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THE THEOREM OF DESARGUES IN PLANES WITH ANALOGUES TO EUCLIDEAN ANGULAR BISECTORS

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1. Introduction.

In the Euclidean plane one can bisect an angle of a triangle by joining the vertex of the angle to the point which divides the opposite side in the ratio of adjacent sides. This construction uses only distances and it does not depend explicitly on angular measure. Making this simple but ingenious observation H. Busemann [3] characterized the Minkowski planes

(i) among all Desarguesian planes, and also,
(ii) among all straight planes with differentiable circles,

by the following property: (P) "Inside a nonstraight convex angle with legs N_1, N_2 and vertex v there is a ray M with origin v such that any segment $T(a_1, a_2)$ with $a_i \in N$, $i=1,2$, $a_i \neq v$, intersects M in a point $b=b(a_1,a_2)$ for which the distances satisfy $va_1 : va_2 = ba_1 : ba_2$". The question as to whether the differentiability hypothesis is necessary in the second characterization was left as an open question. It is the purpose of this paper to show that the hypothesis of differentiable circles is indeed *not* needed.

We use the same terminology and notation as in [2] and [3]. Thus R is a straight plane with distance xy satisfying the property (P). We are going to prove that R is Desarguesian. From this it follows by characterization (i) above, proved by Busemann in [3], that R is in fact Minkowskian.

Our proof is divided into three short paragraphs which follow. The main idea is to show that the group of dilatations with an arbitrary centre is transitive on R.

2. The fundamental axioms of affine geometry.

We take the set of points of R as the set $\mathscr{P}$, the set of metric straight lines of R as the set $\mathscr{L}$ and the incidence relation I given by set inclusion i.e. for $p \in \mathscr{P}$ and $L \in \mathscr{L}$ we say $p\ I\ L$ if and only if $p \in L$. In this section we briefly note that

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the "incidence structure" $(\mathscr{P}, \mathscr{L}, I)$ satisfies the fundamental axioms of affine geometry:

(i) If $a \neq b$ there exists one and only one line $L(a,b)$ incident with a and b. This is true because R is a straight space, see [2, p. 38, also ° 10.7 and ° 11.1].

(ii) There are three non collinear points. This follows from the fact that the topological dimension of R is 2.

(iii) Given a point p and a line L with p not on L there exists a unique line L' containing p such that L and L' do not meet. This follows from the following two propositions:

PROPOSITION 1. *The property* P *implies the following property* (P′): *The parallel axiom holds. For two distinct parallel lines L_1 and L_2 there is a line L (parallel to L_i) which contains the centres of all segments $T(p_1, p_2)$ with $p_i \in L_i$, $i=1,2$.*

This is proved by Busemann [3, p. 6]. Since, however, the meaning of "parallel" and "parallel axiom" is different in the context of G-spaces [2, p. 141] we also need the following proposition, again proved by Busemann [2, ° 23.7, p. 141]. (This proposition shows that in straight planes parallel axiom is the same as the usual one.)

PROPOSITION 2. *The parallel axiom holds in a straight 2-dimensional space if and only if for a given line L and a given point p not on L exactly one line through p exists which does not intersect L.*

These propositions imply that in our straight plane R "being parallel" is an equivalence relation. (Compare with ° 23.6 of [2, p. 141]).

We have thus proved that R with the incidence structure $(\mathscr{P}, \mathscr{L}, I)$ is an affine plane in the sense of Artin [1, p. 52, 53].

3. Existence of dilatations.

Let L_1 and L_2 be two parallel lines and let α be a number between 0 and 1. For a segment $T(a_1, a_2)$ with $a_i \in L_i$, $i=1,2$ let $m_\alpha = m_\alpha(a_1, a_2)$ be the point on $T(a_1, a_2)$ such that $a_1 m_\alpha = \alpha a_1 a_2$. The second condition in property P′ (see Proposition 1 above) can now be stated as follows: the points $m_{\frac{1}{2}}(a_1, a_2)$, $a_i \in L_i$, $i=1,2$ lie on a line L which is parallel to L_1, L_2.

By "bisecting and doubling" we see that $m_\alpha(a_1, a_2)$, $a_i \in L_i$, $i=1,2$ *lie on a line parallel to L_i* whenever α is a dyadic fraction. As the dyadics form a dense set in $[0,1]$ the same conclusion holds for all α, $0 \leqq \alpha \leqq 1$.

Using the symbol $[abc]$ to denote three points a, b, c such that $ab + bc = ac$ (that is, $b \in T(a,c)$) we see that the above observations yield the following:

LEMMA 1. (i) *If* $[axx']$ *and* $[ayy']$ *and* $ay':ay=ax':ax$ then $L(x,y)$ *is parallel to* $L(x',y')$.

