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INTEGRALITY RELATIONS ON SMOOTH MANIFOLDS

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1. Introduction

Hirzebruch, in his book [11], introduced the important notion of a multiplicative sequence (= m-sequence). This is a function K that assigns to each vector bundle ξ over a complex X a class

$$K(\xi) = \{K_j(\xi)\} \in H^{**}(X;A) = \prod_{i=0}^{\infty} H^i(X;A),$$

where A is some fixed coëfficient ring with unit. There are two possibilities: (i) ξ is a real oriented stable vector bundle, in which case $K_j(\xi) \in H^{4j}(X;A)$; or (ii) ξ is a complex stable vector bundle, with $K_j(\xi) \in H^{2j}(X;A)$. In the first case we call K real, and in the second case, complex. The m-sequence K satisfies three axioms.

(1.1) $$K_0(\xi) = 1 \in H^0(X;A).$$

(1.2) $$K(\xi\oplus\eta) = K(\xi)\cdot K(\eta), \text{ for bundles } \xi \text{ and } \eta \text{ over } X.$$

(1.3) $$\text{If } f\colon X' \to X, \text{ then } K(f^*\xi) = f^*K(\xi) \in H^{**}(X';A).$$

We call A the coefficient ring for K. In most of our applications, $A = Q$, the rational numbers.

Suppose that M is a smooth, closed manifold in the domain of K — i.e., M is oriented, if K is real, or M has a stable almost-complex structure, if K is complex. We then set $K(M) = K(\tau_M)$, where τ_M denotes the tangent bundle of M. If $[M]$ denotes the homology orientation class for M, we set

$$K[M] = \{K(M)\}[M] \in A.$$

We now can give the main definition of this paper. Suppose that $S \subset A$ is a subring, that Y is a fixed space, and that K is an m-sequence. We define a subgroup of $H^{**}(Y;A)$, $S(Y,K)$, as follows:

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(1.4) $S(Y, K)$ = all classes $\theta \in H^{**}(Y; A)$ such that for all smooth manifolds M and maps $f: M \to Y$,

$$\{f^*\theta \cdot K(M)\}[M] \in S \subset A .$$

Of course, in (1.4), we only allow manifolds in the domain of K. (In an appendix, section 10, we consider the extension of this definition to PL and Top manifolds.)

We illustrate the definition with two examples — in each case we have $A = \mathbf{Q}$ and $S = \mathbf{Z}$, the integers.

EXAMPLE 1. Take K to be the Todd sequence, td [11]. This is a complex m-sequence such that for all stably almost-complex manifolds M, $\mathrm{td}\,[M] \in \mathbf{Z}$; see [12] and [17]. By the Riemann–Roch Theorem (as extended by Atiyah–Singer [11]), given any stable complex vector bundle ω on M,

$$\{\mathrm{ch}\,\omega \cdot \mathrm{td}\,(M)\}[M] \in \mathbf{Z} ,$$

where $\mathrm{ch}\,\omega$ denotes the Chern character of ω. Now we may regard ω as a map from M into BU, the classifying space for the stable unitary group, and so we have

$$\mathrm{ch} \in \mathbf{Z}(\mathrm{BU}, \mathrm{td}) ,$$

regarding ch as an element of $H^{**}(\mathrm{BU}; \mathbf{Q})$.

EXAMPLE 2. Let K be the m-sequence L of Hirzebruch [11]. Thus if M is an oriented manifold, by the Hirzebruch signature theorem [11],

$$\mathrm{signature}\, M = \mathrm{L}\,[M] \in \mathbf{Z} .$$

The following result has recently been proved, [13], [22]; let r and s be integers with $0 \leqq r \leqq s$. Then, for any oriented manifold M and class $u \in H^2(M; \mathbf{Z})$,

$$\{\exp\,(s-2r)u\, \mathrm{sech}\, su \cdot \mathrm{L}\,(M)\}[M] \in \mathbf{Z} .$$

Since u can be regarded as a map from M into the Eilenberg–MacLane space $\mathrm{K}\,(\mathbf{Z}, 2)$, we have:

$$\exp\,(s-2r)t\, \mathrm{sech}\, st \in \mathbf{Z}(\mathrm{K}\,(\mathbf{Z}, 2), \mathrm{L}) ,$$

where we set $H^{**}(\mathrm{K}\,(\mathbf{Z}, 2); \mathbf{Q}) = \mathbf{Q}[[t]]$, degree $t = 2$.

Example 2 suggests that we consider $\mathbf{Z}(\mathrm{K}\,(\mathbf{Z}, 2n), K)$, $n \geqq 1$. We compute these groups for m-sequences that take integral values on manifolds — see (2.7). In fact, $\mathbf{Z}(\mathrm{K}\,(\mathbf{Z}, 2), K)$ is the ring of formal power series with integer coefficients, generated by a certain power series determined by K, see (2.2).

To state our result for $K(\mathbf{Z}, 2n)$, with n greater than one, we set

$$\tilde{S}(Y, K) = S(Y, K) \cap \tilde{H}^{**}(Y; A), \tag{1.6}$$

where $\tilde{H}^{**}(Y; A)$ denotes the reduced cohomology of Y. In marked contrast with the case $n=1$ we have (see section 7):

(1.7) For $n \geqq 2$,

$$\tilde{\mathbf{Z}}(K(\mathbf{Z}, 2n), \mathrm{L}) = 0, \qquad \tilde{\mathbf{Z}}(K(\mathbf{Z}, 2n), \mathrm{td}) = 0.$$

Our main emphasis in the paper is to compute $S(Y, K)$ for Y either a classifying space (e.g., BSO (n) or BU (n)) or the Thom complex of the universal bundle over a classifying space (e.g., MSO (n), MU (n)). In the first case a map $M \to Y$ represents a vector bundle over M and so $S(Y, K)$ gives information about characteristic classes of bundles relative to the m-sequence K. If Y is a Thom complex a map $M \to Y$ gives rise to a submanifold of M (with a certain type of normal bundle) and so $S(Y, K)$ relates the normal characteristic classes of this submanifold to K. In sections 3–6 we compute $S(\mathrm{B}G, K)$ and $S(\mathrm{M}G, K)$ for various Lie groups G and m-sequences K. The paper concludes with four appendices: section 8, power series; section 9, the Â-sequence; section 10, PL and Top manifolds; section 11, bordism.

We conclude this section by noting three simple properties of $S(Y, K)$.

(1.8) THEOREM. (i) *$S(Y, K)$ is an S-submodule of $H^{**}(Y; A)$.*
(ii) *$f^* S(Y, K) \subset S(Y', K)$ given any map $f \colon Y' \to Y$.*
(iii) *$S'(Y, K) \subset S(Y, K)$, given any subring S' of S.*

These follow at once from (1.4).

REMARK (added March 29, 1976): I am indebted to P. Gilmer for pointing out to me that some of the material in this paper overlaps with work of P. Conner, described in a brief research announcement some time ago (Bull. Amer. Math. Soc. 69 (1963), 276–279). In particular, definition (1.4) is given there; also Theorem (2.6), Corollary (4.4) and Theorem (11.1) are stated there without proof.

2. Complex projective space.

Let K be a fixed m-sequence with coefficient domain A. One of our goals is to compute $S(\mathrm{B}G, K)$, where $S \subset A$ and where $\mathrm{B}G$ denotes the classifying space for a Lie Group G. A general principle in dealing with Lie groups is: restrict to the maximal torus. Thus in this section we compute $S(\mathrm{B}T, K)$, where T is an arbitrary torus group.

Hirzebruch showed that every m-sequence K (real or complex) is completely determined by a power series $C^K \in A[[t]]$. Namely,

$$(2.1) \qquad C^K(t) = K(\omega) \in H^{**}(P_\infty; A) = A[[t]] .$$

Here P_∞ denotes the infinite complex projective space and ω is the canonical complex line bundle (by restriction ω is the normal bundle of P_n in P_{n+1}, $n \geqq 1$). Note that if K is real, then C^K is a series in *even* powers of t. Following Hirzebruch we call C^K the characteristic power series for K.

We define

$$(2.2) \qquad R^K(t) = \frac{t}{C^K(t)} \in A[[t]];$$

we call R^K the reciprocal series for K. If K is real, R^K is a series in odd powers of t.

(2.3) EXAMPLES. (i) If $K = \mathrm{L}$, then

$$C^{\mathrm{L}} = \frac{t}{\tanh t}, \qquad R^{\mathrm{L}} = \tanh t \quad (=T) .$$

(ii) If $K = \mathrm{td}$,

$$C^{\mathrm{td}} = \frac{t}{(1-e^{-t})}, \qquad R^{\mathrm{td}} = 1-e^{-t} \quad (=E) .$$

We now restrict attention to an important type of m-sequence. Let $S \subset A$ be a subring. We say that K is S-integral if

$$(2.4) \qquad K[M] \in S, \text{ for every } M \text{ in the domain of } K .$$

(Note that L and td are Z-integral. Moreover, since the complex and oriented cobordism rings (mod torsion) are integral polynomial rings [20], there is an infinite number of distinct Z-integral m-sequences.)

To state our main result, let P_d denote complex projective d-space, $d \geqq 1$. Also, given indeterminates $t_1, \ldots, t_n$ set

$$(2.5) \qquad R_i = R^K(t_i) \in \mathrm{Q}[[t_1, \ldots, t_n]], \qquad 1 \leqq i \leqq n,$$

where R^K is the reciprocal series for K.

(2.6) THEOREM. *Let K be an S-integral multiplicative sequence. Then, given positive integers $d_1, \ldots; d_n$,*

$$S(P_{d_1} \times \ldots \times P_{d_n}, K) = S[[R_1, \ldots, R_n]]/(t_1^{e_1}, \ldots, t_n^{e_n}), \qquad e_i = d_i + 1 .$$

We use here the fact that

$$H^{**}(P_{d_1} \times \ldots \times P_{d_n}; A) = A[[t_1,\ldots,t_n]]/(t_1^{e_1},\ldots,t_n^{e_n}) .$$

Let $T(n)$ denote the n-dimensional torus group. Then $\mathrm{B}\,T(n) = P_\infty \times \ldots \times P_\infty$ (n factors). Thus, passing to the limit in (2.6) we obtain

(2.7) COROLLARY. *Let K be an S-integral* m-*sequence. Then, for $n \geqq 1$,*

$$S(\mathrm{B}\,T(n), K) = S[[R_1,\ldots,R_n]] \subset A[[t_1,\ldots,t_n]] .$$

We develop some preliminary material before proving (2.6). Let M be an oriented manifold and N an oriented codimension 2 submanifold, with embedding $j\colon N \subset M$. Suppose that N is dual to $u \in H^2(M; \mathbf{Z})$. We prove

(2.8) LEMMA. *Let $\theta \in H^{**}(M; A)$. Then,*

$$\{j^*\theta \cdot K(N)\}[N] = \{\theta \cdot R(u) \cdot K(M)\}[M] ,$$

where R is the reciprocal series for K.

