(G)$ as a subscheme of $\boldsymbol{P}(F)$ via the closed immersion defined by the surjection. The construction of the previous section makes it straight forward to determine the Chow ring of the monoidal transformation of $\boldsymbol{P}(F)$ with center on $\boldsymbol{P}(G)$ in terms of the Chow rings of $\boldsymbol{P}(G)$, $\boldsymbol{P}(F)$ and the exceptional locus (see e.g. [2, § 13, Lemma 19, p. 128] where the necessary computations are performed in the corresponding Grothendieck rings). We shall in the following, restrict our attention to the case of Example 4 of the previous section and perform the few computations we need in our treatment of secant schemes. We keep the notation and definitions of section two. For convenience we assume that the ground field k is algebraically closed. We fix an ideal J in $\mathcal{O}_P$ and assume that the closed subscheme X defined by J is smooth over $\operatorname{Spec} k$, and of pure codimension p in P. To avoid inconveniences later on we assume that $p < \dim P = n$. Note that with the above assumptions, the $\mathcal{O}_X$-module J/J^2 is locally free of rank equal to p ([1, III, Theorem 5.8]). It's dual $\mathcal{O}_X$-module $(J/J^2)^\vee$ is called the normal sheaf of the subscheme X in P and is denoted by $N(J)$.

We shall denote by $B_\Delta(X \times X)$ the monoidal transformation of the product scheme $X \times X$ with center on the diagonal $\Delta(X)$. Moreover, we denote by $B(X \times X)$ the closure in B of the preimage of $(X \times X - \Delta(X))$ by the isomorphism $f \mid f^{-1}(P \times P - \Delta(P))$. It follows from Lemma 1 that $B(X \times X)$ is isomorphic to $B_\Delta(X \times X)$ by the isomorphism $B \cong B_\Delta(P \times P)$ of the diagram (****). The exceptional locus of the monoidal transformation is clearly $\boldsymbol{P}(\Omega_X^1)$ (see [1, VI, Proposition (1.13)]) and is a subscheme of $T = \boldsymbol{P}(\Omega_P^1)$ by the composite map of the closed immersion $w\colon \boldsymbol{P}(\Omega_X^1) \to \boldsymbol{P}(\Omega_P^1 \mid X)$ defined by the canonical surjection $(\Omega_P^1 \mid X) \to \Omega_X^1$ of Kähler differentials, with the projections $b\colon \boldsymbol{P}(\Omega_P^1 \mid X) = X \times_P \boldsymbol{P}(\Omega_P^1) \to \boldsymbol{P}(\Omega_P^1)$ onto the second factor.

$$\begin{tikzcd}
\boldsymbol{P}(\Omega_X^1) \arrow[r, "w"] & \boldsymbol{P}(\Omega_P^1 \mid X) \arrow[r, "b"] \arrow[d] & \boldsymbol{P}(\Omega_P^1) \arrow[d, "g"] \\
& X \arrow[r] & P
\end{tikzcd}$$

We denote by $A(Y)$ the Chow ring of a nonsingular quasi-projective scheme Y. The direct and inverse image maps $A(Z) \to A(Y)$ and $A(Y) \to A(Z)$ associated to a morphism $a\colon Z \to Y$ of nonsingular quasi-projective schemes Y and Z we denote by $a_!$ and $a^!$. Moreover, we denote the rational equivalence class, in $A(Y)$, of a subscheme V of Y by $\gamma(V)$.

Let E denote a locally free sheaf on Y on rank m. The Chern classes of E in $A(Y)$ we denote by $c_i(E)$ $i = 0, 1, \ldots, n$ and the Chern polynomial $c_0(E) + c_1(E)t + \ldots + c_m(E)t^m$ by $c_t(E)$. It is well known that the Chow ring $A(\boldsymbol{P}(E))$ of the projective bundle $\boldsymbol{P}(E)$ over Y is a free $A(Y)$ module with a basis $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^m$ where $\varepsilon = c_1(L_P)$ (see [11, Exposé 4, sections 2, 4, 5 and 6]). We shall, in the following, identify $A(Y)$ with a subring of $A(\boldsymbol{P}(E))$ (via the inverse image of the structure morphism), thus applying the projection formula without mention.

We denote the images of $c_i(N(J))$ by the direct image $A(X) \to A(P)$ of the inclusion of X in P defined by J by $n_i c_i(L_P)^{p+i}$ where n_i is an integer and p is the codimension of X in P.