(ii) *If* $[xax']$ *and* $[yay']$ *and* $ax:ax'=ay:ay'$ *then* $L(x,y)$ *is parallel to* $L(x',y')$.

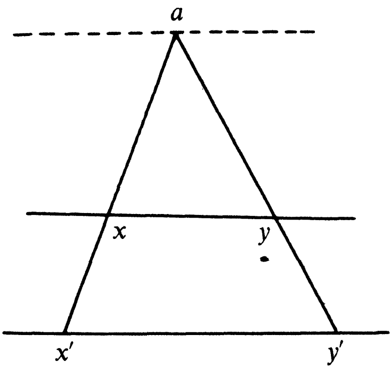

Figure 1

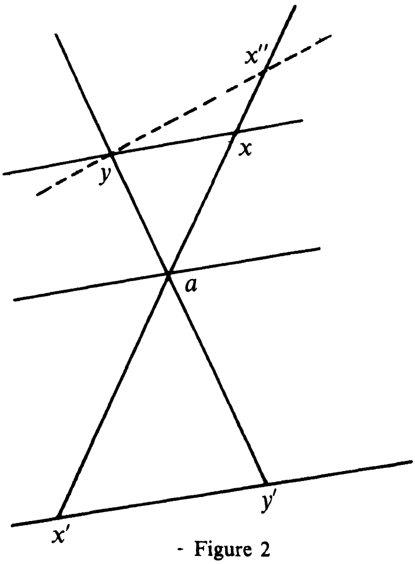

Figure 2

To prove (i) we draw a line through a parallel to $L(x',y')$ and consider division of segments $T(a,x')$ and $T(a,y')$. To prove (ii) we draw a line through y parallel to $L(x',y')$ and let this line meet $L(a,x')$ in x''. By comparing ratios and considering the line through a parallel to $L(x',y')$ we show $x''=x$. This proves (ii).

We are now ready to show the existence of dilatations.

Let z be any point of R and let x,x' be points on a line through z. We show that there is an incidence preserving bijection which takes z to z, x to x' and which takes a line to a parallel line. Such a bijection is an affine collineation called a "dilatation" (sometimes "homothety" or "similarity") with centre z.

Let $zx'=\lambda zx$. We send a point p to p' in such a way that

(i) $zp'=\lambda zp$
(ii) p,p' lie on a line through z
(iii) z lies between p and p' if and only if z lies between x and x'.

Given p these conditions uniquely determine p'.

We show that if p,q,r lie on a line L so do their images. We may assume p,q,r do not lie on a line through z because if they do then p',q',r' also lie on this same line by the definition of the mapping. As $zp:zp' = zq:zq' = zr:zr'$ we have, by the lemma 1 proved above, that $L(p, q)$ is parallel to $L(p', q')$ and $L(q, r)$ is parallel to $L(q', r')$. As $L(p, q)$ and $L(q, r)$ are the same line and since $L(p', q')$ and $L(q', r')$ have the point q' in common, this shows that $L(p', q') = L(q', r')$. Hence p', q', r' lie on a line parallel to the line containing p, q, r.

We have thus proved the existence of the dilatation with centre z and taking x to x'. As z was an arbitrary point we have proved:

LEMMA 2. *The group of dilatations with centre z is linearly transitive on R for every choice of point z.*

4. Proof of the main results.

Standard theory of affine planes now applies and, in particular, from Theorem 2.16 of [1, p. 71] we see that the affine theorem of Desargues holds in our space. Combining this with the results of the previous sections we have the following:

THEOREM 1. *A straight plane R satisfying the property* P *carries an incidence structure of an affine plane whose points are points of R, whose lines are lines of R and whose incidence relation is derived from set inclusion. In this affine plane the affine theorem of Desargues holds true.*

It is well known that this implies that our geometry is coordinatized by a skew field, see, for example [1, Chapter 2]. This skew field must be the field of real numbers in our case because our lines are metric straight lines which are isometric to, and hence models of, the real line. This, together with theorem 1 of Busemann [3, p. 7] gives us the following:

THEOREM 2. *A straight plane satisfying the property* P *is a Minkowski plane.*

This shows that the hypothesis of differentiable circles in theorem 2 of [3, p. 9] is not necessary.

REFERENCES

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2. H. Busemann, *The Geometry of Geodesics* (Pure and Applied Mathematics 6), Academic Press, New York, 1955.
3. H. Busemann, *Planes with analogues to Euclidean angular bisectors*, Math. Scand. 36 (1975), 5–11.

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