To see this, let ν denote the normal bundle to N in M. Then ν is a complex line bundle and so $\nu = j^*\xi$, where ξ is the complex line bundle over M with first Chern class u. Since $\tau_N \oplus \nu = j^*\tau_M$, we have, by (1.2), (2.1), and (2.2),

$$K(N) = j^*(K(M) \cdot K(\xi)^{-1}) = j^*\left(K(M) \cdot \frac{R(u)}{u}\right).$$

Let $j_*\colon H^{**}(N; A) \to H^{**}(M; A)$ denote the Gysin homomorphism (of degree $+2$) defined by j. Thus (see [9], [11]),

$$\{j^*\theta K(N)\}[N] = j_*\{j^*\theta \cdot K(N)\}[M] = j_*\left\{j^*\left(\theta \cdot K(M) \cdot \frac{R(u)}{u}\right)\right\}[M]$$

$$= \left\{\left(\theta \cdot K(M) \cdot \frac{R(u)}{u}\right) \cdot u\right\}[M] = \{\theta \cdot R(u) \cdot K(M)\}[M] ,$$

as claimed.

PROOF OF THEOREM (2.6). We first show

(i) $S[[R_1,\ldots,R_n]]/(t_1^{e_1},\ldots,t_n^{e_n}) \subset S(P_{d_1} \times \ldots \times P_{d_n}, K)$.

Note that since K *is* S-integral, $1 \in S(P_{d_1} \times \ldots \times P_{d_n}, K)$. Now let θ be any element of $S[[R_1,\ldots,R_n]]/(t_1^{e_1},\ldots,t_n^{e_n})$ such that $\theta \in S(P_{d_1} \times \ldots \times P_{d_n}, K)$. We show that

$$\theta \cdot R_i \in S(P_{d_1} \times \ldots \times P_{d_n}, K), \quad \text{for } 1 \leqq i \leqq n .$$

This will prove (i).

Let M be an oriented manifold with $u_1,\ldots,u_n \in H^2(M; \mathbf{Z})$. We are to show that

$$\{\theta(u_1,\ldots,u_n)\cdot R(u_i)\cdot K(M)\}[M] \in S\,,$$

for $1 \leqq i \leqq n$. Regard u_i as a map $M \to P_{d_i}$ and make u_i transverse regular to $P_{d_i-1} \subset P_{d_i}$. Set $N_i = u_i^{-1}(P_{d_i-1})$; then N_i is dual to u_i. Let $j\colon N_i \subset M$ denote the inclusion. By (2.8),

$$\{j^*\theta(u_1,\ldots,u_n)\cdot K(N_i)\}[N_i] = \{\theta(u_1,\ldots,u_n)\cdot R(u_i)\cdot K(M)\}\cdot[M]\,.$$

But by hypothesis, $\theta \in S(P_{d_1} \times \ldots \times P_{d_n}, K)$ and so

$$\{j^*\theta(u_1,\ldots,u_n)\cdot K(N_i)\}[N_i] \in S\,,$$

which completes the proof of (i).

To complete the proof of Theorem (2.6), we prove

(ii) $S(P_{d_1} \times \ldots \times P_{d_n}, K) \subset S[[R_1,\ldots,R_n]]/(t_1^{e_1},\ldots,t_n^{e_n})$.

We adopt the following notation: given variables $x_1,\ldots,x_n$ in any ring and given an ordered set of n non-negative integers $I = (i_1,\ldots,i_n)$, we set $x(I) = x_1^{i_1}\ldots x_n^{i_n}$.

Since the series R_i begins with t_i we have (see section 8),

$$A[[R_1,\ldots,R_n]]/(t_1^{e_1},\ldots,t_n^{e_n}) \approx A[[t_1,\ldots,t_n]]/(t_1^{e_1},\ldots,t_n^{e_n})\,.$$

Thus, given any element $\theta \in S(P_{d_1} \times \ldots \times P_{d_n}, K)$, we may write

$$\theta = \sum_I a(I)R(I)\,,$$

where each $a(I) \in A$. To prove (ii) we need simply show that in fact each coefficient $a(I)$ lies in $S \subset A$. We do this by an inductive argument on the degree of I, where by definition, degree $I = i_1 + \ldots + i_n$.

Let $I_0 = (0,0,\ldots,0)$. We see that $a(I_0) \in S$ by mapping a point into $P_{d_1} \times \ldots \times P_{d_n}$. Suppose inductively we have proved that for some integer $q > 0$, all coefficients $a(I) \in S$, when degree $I < q$. Let $\theta' = \sum_{I'} a(I')R(I')$, where the sum is over all I' with $\deg I' < q$. Then, $a(I') \in S$ and so, by (i) and (1.8), $\theta' \in S(P_{d_1} \times \ldots \times P_{d_n}, K)$. Consequently,

$$\theta'' = \theta - \theta' \in S(P_{d_1} \times \ldots \times P_{d_n}, K);$$

moreover, $\theta'' = \sum_{I''} a(I'')R(I'')$, where $\deg I'' \geqq q$. Let $J = (j_1,\ldots,j_n)$ be a sequence with $\deg J = q$. We show that $a(J) \in S$, which will complete the inductive step.

We may assume $j_i \leqq d_i$, $1 \leqq i \leqq n$, for otherwise $R(J) = R_1^{j_1} \ldots R_n^{j_n} = 0$. Let l: $P_{j_1} \times \ldots \times P_{j_n} \subset P_{d_1} \times \ldots \times P_{d_n}$ denote the inclusion, and let $v_i \in H^2(P_{j_i}; \mathbf{Z})$ denote the canonical generator. Then,

$$\begin{aligned}\{l^*\theta'' \cdot K(P_{j_1} \times \ldots \times P_{j_n})\}[P_{j_1} \times \ldots \times P_{j_n}] \\ &= \{a(J) v_1^{j_1} \ldots v_n^{j_n}(1+\ldots)\ldots(1+\ldots)\}[P_{j_1}] \times \ldots \times [P_{j_n}] \\ &= a(J)\,,\end{aligned}$$

since $\{v_i^{j_i}\}[P_{j_i}] = 1$. But by hypothesis on θ'',

$$\{l^*\theta'' \cdot K(P_{j_1} \times \ldots \times P_{j_n})\}[P_{j_1} \times \ldots \times P_{j_n}] \in S\,,$$

and so $a(J) \in S$, which completes the inductive step and hence the proof of Theorem (2.6).

REMARK. I am indebted to E. Rees for helpful comments which simplified the proof of (ii) above.

3. Restriction to the maximal torus.

In this section we study $S(\mathrm{B}\,G, K)$ where G is a compact connected Lie group. Our key definition is this: given G and $S \subset A$, we say that an m-sequence K is *(G, S)-regular* if there is a maximal torus T of G such that

$$j^*S(\mathrm{B}\,G, K) = j^*H^{**}(\mathrm{B}\,G; A) \cap S(\mathrm{B}\,T, K)\,, \tag{3.1}$$

where $j\colon \mathrm{B}\,T \to \mathrm{B}\,G$ is induced by $T \subset G$.

Recall [4] that when A is a field of characteristic zero, j^* is injective and $H^{**}(\mathrm{B}\,G; A)$ is a formal power series ring. Since we know $S(\mathrm{B}\,T, K)$ by (2.7), (3.1) then reduces the calculation of $S(\mathrm{B}\,G, K)$ to the formal algebraic problem of computing the intersection of two power series subrings of $A[[t_1, \ldots, t_n]]$ (assuming $\dim T = n$). In sections 4 and 5 we give examples of such calculations.

As in section 1 let td denote the Todd sequence, with coefficient ring $\mathbf{Q}$. Our first result is:

(3.2) THEOREM. *Let S be any subring of the rationals. Then for any compact connected Lie group G,* td *is (G, S)-regular.*

Note that in definition (3.1) one always has, by (1.8),

$$j^*S(\mathrm{B}\,G, K) \subset j^*H^{**}(\mathrm{B}\,G; A) \cap S(\mathrm{B}\,T, K)\,.$$

Thus to prove that an m-sequence K is (G, S)-regular we need only show:

(3.3) Given any class θ in $H^{**}(\mathrm{B}\,G; A)$ such that $j^*\theta \in S(\mathrm{B}\,T, K)$, then for any manifold M (with $K(M)$ defined) and any map $f\colon M \to \mathrm{B}\,G$,

$$\{f^*\theta \cdot K(M)\}[M] \in S\,.$$

We show that (3.3) holds for $K = \mathrm{td}$ (and $S \subset \mathbf{Q}$), which will prove (3.2).

To begin with assume simply that M is an oriented manifold and f a map $M \to \mathrm{B}\,G$. Recall that up to homotopy type the map j can be replaced by a fiber map $\pi\colon \mathrm{B}\,T \to \mathrm{B}\,G$, with fiber G/T. By using the theory of Steenrod [19, § 19.6, § 7.4], and by taking finite dimensional approximations, we may assume that

$$G/T \xrightarrow{i} \mathrm{B}\,T \xrightarrow{\pi} \mathrm{B}\,G$$

is a smooth fiber bundle. Also, by a suitable homotopy, we may take f to be a smooth map. We then have a smooth G/T bundle induced over M by f, giving a commutative diagram

(3.4)
$$\begin{array}{ccc} G/T & = & G/T \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle i} \\ \hat{M} & \xrightarrow{l} & \mathrm{B}\,T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & \mathrm{B}\,G\,. \end{array}$$

Choose a Riemannian metric on $\hat{M}$ and define β_F to be the bundle orthogonal to $p^*\tau_M$ in $\tau_{\hat{M}}$: β_F is called the bundle along the fiber (see [5], [6]). By [5] we remark that β_F can be given a complex structure.

We now prove (3.3) for $K = \mathrm{td}$. Assume then that M is a stably almost-complex manifold. Then τ_M is stably complex; since $\tau_{\hat{M}} = p^*\tau_M \oplus \beta_F$, we see that $\hat{M}$ is stably almost-complex.