LEMMA 6. *Let $E \to L$ be a homomorphism from a locally free $\mathcal{O}_X$-module E of rank m to an invertible sheaf L. Assume that the scheme of zeroes Y of the homomorphism is of pure codimension m in X. Then the rational equivalence class $\gamma(Y)$ of Y in $A(X)$ is equal to $c_m(E^\vee \otimes L)$, where $E^\vee$ denotes the dual $\mathcal{O}_X$-module of the module E.*

For a proof see [3, Remark following Theorem 2, p. 153].

LEMMA 7. *With the above notation, the following equations hold.*

(i) *Let x denote an element of $A(T)$. Then*

$$j_!(x) = x \cdot c_1(\lambda^* L_T^{-1} \otimes L_B) = x(c_1(L_B) - c_1(L_T))$$

and $j^!(c_1(L_B)) = 0$.

(ii) *Put* $(n+1) = \dim_k V$. *Then*

$$f^! c_1(L_{P\times P})^i = (c_1(L_B) + c_1(L_P))^i \quad \textit{for all } i\,,$$

$$f_! f^! = \mathrm{id}_{A(P\times P)}$$

and for $i = 0, 1, \ldots, (n-1)$ *we have*

$$f_! c_1(L_T)^i = (c_1(L_{P\times P}) - c_1(L_P))^i\,.$$

(iii) *The formula*

$$c_1(L_B)^i = c_1(L_T)^{i-1} c_1(L_B)\,,$$

holds for $i = 1, 2, \ldots$.

PROOF. (i) From Lemma 5 (i) together with Lemma 6 we conclude that

$$j_!(T) = \gamma(T) = c_1((L_T)_B^{-1} \otimes L_B)$$

where $(L_T)_B = \lambda^* L_T$. The first formula of assertion (i) is thus an immediate consequence of the projection formula applied to j. In order to prove the second formula of assertion (i), it is sufficient to note that j is defined by the surjection $M \to \mathcal{O}_T$ of $\mathcal{O}_T$-modules and consequently that $j^* L_B = \mathcal{O}_T$.

(ii) By Lemma 5 (ii) we have that

$$f^* L_{P\times P} = ((L_P)_T)_B \otimes L_B = \lambda^* g^* L_P \otimes L_B\,.$$

The first quality of (ii) is an immediate consequence of this formula.

The second equality of (ii) is an immediate consequence of the facts that f is a birational morphism (being a monoidal transformation) and that the Chow ring gives rise to a graded intersection theory.

To prove the third equality of (ii), we rewrite the expression $f_! c_1(L_T)^i$ as

$$f_!\big(c_1(L_T)^{i-1}(c_1(L_T) - c_1(L_B))\big) + f_!\big(c_1(L_T)^{i-1}(c_1(L_B) + c_1(L_B))\big)$$
$$- f_!\big(c_1(L_T)^{i-1} c_1(L_P)\big)\,.$$

Computing each of the three terms separately, we obtain

$$f_!\big(c_1(L_T)^{i-1}(c_1(L_T) - c_1(L_B))\big) = f_! j_!\big(c_1(L_T)^{i-1}\big)$$

by formula (i) of Lemma 7 and $f_! j_!(c_1(L_T)^{i-1}) = d_! g_!(c_1(L_T)^{i-1})$ which is zero for $i=1,\ldots,(n-1)$ (see [11, Exposé 4, sections 2, 4, 5 and 6]). Moreover, we have the equations

$$\begin{aligned} f_!(c_1(L_T)^{i-1}(c_1(L_B)+c_1(L_T))) &= f_!(c_1(L_T)^{i-1}f^!c_1(L_{P\times P})) \\ &= f_!(c_1(L_T)^{i-1})c_1(L_{P\times P}) \end{aligned}$$

by the first formula of Lemma 7 (ii) and the projection formula. Finally, we have that

$$f_!(c_1(L_T)^{i-1}c_1(L_P)) = f_!(c_1(L_T)^{i-1})\cdot c_1(L_P)$$

by the projection formula. In sum, we have proved for $i=1,\ldots,(n-1)$, the formula

$$f_!(c_1(L_T)^i) = f_!(c_1(L_T)^{i-1})(c_1(L_{P\times P})-c_1(L_P)) .$$

The last formula of (ii) follows from this formula by induction on i.

To prove the formula (iii) we recall that by the definition of Chern classes ([3, § 3, p. 144]), the following relation is satisfied

$$c_1(L_B)^2-c_1(M)c_1(L_B)+c_2(M) = 0 .$$

However, it follows from the exact sequence $0\to L_T\to M\to \mathcal{O}_T\to 0$ of the diagram (***) that $c_1(M)=c_1(L_T)$ and $c_2(M)=0$. Consequently, we have the formula $c_1(L_B)^2=c_1(L_T)c_1(L_B)$ which is the case $i=2$ of assertion (iii). The general case follows immediately by induction on i.