Let θ be a class in $H^{**}(\mathrm{B}\,G; \mathbf{Q})$ such that $\pi^*\theta (= j^*\theta) \in S(\mathrm{B}\,T, \mathrm{td})$. Our goal is to show:

(*)
$$\{f^*\theta \cdot \mathrm{td}\,(M)\}[M] \in S\,.$$

Let $p_*\colon H^{**}(\hat{M}; \mathbf{Q}) \to H^{**}(M; \mathbf{Q})$ denote "integration along the fiber" ([5], [6]). We need the following result, which can be deduced from §§ 22.2, 22.5 of [5] — see especially equation (10), § 22.5.

(**) The complex structure on β_F can be chosen so that $p_*(\mathrm{td}\,\beta_F) = 1 \in H^0(M; \mathbf{Z})$.

Since $\tau_{\hat{M}} = p^*\tau_M \oplus \beta_F$ we have $\operatorname{td}(\hat{M}) = p^*(\operatorname{td} M)\cdot \operatorname{td}\beta_F$. Thus

$$\begin{aligned}\{p^*f^*\theta\cdot \operatorname{td}(\hat{M})\}[\hat{M}] &= p_*\{p^*f^*\theta\cdot \operatorname{td}(\hat{M})\}[M] \\ &= p_*\{p^*(f^*\theta\cdot \operatorname{td} M)\cdot \operatorname{td}(\beta_F)\}[M] \\ &= \{f^*\theta\cdot \operatorname{td} M\cdot p_*(\operatorname{td}\beta_F)\}[M] \\ &= \{f^*\theta\cdot \operatorname{td} M\}[M]\,.\end{aligned}$$

But $p^*f^* = l^*\pi^*$, in (3.4), and by hypothesis, $\pi^*\theta \in S(\mathrm{B}\,T, \operatorname{td})$. Thus,

$$\{p^*f^*\theta\cdot \operatorname{td}\hat{M}\}[\hat{M}] = \{l^*(\pi^*\theta)\cdot \operatorname{td}\hat{M}\}[\hat{M}] \in S\,,$$

and so $\{f^*\theta\cdot \operatorname{td} M\} \in S$, as desired. This completes the proof of Theorem 3.2.

For real m-sequences we have a rather more general result.

(3.6) THEOREM. *Let $S \subset \mathbf{Q}$ be any subring containing the integers, and let K be a real* m*-sequence that is S-integral. Then, for $n \geqq 2$, K is* $(\mathrm{U}(n), S)$*-regular.*

Again we need simply prove (3.3) for K. Let M be a smooth oriented manifold and f a map $M \to \mathrm{BU}(n)$. We again use diagram (3.4), taking $G = \mathrm{U}(n)$, $T = T(n)$; π now becomes the "standard" inclusion $\mathrm{B}\,T(n) \subset \mathrm{BU}(n)$.

In order to prove (3.3) (for $G = \mathrm{U}(n)$) we need more information about the bundle along the fiber, β_F. Recall the map l in diagram (3.4), $l\colon \hat{M} \to \mathrm{B}\,T(n)$. Let $\omega_1, \ldots, \omega_n$ be the canonical complex line bundles over $\mathrm{B}\,T(n)$, and set $\xi_i = l^*\omega_i$, $1 \leqq i \leqq n$. By Theorem (13.1.1) of [11], one has

$$\beta_F \approx \sum_{i>j} \xi_i \otimes \xi_j^{-1}\,. \tag{3.7}$$

Let $\lambda_i = c_1(\xi_i)$, the first Chern class. Then, $c_1(\xi_i \otimes \xi_j^{-1}) = \lambda_i - \lambda_j$, see [11]. Let $R \in \mathbf{Q}[[t]]$ denote the reciprocal series for K (see 2.2). Set

$$A[\hat{M}] = \left\{p^*f^*\theta\cdot K(\hat{M})\cdot \prod_{i>j} R(\lambda_i - \lambda_j)\right\}[\hat{M}]\,. \tag{3.8}$$

We prove (cf. (3.3))

$$A[\hat{M}] = n!\,\{f^*\theta\cdot K(M)\}[M]\,. \tag{3.9}$$

Since $\tau_{\hat{M}} = p^*\tau_M \oplus \beta_F$, we have by (3.7),

$$K(\hat{M}) = p^*K(M)\cdot K(\beta_F) = p^*K(M)\cdot \prod_{i>j} (\lambda_i - \lambda_j)/R(\lambda_i - \lambda_j)\,,$$

and so

$$A[\hat{M}] = \Big\{p^*(f^*\theta\cdot K(M))\cdot \prod_{i>j} (\lambda_i-\lambda_j)\Big\}[\hat{M}]$$

$$= p_*\Big\{p^*(f^*\theta\cdot K(M))\cdot \prod_{i>j} (\lambda_i-\lambda_j)\Big\}[M]$$

$$= \Big\{f^*\theta\cdot K(M)\cdot p_*\Big(\prod_{i>j} (\lambda_i-\lambda_j)\Big)\Big\}[M] .$$

Let $q=n(n-1)/2$, so that β_F is a complex q-plane bundle. By (3.7) we have:

$$c_q(\beta_F) = \prod_{i>j} (\lambda_i-\lambda_j) .$$

But by [5] one sees that $k^*(\beta_F)$ is the complex tangent bundle to G/T, where $k\colon G/T \to \hat{M}$ in (3.4). Thus,

$$k^*c_q(\beta_F)[G/T] = \chi(G/T) = n! ,$$

by [5]; and so, by [5], $p_*c_q(\beta_F)=n!\cdot 1 \in H^0(M; \mathbf{Z})$. Consequently,

$$A[\hat{M}] = \{f^*\theta\cdot K(M)\cdot p_*c_q(\beta_F)\}[M] = n!\{f^*\theta\cdot K(M)\}[M] ,$$

as claimed.

On the other hand, we will prove

(3.10) $$A[\hat{M}] = n!\, s, \quad \text{for some } s\in S .$$

Combining (3.10) and (3.9) we see that $\{f^*\theta\cdot K(M)\}\in S$, which proves (3.3) and hence Theorem (3.6).

We assemble several facts before proving (3.10).

(3.11) LEMMA. *There is a class $\hat{\Lambda}\in H^{**}(M; \mathbf{Q})$ such that $p^*\hat{\Lambda}=K(\beta_F)$.*

PROOF. In $\mathrm{B}\,T(n)$, set $\zeta=\sum_{i>j}\omega_i\otimes\omega_j^{-1}$, so that $\beta_F=l^*\zeta$. If we show that

$$K(\zeta)\in \pi^*H^{**}(\mathrm{BU}\,(n); \mathbf{Q}) ,$$

this will prove (3.11). But $K(\zeta)=\prod_{i>j}(t_i-t_j)/R(t_i-t_j)$, where $t_i=c_1(\omega_i)$. Since $t/R(t)$ is an even function, $K(\zeta)$ is invariant with respect to permutations of the t_i's. Thus, $K(\zeta)\in \operatorname{Image}\pi^*$, as claimed, proving (3.11).

In $H^*(\mathrm{B}\,T(n); \mathbf{Q})$, set $R_i=R(t_i)$, $1\leqq i\leqq n$. We prove

(3.12) LEMMA. *There is a class $B\in H^{**}(\mathrm{BU}\,(\mathrm{n}); \mathbf{Q})$ such that*

$$\pi^*(B)\in S[[R_1,\ldots,R_n]] \quad \textit{and} \quad \prod_{i>j} R(t_i-t_j) = \prod_{i>j} (R_i-R_j)\cdot \pi^*B .$$

PROOF. We use the following result from [21]: since K is S-integral there is a series $\Psi(x,y) \in S[[x,y]]$ such that

(i) $\Psi(x,y)=\Psi(y,x)$,
(ii) $\Psi(-x,y)=\Psi(x,-y)$,
(iii) $R(x+y)=(Rx+Ry)\cdot\Psi(Rx,Ry)$.

Thus by (iii), (since R is an odd function),

$$\prod_{i>j} R(t_i-t_j) = \sum_{i>j} (R_i-R_j)\cdot\prod_{i>j} \Psi(R_i, -R_j) .$$

But by (i)–(ii) above, $\prod_{i>j}\Psi(R_i, -R_j)$ is invariant with respect to permutations of the t_i's and so belongs to image π^*. Finally, as noted above,

$$\prod \Psi(R_i, -R_j) \in S[[R_1,\ldots,R_n]] ,$$

which completes the proof of (3.12).

At the end of the section we prove:

(3.13) There are classes $C, D \in S[[R_1,\ldots,R_n]]$ such that

$$\prod_{i>j} (R_i-R_j) = C+n!\,D ,$$

and

$$p_*(l^*C) = 0 .$$

PROOF OF (3.10). Combining (3.8), (3.11), (3.12), and (3.13), we have:

$$(**) \qquad \begin{aligned} A[\hat{M}] &= \{p^*(f^*\theta\cdot K(M)\cdot\hat{A}\cdot f^*(B))\cdot l^*C\}[\hat{M}] \\ &\quad +n!\,\{l^*(\pi^*(\theta\cdot B)\cdot D)\cdot K(\hat{M})\}[\hat{M}] . \end{aligned}$$

But

$$\begin{aligned} \{p^*(f^*\theta\cdot K(M)\cdot\hat{A}\cdot f^*(B))\cdot l^*C\}[\hat{M}] &= p_*\{p^*(f^*\theta\cdot K(M)\cdot\hat{A}\cdot f^*(B))l^*C\}[M] \\ &= \{f^*(\theta\cdot B)\cdot K(M)\cdot\hat{A}\cdot p_*l^*C\}[M] = 0 , \end{aligned}$$

by (3.13). On the other hand, since $\pi^*\theta \in S(\mathrm{B}\,T(n), K)$ and D, $\pi^*B \in S[[R_1,\ldots,R_n]]$, it follows from (2.7) that $\pi^*(\theta\cdot B)\cdot D \in S(\mathrm{B}\,T(n), K)$, and so

$$\{l^*(\pi^*(\theta\cdot B)\cdot D)K(\hat{M})\}[\hat{M}] = s \in S .$$

Hence, by (**), $A[\hat{M}]=n!\,s$, as claimed, which proves (3.10).

PROOF OF (3.13). We use the ideas developed by Hirzebruch in § 14 of [11].

Let

$$A_n = Q[[t_1,\ldots,t_n]], \qquad B_n = Q[[\sigma_1,\ldots,\sigma_n]] \subset A_n \,.$$

Then A_n is a free module over B_n, with base elements all monomials $t_1^{a_1},\ldots,t_{n-1}^{a_{n-1}}$ such that $0 \leqq a_i \leqq n-i$, $1 \leqq i \leqq n-1$. Thus any element $P \in A_n$ can be uniquely written

$$P = \sum_{0 \leqq a^i \leqq n-i} \varrho_{a_1,\ldots,a_{n-1}} t_1^{a_1} \ldots t_{n-1}^{a_{n-1}} \qquad \text{where } \varrho_{a_1,\ldots,a_{n-1}} \in B_n \,.$$

Define the "indicator" $\varrho(P)$ by

$$\varrho(P) = (-1)^{n(n-1)/2} \varrho_{n-1,\ldots,1} \,.$$

As before, let $R_i = R(t_i) \in A_n$. We prove:

(3.14) LEMMA. *Let $P \in A_n$. If $\varrho(P) = \alpha \cdot 1 \in B_n$, $\alpha \in Q$, then*

$$P(R_1,\ldots,R_n) = P' + \alpha \cdot R_1^{n-1} R_2^{n-2} \ldots R_{n-1}$$

where $\varrho(P')=0$. Moreover, if $P \in S[[t_1,\ldots,t_n]]$, then $P' \in S[[R_1,\ldots,R_n]]$.

We adopt the following notation. Let $J = (a_1,\ldots,a_{n-1})$ denote an "admissible" sequence as above: that is, $0 \leqq a_i \leqq n-i$. Set

$$t(J) = t_1^{a_1} \ldots t_{n-1}^{a_{n-1}} \,,$$

so that $P \in A_n$ can be written

$$P = \sum_J \varrho_J t(J) \,,$$

summed over all admissible J. Now replace t_i by R_i — we then have

$$P(R_1,\ldots,R_n) = \sum_J \varrho_J(R) \cdot R(J) \,,$$

where $\varrho_J(R)$ is obtained from ϱ_J by replacing each t_i with R_i. Notice that we continue to have $\varrho_J(R) \in B_n$. By hypothesis on P, in (3.14), $t_1^{n-1} \ldots t_{n-1}$ has coefficient α in P. Thus $R_1^{n-1} \ldots R_{n-1}$ has coefficient α in $P(R_1,\ldots,R_n)$. So to prove (3.14) we need simply show:

$$\text{if } J \neq (n-1, n-2,\ldots,1), \text{ then } \varrho(R(J)) = 0$$

To see this, note that since $J \neq (n-1,n-2,\ldots,1)$, there are distinct integers i, j, $1 \leqq i, j \leqq n-1$, such that $a_i = a_j$, where $J = (a_1,\ldots,a_{n-1})$. Therefore $R(J)$ remains invariant when we interchange t_i and t_j, and so by 14.1.2 of [11], $\varrho(R(J))=0$, which proves (3.14).

To prove (3.13), take $P = \prod_{i>j}(t_i - t_j) \in A_n$. On page 108 of [11], (see equation (4)) Hirzebruch shows that

$$\varrho\left(\prod_{i>j}(t_i-t_j)\right) = n!\,,$$

and so by (3.14) there are classes $C, D \in S[[R_1,\ldots,R_n]]$ with

$$\prod_{i>j}(R_i-R_j) = C+n!D \quad \text{and} \quad \varrho(C) = 0\,.$$

Consider now the fibration

$$G/T \to \mathrm{B}\,T \xrightarrow{\pi} \mathrm{B}\,G\,.$$

Then, $H^{**}(\mathrm{B}\,T;\mathrm{Q})=A_n$, $H^{**}(\mathrm{B}\,G;\mathrm{Q})=B_n$, assuming $\dim T=n$. Moreover, from [5, § 20] one sees that

$$\varrho = \pi_* = \text{integration along the fiber}\,.$$

Thus, from our commutative diagram (3.4) and by [6] we have:

$$p_* l^*(C) = f^*\pi_*(C) = f^*\varrho(C) = 0\,,$$

which completes the proof of (3.13).

4. Characteristic classes for m-sequences.

In this section we consider m-sequences that are $(\mathrm{U}(n), S)$-regular and obtain an explicit formula for $S(\mathrm{BU}(n), K)$. To do this we introduce the notion of (complex) characteristic classes for m-sequences.

As in section 3, let $T(n)$, $n \geqq 1$, denote the n-torus and let $j_n\colon \mathrm{B}T(n) \to \mathrm{BU}(n)$ be the standard map. We now take cohomology with coefficients in some fixed ring A, which we omit in our notation. Then,

$$H^{**}(\mathrm{B}\,T(n)) = A[[t_1,\ldots,t_n]], \quad \deg t_i=2\,,$$

$$j_n^* H^{**}(\mathrm{BU}(n)) = A[[\sigma_1,\ldots,\sigma_n]]\,,$$

where σ_i denotes the ith elementary symmetric function in $t_1,\ldots,t_n$.

Suppose that K is any m-sequence (real or complex) with reciprocal sequence R. Set $R_i=R(t_i) \in A[[t_1,\ldots,t_n]]$, $1 \leqq i \leqq n$. We define

(4.1) $\sigma_i(K)=i$th elementary symmetric function in $R_1,\ldots,R_n$, $1 \leqq i \leqq n$.

Notice that $\sigma_i(K) \in A[[\sigma_1,\ldots,\sigma_n]]$, and so we can define $c_i(K) \in H^{**}(\mathrm{BU}(n))$ by

(4.2) $$j_n^* c_i(K) = \sigma_i(K)\,.$$

We use here the fact that j_n^* is injective with A-coefficients, since $H^{**}(\mathrm{B}\,T(n); \mathbf{Z})$ and $H^{**}(\mathrm{BU}\,(n); \mathbf{Z})$ are torsion-free and j_n^* is injective with $\mathbf{Z}$-coefficients.

We call $c_i(K)$ the ith Chern class of K. (Compare [3] and [20]). Note that

$$c_i(K) = \overline{c_i} + \text{higher terms}\,,$$

where $\overline{c_i}$ denotes the image of the ordinary Chern class by the coefficient homomorphism $\mathbf{Z} \to A$.

We now can state our result.

(4.3) THEOREM. *Let K be an* m*-sequence that is* $(\mathrm{U}\,(n), S)$*-regular for some subring S of A. Suppose that K is S-integral. Then,*

$$S(\mathrm{BU}\,(n), K) = S[[c_1(K), \ldots, c_n(K)]]\,.$$

By (3.2) and (3.6) we have

(4.4) COROLLARY. *Let S be a subring of the rationals containing the integers. Then*

$$\text{(i)}\quad S(\mathrm{BU}\,(n), \mathrm{td}) = S[[c_1(\mathrm{td}), \ldots, c_n(\mathrm{td})]]\,,$$

$$\text{(ii)}\quad S(\mathrm{BU}\,(n), K) = S[[c_1(K), \ldots, c_n(K)]]\,.$$

where K is any real m*-sequence that is S-integral.*

Note that (i) can also be deduced from the Riemann–Roch Theorem [11] and the Stong–Hattori Theorem [20], [10].

Chern classes for K can be computed as follows: given an ordered sequence of integers $I = (i_1, \ldots, i_r)$, $r \leqq n$, let s_I denote the unique polynomial in $\mathbf{Z}[[\sigma_1, \ldots, \sigma_n]]$ such that

$$s_I(\sigma_1, \ldots, \sigma_n) = \sum t_1^{i_1} \ldots t_r^{i_r}\,.$$

If R is the reciprocal series for K, write $R = \sum_{1 \leqq i} a_i t^i$, where $a_i \in A$. Then, for $r \geqq 1$,

$$c_r(K) = \sum_{1 \leqq i_1 \leqq \ldots \leqq i_r} (a_{i_1} \ldots a_{i_r}) s_{i_1, \ldots, i_r}\,. \tag{4.5}$$

Here we write $A[[\sigma_1, \ldots, \sigma_n]] = \mathbf{Z}[[\sigma_1, \ldots, \sigma_n]] \otimes A$; also, each s_I can be expressed in terms of the (ordinary) Chern classes (e.g., see [11], [18]).

PROOF OF THEOREM 4.3. By (3.1) and (2.7), since K is $(\mathrm{U}\,(n), S)$-regular,

$$\begin{aligned} j_n^* S(\mathrm{BU}\,(n), K) &= j_n^* H^{**}(\mathrm{BU}\,(n)) \cap S(\mathrm{B}\,T(n), K) \\ &= A[[\sigma_1, \ldots, \sigma_n]] \cap S[[R_1, \ldots, R_n]]\,. \end{aligned}$$

By (8.2) of the appendix

$$A[[\sigma_1,\ldots,\sigma_n]] = A[[\sigma_1(K),\ldots,\sigma_n(K)]] .$$

Also by (8.4) (taking $\varphi_i = t_i = R_i$, $1 \leqq i \leqq n$), we see that

$$A[[\sigma_1(K),\ldots,\sigma_n(K)]] \cap S[[R_1,\ldots,R_n]] = S[[\sigma_1(K),\ldots,\sigma_n(K)]] .$$

Since $j_n^* S[[c_1(K),\ldots,c_n(K)]] = S[[\sigma_1(K),\ldots,\sigma_n(K)]]$, we have

$$S(\mathrm{BU}(n), K) = S[[c_1(K),\ldots,c_n(K)]] ,$$

as claimed. This completes the proof of (4.3).

5. Real characteristic classes.

We define Pontrjagin classes and an Euler class for m-sequences. Using these we compute $S(\mathrm{BSO}(n), K)$. Throughout the section we take $A = \mathrm{Q}$, the rationals.

We adopt the following notation. Let n be a fixed positive integer and set $q = 2n + \varepsilon$, $\varepsilon = 0$ or 1. Let $\hat{j}_q \colon \mathrm{B}\,T(n) \to \mathrm{BSO}(q)$ denote the map induced by the standard embedding of the maximal torus. Then,

$$\hat{j}_q^* H^{**}(\mathrm{BSO}(q)) = \mathrm{Q}[[\hat{\sigma}_1,\ldots,\hat{\sigma}_n]], \qquad q = 2n+1 \tag{5.1}$$

$$= \mathrm{Q}[[\hat{\sigma}_1,\ldots,\hat{\sigma}_{n-1},\sigma_n]], \qquad q = 2n ,$$

where $\hat{\sigma}_i$ denotes the ith elementary symmetric function in $t_1^2,\ldots,t_n^2$.

Suppose now that K is a m-sequence with reciprocal series R. Define $\bar{R} \in \mathrm{Q}[[t]]$ by

$$\bar{R}(t) = -R(-t) .$$

Note that $R\bar{R}$ and $(R+\bar{R})/2$ are both *even* functions, and that

$$R\bar{R} = t^2 + \ldots, (R+\bar{R})/2 = t + \ldots .$$

Also, if R is an *odd* function (e.g., if K is a real m-sequence) then

$$R\bar{R} = R^2, \quad (R+\bar{R})/2 = R .$$

As usual, set $R_i = R(t_i)$ for $1 \leqq i \leqq n$; we define

(5.2) $\hat{\sigma}_i(K) =$ ith elementary symmetric function in $R_1\bar{R}_1,\ldots,R_n\bar{R}_n$.