LEMMA 8. *With the above notation, the following equality holds in* $A(\boldsymbol{P}(\Omega_P^1\,|\,X))$,

$$w_!\gamma(\boldsymbol{P}(\Omega_X^1)) = c_p(N(J)\otimes b^*L_T) .$$

Moreover, we have that

$$c_p(N(J)\otimes b^*L_T) = \sum_{i+j=p} c_1(b^*L_T)^i c_j(N(J)) .$$

PROOF. On X, there is a canonical exact sequence of locally free $\mathcal{O}_X$-modules (see [1, VII, Theorem 5.8]),

$$0\to J/J^2\to \Omega_P\,|\,X\to \Omega_X\to 0 .$$

The first part of the lemma follows from Lemma 2 and Lemma 6 applied to this sequence.

The second part of the lemma follows from the first part and a well known formula for the highest Chern class of a locally free sheaf tensorized by an invertible sheaf.

COROLLARY 9. *With the above notation, the following formula holds in* $A(T)$,

$$b_! w_! \gamma(\boldsymbol{P}(\Omega_X^1)) = \sum_{i+j=p} n_j c_1(L_T)^i c_1(L_P)^{p+j} .$$

PROOF. We want to compute $b_! w_! \gamma(\boldsymbol{P}(\Omega_X^1))$. By Lemma 8 and the projection formula we obtain

$$b_! w_! \gamma(\boldsymbol{P}(\Omega_X^1)) = \sum_{i+j=a} b_!(c_1(b^* L_T)^i c_j(N(J)))$$

$$= \sum_{i+j=a} c_1(L_T)^i b_! c_j(N(J)) .$$

The formula of the corollary follows since, with the above notation and identifications, we clearly have

$$b_! c_j(N(J)) = n_j c_1(L_P)^{p+j} .$$

LEMMA 10. *With the above notation, the following equality holds in* $A(T)$,

$$b_! w_! \gamma \boldsymbol{P}(\Omega_X^1) = j^! \gamma(B(X \times X)) .$$

PROOF. The scheme $\boldsymbol{P}(\Omega_X)$ is the exceptional locus of the monoidal transformation $B_\Delta(X \times X)$ and consequently it is isomorphic, by the composite morphism bw, to the scheme theoretic inverse image of the scheme $B(X \times X)$ by the morphism j. Consequently, Lemma 10 follows from the following well known result.

LEMMA 11. *Let* $a: Z \to Y$ *be a morphism of irreducible schemes, both quasi-projective and smooth over* Spec k. *Moreover, let* V *be an integral subscheme of* Y. *Assume that* Z *is of codimension one in* Y *and that the scheme theoretic intersection* $a^{-1}(V)$ *of* Z *and* Y *is of pure codimension one in* Z. *Then the formula* $a^!(\gamma(V)) = \gamma(a^{-1}(V))$ *holds in* $A(Z)$.

PROOF. Let q be a generic point of $a^{-1}(V)$. Since Z is of codimension one in Y there is an exact sequence

$$0 \to \mathcal{O}_{Y,q} \xrightarrow{e} \mathcal{O}_{Y,q} \to \mathcal{O}_{Z,q} \to 0$$

where e is an element of the local ring $\mathcal{O}_{Y,q}$. Tensoring the exact sequence by $\mathcal{O}_{V,p}$ we obtain the sequence

$$0 \to \mathcal{O}_{V,q} \xrightarrow{\bar{e}} \mathcal{O}_{V,q} \to \mathcal{O}_{a^{-1}(V),q} \to 0$$

which is exact because V is integral. Consequently, we have that

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,q}}(\mathcal{O}_{Z,q}, \mathcal{O}_{V,q}) = 0 \quad \text{for } i>0$$

and since $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Z, \mathcal{O}_V) = \mathcal{O}_{a^{-1}(V)}$ the lemma follows.

LEMMA 12. *The rational equivalence class of $X \times X$ in the ring $A(P \times P)$ is equal to $d^2 c_1(L_P)^p c_1(L_{P\times P})^p$, here p and d are the codimension and the degree of X in P.*

PROOF. Clearly the product $(X \times X)$ is the transversal intersection of the subschemes $(X \times P)$ and $(P \times X)$ of the product $P \times P$, hence $\gamma(X\times X) = \gamma(P\times X)\cdot\gamma(X\times P)$. Moreover, recall that $\boldsymbol{P}(V_P) = P \times P$ is considered as a projective bundle over P via the projection onto the second factor. Consequently, $\gamma(P\times X) = dc_1(L_P)^p$. Finally, we have that

$$\gamma(X \times P) = dp^!{}_1 c_1(L_P)^p$$

where p_1 is the projection onto the first factor and clearly we have that $p_1^* L_P = L_{P\times P}$.

As a result of the preliminary computations, we obtain the following theorem which is the main result of this article.

THEOREM 12. *With the above notation we have the following two expressions for the rational equivalence class of $B(X \times X)$ in $A(B)$,*

(i) $f^! \gamma(X\times X) - (c_1(L_B) - c_1(L_T)) \sum\limits_{i+j=p-1} n_j c_1(L_T)^i c_1(L_p)^{p+j}$

(ii) $\sum\limits_{i+j=p} n_j c_1(L_T)^i c_1(L_P)^{p+j} + c_1(L_B)\left(\sum\limits_{i+j=p-1} (d^2\binom{p}{j} - n_j) c_1(L_T)^i c_1(L_P)^{p+j}\right)$

where d is the degree of X in P and p is the codimension of X in P.

PROOF. Since the morphism f is an isomorphism outside of the exceptional locus T, the subscheme $B(X \times X)$ and $f^{-1}(X \times X)$ agree outside of T. Consequently, the element $\gamma(B(X\times X)) - f^! \gamma(X\times X)$ of the ring $A(B)$ will lie in the image of $A(T)$ by $j_!$. However, we consider $A(T)$ as a subring of $A(B)$ by the homomorphism $\lambda^!$ and since $j^!\lambda^!$ is the identity of $A(T)$ we may write

$$\gamma(B(X\times X)) - f^! \gamma(X \times X) = j_! x$$

with $x \in A(T)$. To determine x we recall that $j_!(x) = x(c_1(L_B) - c_1(L_T))$ and that

$$j^!(x(c_1(L_B) - c_1(L_T))) = -xc_1(L_T)$$

by Lemma 7 (i). On the other hand, we have that

$$j^! \gamma(B(X \times X)) = b_! w_! \gamma(\boldsymbol{P}(\Omega_X^1))$$

by Lemma 10 and that

$$j^! f^! \gamma(X \times X) = j^! f^! d^2 c_1(L_P)^p c_1(L_{P \times P})^p$$

by Lemma 12. Consequently, we have the formula

$$j^! f^! \gamma(X \times X) = d^2 j^! c(L_P)^p (c_1(L_B) + c_1(L_P))^p = d^2 c_1(L_P)^{2p} ,$$

by Lemma 7 (i) and (ii). We conclude that

$$b_! w_! \gamma(\boldsymbol{P}(\Omega_X^1)) - d^2 c_1(L_P)^{2p} = -x c_1(L_T) ,$$

and using Corollary 9, we obtain the formula

$$\sum_{i+j=p} n_j c_1(L_T)^i c_1(L_P)^{p+j} - d^2 c_1(L_P)^{2p} = -x c_1(L_T) ,$$

in $A(T)$. However, the elements $c_1(L_T)^i$ $i=0,1,\ldots,(n-1)$ form a basis for the $A(P)$ module $A(T)$ and clearly x and $c_1(L_P)$ are both in the ring $A(P)$. Consequently, we have that $n_p c_1(L_P)^{2p} = d^2 c_1(L_P)^{2p}$ and that

$$x = - \sum_{i+j=p-1} n_j c_1(L_T)^i c_1(L_P)^{p+j} .$$

Having determined x, the proof of formula (i) is complete.

In the proof of formula (i) we used the formula

$$\begin{aligned} f^! \gamma(X \times X) &= d^2 c_1(L_P)^p (c_1(L_B) + c_1(L_P))^p \\ &= d^2 c_1(L_P)^p \sum_{i+j=p} \binom{p}{i} c_i(L_B)^i c_1(L_P)^j . \end{aligned}$$

From this formula together with the formula (iii) of Lemma 7 we obtain the expressions

$$\begin{aligned} f^! \gamma(X \times X) &= d^2 c_1(L_P)^p c_1(L_B) \sum_{i+j=p-1} \binom{p}{i+1} c_1(L_T)^i c_1(L_P)^j + d^2 c_1(L_P)^{2p} \\ &= d^2 c_1(L_B) \sum_{i+j=p-1} \binom{p}{j} c_1(L_T)^i c_1(L_P)^{p+j} + d^2 c_1(L_P)^{2p} . \end{aligned}$$

Substituting these expressions into formula (i) of the theorem and reordering, we obtain (applying the equality $n_p c_1(L_P)^{2p} = d^2 c_1(L_P)^{2p}$ noted in the first part of the proof) formula (ii).