Thus, $\hat{\sigma}_i(K) \in \mathrm{Q}[[\hat{\sigma}_1,\ldots,\hat{\sigma}_n]]$ and so we can define $p_i(K) \in H^{**}(\mathrm{BSO}(q))$ by

$$\hat{j}_q^* p_i(K) = \hat{\sigma}_i(K), \quad 1 \leqq i \leqq n . \tag{5.3}$$

Similarly, when $q=2n$, we define $\chi_{2n}(K) \in H^{**}(\mathrm{BSO}\,(q))$ by

$$\hat{j}_{2n}^{*}\chi_{2n}(K) = \prod_{i=1}^{n} (R_i+\bar{R}_i)/2 .$$

Notice that if R is odd, then

$$\hat{j}_{2n}^{*}\chi_{2n}(K) = \sigma_n(K) .$$

We now prove

(5.4) THEOREM. *Let K be an* m*-sequence that is* $(\mathrm{SO}\,(q), S)$*-regular and* S*-integral for some subring S of the rationals. Suppose moreover that*

$$(*) \qquad \bar{R} \in S[[R]] ,$$

and that

$$(**) \qquad (R+\bar{R})/2 \in S[R]], \quad \textit{if } q \textit{ is even} .$$

Then,

$$S(\mathrm{BSO}\,(2n), K) = S[[p_1(K),\dots,p_{n-1}(K),\chi_{2n}(K)]], \qquad q=2n ,$$
$$S(\mathrm{BSO}\,(2n+1), K) = S[[p_1(K),\dots,p_n(K)]], \qquad q=2n+1 .$$

Note that when K is a real m-sequence, $\bar{R}=R$ and hence $(*)$ and $(**)$ are trivially satisfied for any $S \subseteqq \mathbf{Q}$ such that $\mathbf{Z} \subseteqq S$. We give a second instance where these hypotheses are satisfied.

(5.5) EXAMPLE. Ler $E=1-e^{-t}$, and let S be any subring of $\mathbf{Q}$ containing $\mathbf{Z}$. Then, $\bar{E} \in S[[E]]$. Moreover, if $\frac{1}{2} \in S$, then $(E+\bar{E})/2 \in S[[E]]$.

PROOF. We have $e^{-t}=1-E$ and so $e^{t}=(1-E)^{-1}$. Thus, $\bar{E}=e^{t}-1=E\cdot(1-E)^{-1}=\sum_{1\leqq i}E^{i}$, from which the result follows.

Combining Theorems (3.2) and (5.4), with Example (5.5), we obtain:

(5.6) COROLLARY. *Let S be a subring of the rationals containing the integers. Then, for $n \geqq 1$,*

$$S(\mathrm{BSO}\,(2n+1), \mathrm{td}) = S[[p_1(\mathrm{td}),\dots,p_n(\mathrm{td})]] .$$

Moreover, if $\frac{1}{2} \in S$, then

$$S(\mathrm{BSO}\,(2n), \mathrm{td}) = S[[p_1(\mathrm{td}),\dots,p_{n-1}(\mathrm{td}),\chi_{2n}(\mathrm{td})]] .$$

Before proving (5.4) we develop some preliminary material. Let l_q: BSO $(q) \to$ BU (q) denote the natural inclusion, and let i_q: B $T(n) \subset$ B $T(q)$ denote the mapping on tori induced by l_q. Recall (see [5]),

$$\left.\begin{aligned} i_q^* t_{2r-1} &= t_r \\ i_q^* t_{2r} &= -t_r \end{aligned}\right\} \quad 1 \leqq r \leqq n\,,$$

$$i_{2n+1}^* t_{2n+1} = 0, \qquad \text{if } q = 2n+1$$

Consequently,

$$\begin{aligned} i_q^* R_{2r-1} &= R_r \\ i_q^* R_{2r} &= -\bar{R}_r, \quad 1 \leqq r \leqq n\,, \end{aligned}$$

$$i_{2n+1}^* R_{2n+1} = 0, \qquad \text{if } q = 2n+1\,.$$

This implies that

$$\begin{aligned} l_q^* c_{2i}(K) &= (-1)^i p_i(K), \quad 1 \leqq i \leqq n\,, \\ l_q^* c_{2i-1}(K) &= 0, \qquad\qquad 1 \leqq i \leqq n+1\,. \end{aligned}$$

Proof of Theorem (5.4). We consider first the case $q = 2n+1$. By hypothesis, and by (3.1) and (2.7),

$$\hat{j}_{2n+1}^* S(\mathrm{BSO}\,(2n+1), K) = \mathrm{Q}[[\hat{\sigma}_1, \ldots, \hat{\sigma}_n]] \cap S[[R_1, \ldots, R_n]]\,.$$

Now by (8.3) (see Appendix)

(A) $$\mathrm{Q}[[\hat{\sigma}_1, \ldots, \hat{\sigma}_n]] = \mathrm{Q}[[\hat{\sigma}_1(K), \ldots, \hat{\sigma}_n(K)]]\,.$$

Note that $\hat{\sigma}_i(K) = \hat{\sigma}_i(R\bar{R})$, in the notation of section 8, and that by hypothesis (*), $\hat{\sigma}_i(K) \in S[[R_1, \ldots, R_n]]$. Therefore by (8.4),

(B) $$\begin{aligned} \mathrm{Q}[[\hat{\sigma}_1(K), \ldots, \hat{\sigma}_n(K)]] \cap S[[R_1, \ldots, R_n]]\ \mathrm{Q}[[\hat{\sigma}_1(R\bar{R}), \ldots, \hat{\sigma}_n(R\bar{R})]] \cap \\ \cap S[[R_1, \ldots, R_n]] \\ = S[[\hat{\sigma}_1(K), \ldots, \hat{\sigma}_n(K)]]\,. \end{aligned}$$

But by (5.3), $S[[\hat{\sigma}_1(K), \ldots, \hat{\sigma}_n(K)]] = \hat{j}_{2n+1}^* S[[p_1(K), \ldots, p_n(K)]]$. Therefore, since $\hat{j}_{2n+1}^*$ is injective, we have

$$S(\mathrm{BSO}\,(2n+1), K) = S[[p_1(K), \ldots, p_n(K)]]\,,$$

as claimed.

When $q = 2n$ the proof is similar, using now the fact that $\sigma_n(K) = \sigma_n((R + \bar{R})/2)$; and so by (**), $\sigma_n(K) \in S[[R_1, \ldots, R_n]]$. Thus we may again use (8.4). We leave the details to the reader.

In order to apply Theorem (5.4) we need to know when an m-sequence K is $(\mathrm{SO}(q), S)$-regular. We prove

(5.7) THEOREM. *Let K be a real* m-*sequence that is* $(\mathrm{U}(q), S)$-*regular for some subring $S \subseteqq \mathrm{Q}$ and some integer $q \geqq 2$. Suppose also that K is S-integral. Then, K is* $(\mathrm{SO}(q), S)$-*regular.*

PROOF. By (3.3) we need simply show

$$\hat{j}_q^* H^{**}(\mathrm{BSO}(q)) \cap S(\mathrm{B}\,T(n), K) \subset \hat{j}_q^* S(\mathrm{BSO}(q), K). \tag{5.8}$$

Consider the following commutative diagram, where the maps are those defined earlier. ($q = 2n+\varepsilon$, $\varepsilon = 0$ or 1).

$$\begin{tikzcd}
\mathrm{B}\,T(q) \arrow[r, "j_q"] & \mathrm{BU}(q) \\
\mathrm{B}\,T(n) \arrow[u, "i_q"] \arrow[r, "\hat{j}_q"'] & \mathrm{BSO}(q) \arrow[u, "l_q"']
\end{tikzcd}$$

Take first the case q *odd*. Then

$$l_q^* H^{**}(\mathrm{BU}(q)) = H^{**}(\mathrm{BSO}(q)) \quad \text{and} \quad i_q^* S(\mathrm{B}\,T(q), K) = S(\mathrm{B}\,T(n), K).$$

Since K is $(\mathrm{U}(q), S)$-regular and since (by (1.8)) $l_q^* S(\mathrm{BU}(q), K) \subset S(\mathrm{BSO}(q), K)$, we see that (5.8) is satisfied. (We need (A) and (B) above to show that

$$i_q^* j_q^* H^{**}(\mathrm{BU}(q)) \cap i^* S(\mathrm{B}\,T(q), K) = i_q^* (j_q^* H^{**}(\mathrm{BU}(q)) \cap S(\mathrm{B}\,T(q), K)).$$

Suppose then that q is even, $q = 2n$. Then, by the same argument as above, we have:

$$\mathrm{Q}[[\hat{\sigma}_1, \ldots, \hat{\sigma}_{n-1}, \hat{\sigma}_n]] \cap S[[R_1, \ldots, R_n]] \subset \hat{j}_{2n}^* S(\mathrm{BSO}(2n), K).$$

Now $\hat{j}_{2n}^* H^{**}(\mathrm{BSO}(2n)) = \mathrm{Q}[[\hat{\sigma}_1, \ldots, \hat{\sigma}_{n-1}, \sigma_n]]$, and $\hat{\sigma}_n = \sigma_n^2$. Also, by (8.4) (see appendix),

$$\mathrm{Q}[[\hat{\sigma}_1, \ldots, \hat{\sigma}_{n-1}, \sigma_n]] = \mathrm{Q}[[\hat{\sigma}_1(K), \ldots, \hat{\sigma}_{n-1}(K), \sigma_n(K)]],$$

since $\hat{\sigma}_i(K) = \sigma_i(R^2)$, $\sigma_n(K) = \sigma_n(R)$. Using the fact that $\hat{j}_{2n}^* \chi_{2n}(K) = \sigma_n(K)$, the proof of (5.8) follows from:

(5.9) LEMMA. *Let K be a real* m-*sequence. Then, given any class $\theta \in S(\mathrm{BSO}(2n), K)$,*

$$\theta \cdot \chi_{2n}(K) \in S(\mathrm{BSO}(2n), K).$$

PROOF OF (5.9). This is a mild generalization of (2.8) and so we only sketch the details.