DEFINITION 13. The scheme theoretic image of $B(X\times X)$ by λ we call the pointed secant bundle of X, relative to the imbedding of X in P defined by the ideal J of $\mathcal{O}_P$ and denote it by Sb (J). Note that since $p_2 f=g\lambda$ with the notation of the diagram (****) we have that g maps Sb (J) onto X.

The scheme theoretic image of $\lambda^{-1}\,\mathrm{Sb}\,(J)$ by the composite morphism $p_1 f\colon B\to P$, where p_1 is the projection of $P\times P$ onto the first factor we call the secant scheme of X in P, relative to the imbedding of X in P defined by the ideal J of $\mathcal{O}_P$, and denote it by Sec (J). In other words, Sec (J) is the transform of Sb (J) by the incidence correspondence between the schemes T and P defined by the subscheme B of the product $T\times P$.

Intuitively, the points of T consists of a point x of P with a line going through it and the fiber of $g\,|\,\mathrm{Sb}\,(J)$ at a point x of X consists of the point x together with all the secants of X passing by the point x. Moreover, the fiber of λ at a point t of T consists of all points of the line through the point $g(t)$ in the tangent direction t. Consequently, Sec (J) consists of all points lying on all secants to X.

LEMMA 14. *Let m denote the dimension of the scheme X. Then the pointed secant bundle* Sb (J) *is of dimension* $(2m-1)$ *when X is a linear subspace of P and of dimension $2m$ when X is non-linear. In the latter case, the morphism $\lambda\,|\,B(X\times X)$ maps $B(X\times X)$ birationally onto* Sb (J) *when X is not a hypersurface in P. When X is a hypersurface of P then the fiber of $\lambda\,|\,B(X\times X)$ over a generic point of* Sb (J) *consists of* $(\deg(X)-1)$ *simple points.*

PROOF. Let t be a rational point in Sb (J) and put $x=g(t)$. The fiber of λ at t is the projective line $\boldsymbol{P}(M(t))$, where $M(t)$ is the restriction to t of the locally free $\mathcal{O}_T$-module M defined by the diagram (***). Denote by h the morphism

$$\boldsymbol{P}(N(t))\to P\times x\cong P$$

obtained by restricting f to $N(t)$. Recall that the morphism f is defined by the surjective map $(V_P\otimes L_P^{-1})_T\to M$ of the diagram (***) together with a "twisting by L_P". Hence, the morphism h is defined by the restriction $V\cong V(t)\to M(t)$ of this surjection to the point t of T. In other words, h maps the projective line $\boldsymbol{P}(M(t))$ isomorphically onto the line in P through x "in the direction" defined by t. We know that the morphism f is an isomorphism outside of T. Consequently, outside of T the intersection of $B(X\times X)$ and $\boldsymbol{P}(M(t))=\lambda^{-1}(t)$ is isomorphic to the intersection of $X\times x\cong X$ with the line $h(\boldsymbol{P}(M(t)))$ in $P\times x\cong P$ outside of x. If X is a linear subspace, then clearly the fiber $\lambda^{-1}(f)$ is contained in $B(X\times X)$ and consequently the dimension of the image of Sb (J) by λ will be one less than the dimension $2m$ of $B(X\times X)$. If X is a hypersurface, then by Bertini's theorem (e.g., [5, Corollary 11, p. 296]) we may choose t in T such that $g(t)$ is in X and such that the line $\boldsymbol{P}(M(t))$ intersect X in deg (X) simple points. Consequently, the fiber of $\lambda \mid B(X \times X)$ at a general point of Sb (X) consists of (deg $(X)-1$) simple points. In particular, if X is not a hyperplane in P then $\dim(\mathrm{Sb}(X)) = \dim(B(X \times X)) = 2m$. Finally, assume that X is neither a hypersurface in P nor a linear subspace. Then, by the following lemma a generic secant intersect X in exactly two simple points. Choose t to be such a secant together with one of the points of intersection. Then the fiber of $\lambda \mid B(X \times X)$ at t consists of one simple point. It easily follows that $\lambda \mid B(X \times X)$ defines a birational map between the schemes $B(X \times X)$ and Sb (X).

LEMMA 15. *Let X be a non-linear integral subscheme of P of codimension strictly greater than one. Assume that X is not contained in any linear subspace of P and that it is non-singular in codimension one. Then a generic secant to X intersect X in exactly two simple points.*

PROOF. A generic linear subspace of P of codimension equal to (dim $(X)-1$) intersect X in a smooth connected curve ([5, 11 Corollary p. 296] and [7, Theorem 7, Chapter VIII, § 6, p. 212]) and since $(n-\dim(X)-1)>2$ and X is not contained in a hyperplane, this curve is not contained in a plane. Since the intersection of a secant to the curve with the curve is the same as the intersection of the secant with X it is sufficient to prove the lemma when X is a curve.

Assume that every secant to the curve X intersect X in at least three points. Let x and y be points on X and assume that the secant through x and y is not tangent to X at any point. We shall show that the tangents to X at the points x and y intersect. Let z be a third point on the secant through x and y. The projection of P onto $\boldsymbol{P}^{n-1}$ with center z maps X onto a curve X'. A general secant to the curve X through x intersect X in distinct points $x, x_1, \ldots, x_s$ with $s \geqq 2$ and maps to a nonsingular point x' of X under the projection. The tangents $t_1, \ldots, t_s$ to the curve X at the points $x_1, \ldots, x_s$ then map onto the tangent to X' at x'. Consequently, the tangents $t_1, \ldots, t_s$ all intersect. In particular a general secant through z has the property that the tangents to X through any two points on the secant, different from z, intersect. Hence this property must hold for all secants through z. In particular, the tangents through x and y intersect. Since the tangents at two points on X in general position intersect it easily follows that the tangents at any two points intersect.

Fix two tangents t_1 and t_2 to the curve X and denote by u their point of intersection. Then either all tangents to the curve X pass through u or at most a finite number of tangents pass through u. In the first case, the curve is called strange and the only nonsingular strange projective curves are the lines and in characteristic two also the plane conics ([9, Appendix to Chapter II, p. 76]). In the second case all but a finite number (and hence all) tangents to X lie in the plane spanned by the lines t_1 and t_2, hence the curve X lies in this plane. Since by assumption X is not contained in a plane we conclude that every secant to the curve X can not intersect X in at least three points.

PROPOSITION 16. *Assume that X is a subscheme of P of degree $d>1$. Then with the above notation, the rational equivalence class of* $\mathrm{Sb}(J)$ *in* $A(T)$ *is equal to*

$$v^{-1}\sum_{i+j=p-1}\left(d^2\binom{p}{j}-n_j\right)c_1(L_T)^i c_1(L_P)^{p+j},$$

where v is one when X is not a hypersurface $(p>1)$ *and* $v=(\deg(X)-1)$ *when X is a hypersurface* $(p=1)$.

PROOF. We have that $\lambda_! 1_{A(B)}=0$ and $\lambda_! c_1(L_B)=1_{A(T)}$ (see [11, Exposé 4, sections 2, 4, 5 and 6]). Hence, applying $\lambda_!$ to the formula (ii) of Theorem 12 we obtain the formula

$$\lambda_!\gamma(B(X\times X)) = \sum_{i+j=p-1}\left(d^2\binom{p}{j}-n_j\right)c_1(L_T)^i c_1(L_P)^{p+j}.$$

The formula of the proposition now follows from the equality $\lambda_!(B(X\times X)) = v\gamma(\mathrm{Sb}(J))$ which holds because of Lemma 14.

COROLLARY 17. *Assume that X is not a linear subspace of P. Then, with the above notation, the following equalities hold in* $A(P\times P)$,

$$\begin{aligned} f_!\lambda^!\gamma(\mathrm{Sb}(J)) &= v^{-1}\sum_{i+j=p-1}\left(d^2\binom{p}{j}-n_j\right)\left(c_1(L_{P\times P})-c_1(L_P)\right)^i c_1(L_P)^{p+j} \\ &= v^{-1}\sum_{i=0}^{p-1}\sum_{j=i}^{p-1}(-1)^{j-i}\left(d^2\binom{p}{p-1-j}-n_{p-1-j}\right)\binom{j}{i}c_1(L_P)^{2p-i-1}c_1(L_{P\times P})^i \\ &= v^{-1}\sum_{i=0}^{p-1}\left(d^2-\sum_{j=i}^{p-1}(-1)^{j-i}\binom{j}{i}n_{p-1-j}\right)c_1(L_P)^{2p-i-1}c_1(L_{P\times P})^i \end{aligned}$$

PROOF. The first equality of Corollary 17 is an immediate consequence of Proposition 16 together with the last formula of Lemma 7, (ii). The second equality follows by a simple reordering. Finally, the third equality follows from the second equality and the formula

$$\sum_{j=i}^{p-1}(-1)^j\binom{p}{p-1-j}\binom{j}{i} = (-1)^i.$$

The last formula follows from the trivial formula

$$\sum_{j=0}^{p-1-i} \binom{p}{p-1-j+i}\binom{-i-1}{j} = 1$$

together with the identity

$$\binom{j}{i} = (-1)^{j-i}\binom{-i-1}{j-i}.$$

4. Applications.

I. (A. Holme [4]). *Embeddings of non-singular algebraic schemes by linear projections.*

We keep the notation of the previous section. In particular, X is an algebraic scheme, smooth over an algebraically closed field k with a fixed embedding into a projective space of dimension n over the field k.