Suppose then that $\theta \in S(\mathrm{BSO}\,(2n), K)$, that M is an oriented manifold and f a map $M \to \mathrm{BSO}\,(2n)$. We need to show that

$$\text{(C)} \qquad \{f^*(\theta\cdot\chi_{2n}(K))\cdot K(M)\}[M] \in S\,.$$

Deform f so that it is smooth; let ξ denote the smooth bundle over M induced by f. Let $s\colon M \to E_\xi$ denote the zero-section of ξ. Deform s to a section $\bar{s}$ transverse regular to $s(M)$ and set $N=\bar{s}^{-1}(s(M))\subset M$. Thus N is an oriented codimension $2n$ submanifold of M. (We assume $\dim M\geqq 2n$; otherwise (C) is trivially true). Let $i\colon N\subset M$ denote the embedding, with normal bundle ν. Note that $\nu=i^*\xi$ and that N is dual to $\chi(\xi)$. By an argument similar to that given for (2.8) we have

$$\{i^*f^*\theta\cdot K(N)\}[N] = \{f^*\theta\cdot K(\xi)^{-1}\cdot K(M)\cdot\chi(\xi)\}[M]\,.$$

Since $\{i^*f^*\theta\cdot K(N)\}[N] \in S$ by hypothesis on θ, the proof of (5.9) is complete when we show

$$\text{(D)} \qquad K(\xi)^{-1}\chi(\xi) = f^*\chi_{2n}(K)\,.$$

We do the proof in the universal example: let γ_{2n} denote the canonical bundle over $\mathrm{BSO}\,(2n)$. By (2.2),

$$\hat{j}_{2n}^*K(\gamma_{2n})^{-1} = \prod_{i=1}^{n}\frac{R^K(t_i)}{t_i}\,.$$

Since $\hat{j}_{2n}^*\chi_{2n}=\prod_{i=1}^n t_i$, we see that

$$\hat{j}_{2n}^*(K(\gamma_{2n})^{-1}\cdot\chi_{2n}) = \prod_{i=1}^{n} R_i = \hat{j}_{2n}^*\chi_{2n}(K)\,.$$

Thus, $K(\gamma_{2n})^{-1}\cdot\chi_{2n}=\chi_{2n}(K)$, which proves (D) and hence Lemma (5.9).

Combining Theorems (5.4) and (5.7) we now have computed $S(\mathrm{BSO}\,(n), K)$ for any real m-sequence that is $(\mathrm{U}\,(n), S)$-regular and S-integral. In particular by Theorem (3.6) we have

(5.10) Corollary. *Let S be any subring of the rationals containing the integers, and let K be a real* m-*sequence that is S-integral* (e.g., $K=L$). *Then, for $n\geqq 1$,*

$$S(\mathrm{BSO}\,(2n), K) = S[[p_1(K),\ldots,p_{n-1}(K),\chi_{2n}(K)]]\,,$$

$$S(\mathrm{BSO}\,(2n+1), K) = S[[p_1(K),\ldots,p_n(K)]]\,.$$

6. Thom Complexes.

So far we have studied $S(Y, K)$ with Y a classifying space. We now consider the case Y a Thom complex.

Suppose then that ξ is a smooth oriented vector bundle with base space B_ξ and total space E_ξ. We denote by T_ξ the Thom complex of ξ. Recall that $\tilde{H}^*(T_\xi)$ is a free module over $H^*(B_\xi)$ on one generator U_ξ (the Thom class) — we take coefficients in a fixed field A.

Now let K be an m-sequence with coefficient domain A. The question we consider is: given $S(B_\xi, K)$, how does one compute $S(T_\xi, K)$? To state our results, we define

$$U_\xi(K) = U_\xi \cdot K(\xi)^{-1} \in H^{**}(T_\xi);$$

We assume that ξ is complex if K is.

Our first result is

(6.1) PROPOSITION. *Let ξ be a smooth oriented bundle with Thom complex T_ξ, and let K be an* m*-sequence. If K is complex assume ξ is also. Then,*

$$\tilde{S}(T_\xi, K) \supset U_\xi(K) \cdot S(B_\xi, K) .$$

PROOF. Let M be a smooth manifold and f a map $M \to T_\xi$. Given $\theta \in S(B_\xi, K)$ we are to show:

$$(*) \qquad \{f^*(U_\xi(K)\cdot\theta)\cdot K(M)\}[M] \in S .$$

Now $T_\xi = E_\xi \cup \infty$, that is, the one-point compactification of E_ξ. Since E_ξ is a smooth manifold the map f can be deformed to a map (which we continue to call f) which is transverse regular to $B_\xi \subset E_\xi$. Set $N = f^{-1}(B_\xi)$, $g = f \mid N$: $N \to B_\xi$, i: $N \subset M$. Then N is an oriented submanifold of M with normal bundle $g^*\xi$. In particular if M is stably almost-complex and ξ is complex, then N is stably almost-complex. Let i_*: $H^*(N) \to H^*(M)$ denote the Gysin homomorphism. Now the Thom isomorphism $H^*(B\xi) \approx \tilde{H}^*(T\xi)$ can also be viewed as a Gysin homomorphism [11]. Thus, given $\theta \in H^{**}(B\xi)$,

$$f^*(U_\xi \cdot \theta) = i_* g^* \theta .$$

Moreover, $K(N) = i^*(K(M)\cdot g^*K(\xi)^{-1})$, and so

$$\begin{aligned}
\{g^*\theta\cdot K(N)\}[N] &= i_*\{g^*(\theta\cdot K(\xi)^{-1}) i^* K(M)\}[M] \\
&= \{i_* g^*(\theta\cdot K(\xi)^{-1})\cdot K(M)\}[M] \\
&= \{f^*(U_\xi\cdot K(\xi)^{-1}\cdot\theta)\cdot K(M)\}[M] \\
&= \{f^*(U_\xi(K)\cdot\theta)\cdot K(M)\}[M] .
\end{aligned}$$

Since $\theta \in S(B\xi, K)$, $\{g^*\theta\cdot K(N)\}[N] \in S$, which proves (*) and hence (6.1).

We now prove two theorems giving sufficient conditions for the inclusion in (6.1) to be an equality.

Following Thom, we write MU (n) (respectively, MSO (n)) for the Thom complex of the canonical bundle over BU (n) (respectively, BSO (n)).

We prove

(6.2) THEOREM. *Let K be an* m*-sequence that is* $(\mathrm{U}(n), S)$*-regular for some subring S of A. Suppose also that K is S-integral. Then,*

$$\tilde{S}(\mathrm{MU}(n), K) = U_n(K)\cdot S(\mathrm{BU}(n), K).$$

Here $U_n(K) = U_{\omega_n}(K)$, where ω_n is the canonical bundle over BU (n).

Let s_n: BU $(n) \to$ MU (n) denote the map given by the zero cross-section. At the end of the section we prove:

$$s_n^* U_n(K) = c_n(K), \tag{6.3}$$

where $c_n(K)$ is defined in section 4. Using this we prove (6.2).

By (6.1) and (6.3), since s_n^* is injective, (6.2) follows from

$$s_n^* \tilde{S}(\mathrm{MU}(n), K) \subset c_n(K)\cdot S(\mathrm{BU}(n), K). \tag{6.4}$$

To prove (6.4), recall that $c_i(K) = \bar{c}_i + \text{higher terms}$, where $\bar{c}_i$ denotes the image of the ordinary Chern class by the coefficient homomorphism $\mathbf{Z} \to A$. Thus,

$$\begin{aligned} s_n^* \tilde{H}^*(\mathrm{MU}(n)) &= \bar{c}_n \cdot A[[c_1, \ldots, c_n]] \\ &= c_n(K)\cdot A[[c_1(K), \ldots, c_n(K)]], \end{aligned}$$

and so

$$s_n^* \tilde{S}(\mathrm{MU}(n), K) \subset c_n(K)\cdot A[[c_1(K), \ldots, c_n(K)]].$$

On the other hand, by (1.8) and (4.3),

$$s_n^* \tilde{S}(\mathrm{MU}(n), K) \subset \tilde{S}(\mathrm{BU}(n), K) = \tilde{S}[[c_1(K), \ldots, c_n(K)]].$$

Thus (6.4) follows, since

$$\begin{aligned} &c_n(K)\cdot A[[c_1(K), \ldots, c_n(K)]] \cap S[[c_1(K), \ldots, c_n(K)]] \\ &\qquad = c_n(K)\cdot S[[c_1(K), \ldots, c_n(K)]] = c_n(K)\cdot S(\mathrm{BU}(n), K). \end{aligned}$$

This completes the proof of (6.4) and hence of Theorem (6.2).

Combining this with Theorems (3.2) and (3.6) we have:

(6.5) COROLLARY. *Let S be a subring of the rationals that contains the integers. Then, for* $n \geqq 1$,

$$\tilde{S}(\mathrm{MU}(n), \mathrm{td}) = U_n(\mathrm{td}) \cdot S[[c_1(\mathrm{td}), \ldots, c_n(\mathrm{td})]]$$

$$\tilde{S}(\mathrm{MU}(n), K) = U_n(K) \cdot S[[c_1(K), \ldots, c_n(K)]] .$$

where K is any real m*-sequence that is S-integral* (e.g., $K = L$).

We now consider MSO (n) and a *real* m-sequence K.

(6.6) THEOREM. *Let K be a real* m*-sequence that is* (SO $(q), S$)*-regular and S-integral for some subring S of the rationals. Then,*

$$\tilde{S}(\mathrm{MSO}(q), K) = U_q(K) \cdot S(\mathrm{BSO}(q), K) .$$

Again we set $U_q(K) = U_{\gamma_q}(K)$, where γ_q is the canonical bundle over BSO (q). Combining (3.6) and (5.7), we have:

(6.7) COROLLARY. *Let S be a subring of the rationals that contains the integers. Then, for* $n \geqq 1$,

$$\tilde{S}(\mathrm{MSO}(2n), L) = U_{2n}(L) \cdot S[[p_1(L), \ldots, p_{n-1}(L), \chi_{2n}(L)]] ,$$

$$\tilde{S}(\mathrm{MSO}(2n+1), L) = U_{2n+1}(L) \cdot S[[p_1(L), \ldots, p_n(L)]] .$$

Note that (6.6) does *not* apply to the m-sequence td. Thus we have:

(6.8) PROBLEM. Compute $S(\mathrm{MSO}(q), \mathrm{td})$.

PROOF OF (6.6). We distinguish two cases: q even and q odd.

CASE I: $q = 2n$. At the end of the section we prove:

$$s_{2n}^* U_{2n}(K) = \chi_{2n}(K) \in H^{**}(\mathrm{BSO}(2n)) , \tag{6.9}$$

where $\chi_{2n}(K)$ is defined in section 4. (Here $s_q \colon \mathrm{BSO}(q) \to \mathrm{MSO}(q)$ is the zerosection). Using (6.9) the proof for case I is similar to that of Theorem (6.2) and so we leave the details to the reader.

CASE II: $q = 2n+1$. Let S^1 denote the circle. Recall that by Atiyah [2] there is a natural map

$$\mu_{2n} \colon S^1 \times \mathrm{MSO}(2n) \to \mathrm{MSO}(2n+1)$$

such that $\mu_{2n}^* U_{2n+1} = \iota_1 \otimes U_{2n}$, where ι_1 generates $H^1(S^1; \mathrm{Z})$. Define

$$\hat{s}_{2n+1} \colon S^1 \times \mathrm{BSO}(2n) \to \mathrm{MSO}(2n+1)$$

to be the composition

$$S^1 \times \mathrm{BSO}(2n) \xrightarrow{1 \times s_{2n}} S^1 \times \mathrm{MSO}(2n) \xrightarrow{\mu_{2n}} \mathrm{MSO}(2n+1)\,.$$

Then, $\hat{s}^*_{2n+1} U_{2n+1} = \iota_1 \otimes \chi_{2n}$. At the end of the section we prove:

(6.10) $$\hat{s}^*_{2n+1} U_{2n+1}(K) = \iota_1 \otimes \chi_{2n}(K)\,.$$

Finally, in the following section we prove:

(6.11) $$S(S^1 \times \mathrm{BSO}(q), K) = H^{**}(S^1; \mathbf{Z}) \otimes_{\mathbf{Z}} S(\mathrm{BSO}(q), K)\,.$$

Assuming these facts we now prove case II of Theorem (6.6). Note first that $\hat{s}^*_{2n+1}$ is an injection, mapping $\tilde{H}^{**}(\mathrm{MSO}(2n+1))$ into $\tilde{H}^{**}(S^1) \otimes \tilde{H}^{**}(\mathrm{BSO}(2n+1))$. Thus by (6.1) and (6.10) to prove case II of (6.6) we need simply show:

(*) $$\hat{s}^*_{2n+1} S(\mathrm{MSO}(2n+1), K) \subset (\iota_1 \otimes \chi_{2n}(K)) \cdot S(\mathrm{BSO}(2n+1), K)\,.$$

Now

$$\begin{aligned} \hat{s}^*_{2n+1} H^{**}(\mathrm{MSO}(2n+1)) &= (\iota_1 \otimes \chi_{2n}) \cdot \mathbf{Q}[[p_1, \dots, p_n]] \\ &= (\iota_1 \otimes \chi_{2n}(K)) \cdot \mathbf{Q}[[p_1(K), \dots, p_n(K)]]\,, \end{aligned}$$

and so

$$\hat{s}^*_{2n+1} S(\mathrm{MSO}(2n+1), K) \subset (\iota_1 \otimes \chi_{2n}(K)) \cdot \mathbf{Q}[[p_1(K), \dots, p_n(K)]]\,.$$

On the other hand, by (1.8), (5.4), and (6.11),

$$\begin{aligned} \hat{s}^*_{2n+1} S(\mathrm{MSO}(2n+1), K) &\subset S(S^1 \times \mathrm{BSO}(2n+1), K) \\ &= H^{**}(S^1; \mathbf{Z}) \otimes_{\mathbf{Z}} S[[p_1(K), \dots, p_n(K)]]\,. \end{aligned}$$

Thus, (*) follows, since

$$\begin{aligned} (\iota_1 \otimes \chi_{2n}(K)) \cdot \mathbf{Q}[[p_1(K), \dots, p_n(K)]] &\cap S[[p_1(K), \dots, p_n(K)]] \\ &= (\iota_1 \otimes \chi_{2n}(K)) \cdot S[[p_1(K), \dots, p_n(K)]] \\ &= (\iota_1 \otimes \chi_{2n}(K)) \cdot S(\mathrm{BSO}(2n+1), K)\,. \end{aligned}$$

This proves (*) and hence Theorem 6.6.

We are left with proving (6.3), (6.9), and (6.10); (recall that (6.11) is proved in section 7.) We start with (6.9). By definition and by equation (D) in section 5,

$$\begin{aligned} s^*_{2n} U_{2n}(K) &= s^*_{2n}(U_{2n}(\gamma_{2n}) \cdot K(\gamma_{2n})^{-1}) \\ &= \chi_{2n}(\gamma_{2n}) \cdot K(\gamma_{2n})^{-1} = \chi_{2n}(K)\,, \end{aligned}$$

which proves (6.9). In exactly the same way, one shows that

$$c_n(\omega_n)\cdot K(\omega_n)^{-1} = c_n(K)\,,$$

in $H^{**}(\mathrm{BU}\,(n))$, and so (6.3) follows as above. Finally, (6.10) follows from (6.9) when we show

$$\mu_{2n}^* U_{2n+1}(K) = i_1 \otimes U_{2n}(K)\,.$$

To see this, let j: BSO $(2n) \to$ BSO $(2n+1)$ denote the natural inclusion. Then, by [2], given $\theta \in H^{**}(\mathrm{BSO}\,(2n+1))$,

$$\mu_{2n}^*(U_{2n+1}\cdot\theta) = (\iota_1 \otimes U_{2n})\cdot j^*\theta\,.$$

Hence,

$$\begin{aligned}\mu_{2n}^* U_{2n+1}(K) &= \mu_{2n}^*(U_{2n+1}\cdot K(\gamma_{2n+1})^{-1})\\ &= (\iota_1 \otimes U_{2n})\cdot j^* K(\gamma_{2n+1})^{-1}\\ &= (\iota_1 \otimes U_{2n})\cdot K(\gamma_{2n})^{-1} = \iota_1 \otimes U_{2n}(K)\,,\end{aligned}$$

which proves (6.10).

Theorems (6.2) and (6.6) might lead one to conjecture that for any bundle ξ, $\tilde{S}(T_\xi, K) = U_\xi \cdot S(B_\xi, K)$. However, this is not the case as we now show.

Let ω denote the complex line bundle over $P_1\,(=S^2)$. Then $T_\omega = P_2$. By (2.6),

$$\mathrm{Z}(P_1, \mathrm{td}) = H^{**}(P_1; \mathrm{Z});$$

also by (2.6), $E(=1-e^{-t}) \in \mathrm{Z}(P_2, \mathrm{td})$. Since $U_\omega = t \in H^2(P_2; \mathrm{Z})$ and $E = t - \frac{1}{2}t^2 + \ldots$, we see that $E \notin U_\omega \cdot \mathrm{Z}(P_1, \mathrm{td})$ and so $\tilde{\mathrm{Z}}(T_\omega, \mathrm{td}) \neq U_\omega \cdot \mathrm{Z}(B_\omega, \mathrm{td})$.

Thus we have

(6.12) Problem. Given S and K, for which bundles ξ does $\tilde{S}(T_\xi, K) = U_\xi \cdot S(B_\xi, K)$?

7. Computations.

We consider in this final section two separate problems: (i) compute $S(\mathrm{K}(\mathrm{Z}, 2n), K)$, $n \geqq 1$; (ii) given $S(Y, K)$, compute $S(S^n \times Y, K)$, where S^n denotes the n-sphere, $n \geqq 1$.

Since $\mathrm{K}(\mathrm{Z}, 2) = P_\infty$, we have computed $S(\mathrm{K}\,(\mathrm{Z}, 2), K)$ by Corollary (2.7) (at least when K is S-integral). To compute $S(\mathrm{K}\,(\mathrm{Z}, 2n), K)$ for $n > 1$ we need several definitions.

Let K denote a fixed m-sequence (real or complex) with the rationals as coefficient domain. In section 2 we associated with K a power series R^K (the reciprocal series) with $R^K \in \mathbf{Q}[[t]]$ and $R^K(t) = t + \ldots$. Define I^K to be the inverse to R^K, in the sense of composition of series; that is, $I^K(R^K(t)) = t$. We call I the *inverse series* for K. By (2.3) we obtain:

(7.1) EXAMPLE.

$$I^{\mathrm{td}} = \sum_{1 \leqq n} \frac{t^n}{n}, \qquad I^{\mathrm{L}} = \sum_{1 \leqq n} \frac{t^{2n-1}}{2n-1}.$$

Now let $\varphi \in \mathbf{Q}[[t]]$ be any series with $\varphi(0) = 0$. We write φ as:

$$\varphi(t) = \sum_{0 \leqq i} \frac{\beta_i}{\gamma_i} t^{i+1},$$

where $\beta_i, \gamma_i \in \mathbf{Z}$ and $(\beta_i, \gamma_i) = 1$. (Note that $\gamma_i = 1$ if $\beta_i = 0$). We say that φ has *infinitely many primes* if, given any positive integer N, there is a prime p and an integer $i(N)$ such that $p > N$ and $p \mid \gamma_{i(N)}$. We say that the m-sequence K has *infinitely many primes* if this is true of its inverse series I^K. Note that by (7.1) *the* m-*sequences* td *and* L *have infinitely many primes.*

We now can state our result.

(7.2) THEOREM. *Let K be a $\mathbf{Z}$-integral* m-*sequence with infinitely many primes. Then, for $n \geqq 2$,*

$$\mathfrak{Z}(\mathrm{K}(\mathbf{Z}, 2n), K) = 0.$$

In particular,

$$\mathfrak{Z}(\mathrm{K}(\mathbf{Z}, 2n), \mathrm{td}) = 0, \qquad \mathfrak{Z}(\mathrm{K}(\mathbf{Z}, 2n), L) = 0.$$

Recall that $H^{**}(\mathrm{K}(\mathbf{Z}, 2n); \mathbf{Q}) = \mathbf{Q}[[t]]$, degree $t = 2n$. Thus we have,

(7.3) COROLLARY. *Given any series $\theta \in \mathbf{Q}[[t]]$ and any integer $n \geqq 2$, there is an oriented manifold $M(\theta, n)$ $(= M)$ and a class $u \in H^{2n}(M; \mathbf{Z})$ such that*

$$\{\theta(u) \cdot \mathrm{L}(M)\}[M] \notin \mathbf{Z}.$$

Similarly, there is a stably almost-complex manifold $N(\theta, n)$ $(= N)$ and a class $v \in H^{2n}(N; \mathbf{Z})$ such that

$$\{\theta(v) \cdot \mathrm{td}(N)\}[N] \notin \mathbf{Z}.$$

REMARK. Theorem (7.2) suggest the following problems: (i) Characterize those spaces Y such that $\mathfrak{Z}(Y, K) = 0$, for all m-sequences K with infinitely many primes. (ii) Calculate $\mathbf{Z}(\mathrm{K}(\mathbf{Z}, 2n), K)$, $n \geqq 2$, for m-sequences that do *not* have infinitely many primes.