The following classical and geometrically obvious result holds.

PROPOSITION 18. *The scheme X can be embedded into a projective space P', of dimension $n' < n$ over k, by a projection from P with center on a linear subspace if and only if there exists a linear subspace L of P of dimension $(n-n'-1)$ which is disjoint from the secant scheme* Sec (J).

In particular, $s(J) = \dim \operatorname{Sec}(J)$ is the smallest integer such that X can be embedded into the projective space of that dimension by a projection from P with center on a linear subspace of P.

For a proof see e.g. [4, IV, Proposition 12.1].

To compute the number $s(J) = \dim \operatorname{Sec}(J)$, we use, following Holme, the following, easily proved observation (for a proof, see [4, II Lemma 7.5].)

LEMMA 19. *With the above notation, let Y be a closed irreducible subvariety of the product $\boldsymbol{P}(V_P) = P \times P$ and write*

$$\gamma(Y) = \sum_{i=0}^{n} a_i c_1(L_{P\times P})^i$$

in $A(\boldsymbol{P}(V_P))$ with a_i in $A(P)$. Then $\dim p_1(S) = s$ *if and only if* $a_0 = \ldots = a_{n-s-1} = 0$ *and* $a_{n-s} \neq 0$.

THEOREM 20. (A. Holme [4, II, Theorem 4.2.]). *With the above notation the scheme X can be embedded in a projective space of dimension s by a projection from P, with center on a linear subspace, if and only if the integers*

$$
\begin{aligned}
2s_i(J) &= \sum_{j=i}^{p-1} (-1)^{j-i}\left(d^2\binom{p}{p-1-j} - n_{p-1-j}\right)\binom{j}{i} \\
&= \sum_{j=0}^{p-1-i} (-1)^{p-1-j-i}\left(d^2\binom{p}{j} - n_j\right)\binom{p-1-j}{i} \\
&= d^2 - \sum_{j=0}^{p-1-i} (-1)^{p-1-j-i}\binom{p-1-j}{i} n_j
\end{aligned}
$$

are zero for $i=0,\dots,n-s-1$ and s_{n-s} is different from zero.

PROOF. Denote by Y the scheme theoretic image by f of the scheme $\lambda^{-1}\,\mathrm{Sb}\,(J)$. Then, since λ is the structure morphism of a projective bundle and f is birational outside of $f^{-1}P$ it is clear that we have the equality $f_!\lambda^!\gamma(\mathrm{Sb}\,(J)) = \gamma(Y)$ in the ring $A(P\times P)$. The theorem is thus an immediate consequence of Corollary 17 together with Lemma 19.

II. (C. A. M. Peters and J. Simonis [8]). *The secants of a nonsingular algebraic scheme passing through a general point.*

Assume that X is not a linear subspace of P. Since the morphism λ is flat with one dimensional integral fibers and the morphism f is birational it follows from Lemma 4 that the scheme $Y=f\lambda^{-1}\,\mathrm{Sb}\,(J)$ is irreducible of dimension $(2m+1)$. Denote by μ the number of points (counted with multiplicity) of the general fiber of the morphism $q\colon Y\to \mathrm{Sec}\,(J)$ of Y onto $\mathrm{Sec}\,(J)$ induced by the projection of $P\times P$ onto the first factor. We call the number $2^{-1}\mu$ the number of secants through a general point of $\mathrm{Sec}\,(J)$ and when $\mathrm{Sec}\,(J)$ is properly contained in P we say that there are no secants through a general point of P.

Clearly μ is never zero and is finite if and only if the equality $\dim(\mathrm{Sec}\,(J)) = (2m+1)$ holds. Intuitively, the points of the fiber of the morphism q over a point y of $\mathrm{Sec}\,(J)$ consists of pairs (y,x) where x is a point of X lying on a secant to X through y. When y is in general position on $\mathrm{Sec}\,(J)$ and X is not contained in a linear subspace of P it follows from Lemma 15 that to each secant l to X through y there correspond exactly two such pairs (y,x_1) and (y,x_2) where x_1 and x_2 are the points of intersection of l and x.

THEOREM 21. *Assume that X is not a linear subspace of P. Denote by e the degree of the secant scheme* $\mathrm{Sec}\,(J)$ *in P and by p the codimension of X in P. With the above notation the following two assertions hold.*

(i) *Assume that* $\dim(\mathrm{Sec}\,(J))=2m+1$. *Then the number of secants to the scheme X passing through a general point of the secant scheme* $\mathrm{Sec}\,(J)$ *is equal to*

$$e^{-1}s_{p-m-1}(J) = 2^{-1}e^{-1}\left(d^2 - \sum_{j=p-m-1}^{p-1} (-1)^{j-p+m+1}\binom{j}{p-m-1}n_{p-1-j}\right)$$

$$= 2^{-1}e^{-1}\left(d^2 - \sum_{j=0}^{m} (-1)^{m-j}\binom{p-1-j}{p-1-m}n_j\right)$$

(ii) (C. A. M. Peters and J. Simonis [8, Theorem 3.4]). *When finite, the number of secants to X passing through a general point of P is equal to*

$$s_0(J) = 2^{-1}\left(d^2 - \sum_{j=0}^{p-1} (-1)^{p-1-j}n_j\right)$$

PROOF. By definition of the direct image map $(p_1)_!: A(P \times P) \to A(P)$ we have that the class $(p_1)_!(\gamma(Y))$ is equal to $\mu\gamma(\operatorname{Sec}(J))$ when $\dim \operatorname{Sec}(J) = (2m+1)$ and is zero when $\dim \operatorname{Sec}(J) < (2m+1)$. Consequently, when $\dim(\operatorname{Sec}(J)) = (2m+1)$ the number $e\mu$ is equal to

$$c_1(L_P)^{2m+1}(p_1)_!(\gamma(Y)) = (p_1)_!(c_1(L_{P\times P})^{2m+1}\cdot\gamma(Y)).$$

We clearly have an equality $\gamma(Y) = f_!\lambda^!\gamma(\operatorname{Sb}(J))$. Consequently, part (i) of the theorem follows from Corollary 17 and the relations $(p_1)_!c_1(L_p)^i = 0$ for $i \neq n$ and $(p_1)_!c_1(L_P)^n = \gamma(P)$.

Clearly, the assertions (i) and (ii) of the theorem are the same when $\dim \operatorname{Sec}(J) = (2m+1) = n$. Consequently, to prove part (ii) of the theorem it is sufficient to prove that the number $s_0(J)$ is zero when $\operatorname{Sec}(J)$ is properly contained in P. However, $\operatorname{Sec}(J)$ is properly contained in P if and only if $\gamma(\operatorname{Sec}(J))c_1(L_P)^n$ is equal to zero. The last equality clearly holds if and only if $\gamma(S)\cdot c_1(L_{P\times P})^n$ is zero (see e.g. Lemma 19). Part (ii) of the theorem is now an immediate consequence of Corollary 17 and the relations $\gamma(Y) = f_!\lambda^!\gamma(\operatorname{Sb}(J))$ and $c_1(L_{P\times P})^i = 0$ for $i > n$.

The connection between the results of Holme and Peters–Simonis is now clear. To obtain Holmes result from the result of Peters–Simonis, we note that by Proposition 18 and Theorem 21 X can be embedded into $\boldsymbol{P}^{n-1}$ by a projection from P if and only if $s_0(J)$ zero. Similarly, for X to be embedded into $\boldsymbol{P}^{n-2}$ by a projection from $\boldsymbol{P}^{n-1}$ it is necessary and sufficient that the integer $s_0(J')$ is zero, where J' is the ideal defining the embedding of X into $\boldsymbol{P}^{n-1}$. The integer $s_0(J')$ is easily seen to be equal to $s_1(J)$. Continuing in this same way successively projecting X into projective spaces $\boldsymbol{P}^{n-3}, \boldsymbol{P}^{n-4}, \ldots$ we obtain the integers $s_2(J), s_3(J), \ldots$ and finally Holmes criterion Theorem 20. Conversely, by the theorem of Holme and Proposition 18, $s_0(J)$ is zero if and only if $\operatorname{Sec}(J)$ is properly contained in P. Hence to obtain the result of Peters–Simonis it is sufficient to identify s_0 with the number of secants through a general point of P

when this number is finite and different from zero. However, as we have seen in section 4 II, this identification follows from the definition, used by Holme, of Sec (J) as the image of $Y = f\lambda^{-1}\,\mathrm{Sb}\,(J)$ by the projection p_1 and from Lemma 15 above.

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