PROOF OF THEOREM (7.2). We use the space $\mathrm{B}\,T(n)$; as in section 2, set

$$H^{**}(\mathrm{B}\,T(n); \mathbf{Z}) = \mathbf{Z}[[t_1,\ldots,t_n]], \quad \deg t_i = 2 .$$

Define $\mu_n\colon \mathrm{B}\,T(n) \to \mathrm{K}(\mathbf{Z}, 2n)$ by $\mu_n^* \iota_{2n} = t_1 \ldots t_n$, where ι_{2n} denotes the fundamental class for $\mathrm{K}(\mathbf{Z}, 2n)$. Then with *rational* coefficients μ_n^* is a monomorphism and

$$\mu_n^* \mathbf{Q}[[t]] = \mathbf{Q}[[(t_1 \ldots t_n)]] \subset \mathbf{Q}[[t_1,\ldots,t_n]] ,$$

where degree $t = 2n$. By (1.8) and (2.7),

$$\mu_n^* \mathbf{Z}(\mathrm{K}(\mathbf{Z}, 2n), K) \subset \mathbf{Z}(\mathrm{B}\,T(n), K) = \mathbf{Z}[[R_1,\ldots,R_n]] ,$$

where R is the reciprocal series for K. Thus to prove (7.2) we need only show:

(*) $$\tilde{\mathbf{Z}}[[R_1,\ldots,R_n]] \cap \tilde{\mathbf{Q}}[[t_1 \ldots t_n]] = 0 .$$

We transform this as follows: let I denote the inverse series for K, so that $I(R(t)) = t$. Then (*) becomes

(**) $$\tilde{\mathbf{Z}}[[u_1,\ldots,u_n]] \cap \tilde{\mathbf{Q}}[[I(u_1) \ldots I(u_n)]] = 0 ,$$

when $u_i = R_i = R(t_i)$.

To prove (**) we need the following result, whose proof is given in the appendix (section 8).

(7.4) LEMMA. *Let $\varphi \in \mathbf{Q}[[t]]$ have infinitely many primes. Then so does φ^r, for $r \geqq 1$.*

Assuming this we prove (**). Let $\psi \in \tilde{\mathbf{Q}}[[t]]$ be a series such that

$$\psi(I(u_1) \ldots I(u_n)) \in \tilde{\mathbf{Z}}[[u_1,\ldots,u_n]] .$$

Write $\psi(t) = \sum_{1 \leq i} a_i t^i$, where $a_i \in \mathbf{Q}$. We are to show that $a_i = 0$ for all $i \geqq 1$. Suppose inductively we have shown that $a_1,\ldots,a_{N-1} = 0$, for some $N \geqq 1$. We show that then $a_N = 0$, which will complete the inductive step. Write $a_N = b_N/c_N$, with $b_N, c_N \in \mathbf{Z}$ and $(b_N, c_N) = 1$. Also, set

$$(I(t))^N = \sum_{0 \leq i} \frac{\delta_i}{\varepsilon_i} t^{N+i} ,$$

where $\delta_i, \varepsilon_i \in \mathbf{Z}$ and $(\delta_i, \varepsilon_i) = 1$. Then,

$$\begin{aligned}
\psi(I(u_1) \ldots I(u_n)) &= a_N (I(u_1) \ldots I(u_N))^N + \text{higher terms} \\
&= a_N \left(u_1^N \ldots u_{n-1}^N \left(\sum_{0 \leq i} \frac{\delta_i}{\varepsilon_i} u_n^{N+i} \right) \right) + \ldots , \\
&= \sum_{0 \leq i} \frac{b_N \delta_i}{c_N \varepsilon_i} u_1^N \ldots u_{n-1}^N u_n^{N+i} + \ldots ,
\end{aligned}$$

where the terms omitted in the last equation all have higher powers of $u_1 \dots u_{n-1}$. Since $\psi(I(u_1)\dots I(u_n)) \in \tilde{\mathbf{Z}}[[u_1,\dots,u_n]]$, this implies that

$$\frac{b_N\delta_i}{c_N\varepsilon_i} \in \mathbf{Z}, \quad \text{for } i \geqq 0 .$$

By hypothesis and by Lemma (7.4), $I(t)^N$ has infinitely many primes and so there is a prime p and an integer s such that

$$p > b_N, \quad p \mid \varepsilon_s .$$

Thus, $\varepsilon_s \neq 1$ and so $\delta_s \neq 0$. Since $(b_N, c_N)=1$ and $(\delta_s, \varepsilon_s)=1$ and since $b_N\delta_s/c_N\varepsilon_s \in \mathbf{Z}$, we must have $b_N=0$, and so $a_N=0$ as claimed. This completes the proof of Theorem (7.2).

We turn now to the problem of computing $S(S^n \times Y, K)$.

(7.5) THEOREM. *Let K be an* m*-sequence with coefficient domain A and let S be a subring of A. Then for any space Y and any positive integer n,*

$$S(S^n \times Y, K) = H^{**}(S^n; \mathbf{Z}) \otimes_{\mathbf{Z}} S(Y, K) .$$

PROOF. Let 1 and ι_n generate the respective free $\mathbf{Z}$-modules $H^0(S^n; \mathbf{Z})$ and $H^n(S^n; \mathbf{Z})$. Then any element θ of $H^{**}(S^n; \mathbf{Z}) \otimes_{\mathbf{Z}} S(Y, K)$ has the form

$$\theta = 1 \otimes \varphi + \iota_n \otimes \psi ,$$

where $\varphi, \psi \in S(Y, K)$. We first show that every such class θ is an element of $S(S^n \times Y, K)$.

By the universal coefficient theorem, we identify $S(S^n \times Y, K)$ with a subring of $H^{**}(S^n; \mathbf{Z}) \otimes_{\mathbf{Z}} H^{**}(Y; A)$. Note that by Theorem (1.8), applied to the projection $S^n \times Y \to Y$,

$$1 \otimes \varphi \in S(S^n \times Y, K) .$$

Thus it remains to show that $\iota_n \otimes \psi \in S(S^n \times Y, K)$. Let M be a smooth oriented manifold (in the domain of K) and let $u\colon M \to S^n$, $f\colon M \to Y$ be maps. We need simply show:

(A) $$\{(u, f)^*(\iota_n \otimes \psi) \cdot K(M)\}[M] \in S .$$

Let $e \in S^n$ be a basepoint; make u transverse regular to e and let $N = u^{-1}(e) \subset M$, with embedding i. Note that N is dual to $u^*\iota_n$ and that N has a stably trivial normal bundle — hence, $K(N) = i^*K(M)$. Also, if M is stably almost-complex, so is N. Thus we have (compare (2.8)),

$$\begin{aligned}\{i^*f^*\psi\cdot K(N)\}[N] &= i_*i^*\{f^*\psi\cdot K(M)\}[M]\\ &= \{f^*\psi\cdot K(M)\cdot u^*\iota_n\}[M]\\ &= \pm\{(u,f)^*(\iota_n\otimes\psi)\cdot K(M)\}[M]\,.\end{aligned}$$

But $\{i^*f^*\psi\cdot K(N)\}[N]\in S$, since $\psi\in S(Y,K)$. Thus, (A) follows, and so

$$H^{**}(S^n;Z)\otimes_Z S(Y,K)\subset S(S^n\times Y,K)\,.$$

To complete the proof of Theorem (7.5) we show:

(B) Every element ω in $S(S^n\times Y,K)$ can be written in the form

$$\omega = 1\otimes\varrho_1+\iota_n\otimes\varrho_2\,,$$

where $\varrho_1,\varrho_2\in \mathrm{S(Y,K)}$.

As remarked above, the element ω can be written in this form with $\varrho_1\varrho_2\in H^{**}(Y;A)$. To prove (B) we need only show $\varrho_1,\varrho_2\in S(Y,K)$.

Let $j\colon Y\to S^n\times Y$ be the map given by $y\to(e,y)$. Since $j^*S(S^n\times Y,K)\subset S(Y,K)$ and since $j^*\omega=\varrho_1$, it follows that $\varrho_1\in S(Y,K)$. Let $\omega'=\omega-1\otimes\varrho_1=\iota_n\otimes\varrho_2$. By (A) and the above, $\omega'\in S(S^n\times Y,K)$. Let M be a manifold and f a map $M\to Y$. Consider the map

$$1\times f\colon S^n\times M\to S^n\times Y\,.$$

Then,

$$\{(1\times f)^*\omega'\cdot K(S^n\times M)\}[S^n\times M]\in S\,,$$

since $\omega'\in S(S^n\times Y,K)$. But $K(S^n\times M)=1\otimes K(M)$ (since S^n is stably parallelizable) and so

$$\begin{aligned}\{(1\times f)^*\omega'\cdot K(S^n\times M\}[S^n\times M] &= \pm\{\iota_n\otimes f^*\varrho_2\cdot K(M)\}[S^n\times M]\\ &= \pm\{f^*\varrho_2\cdot K(M)\}[M]\,.\end{aligned}$$

Thus, $\{f^*\varrho_2\cdot K(M)\}[M]\in S$, and so $\varrho_2\in S(Y,K)$ as claimed. This completes the proof of Theorem (7.5).

Taking $Y=$point, in (7.5), we obtain

(7.6) Corollary. *For* $n\geqq 1$,

$$S(S^n,K) = H^{**}(S^n;S)\,.$$

8. Appendix: Power series.

Let A be a fixed commutative ring with unit. We consider formal power series over A in variables $t_1,\ldots,t_n$, $n\geqq 1$, (see [14, p. 146]). Denote by $\tilde{A}[[t_1,\ldots,t_n]]$ the subring of $A[[t_1,\ldots,t_n]]$ consisting of those power series with constant term zero. Given a series $\varphi\in\tilde{A}[[t]]$, we set, for $1\leqq i\leqq n$,

$$\sigma_i\varphi = i\text{th elementary symmetric function in } \varphi(t_1),\ldots,\varphi(t_n);$$

for example

$$\sigma_1\varphi = \varphi(t_1)+\ldots+\varphi(t_n)\in A[[t_1,\ldots,t_n]]\,.$$

(8.1) LEMMA. *Let $\varphi_1,\ldots,\varphi_n\in\tilde{A}[[t]]$, $\varphi_i\neq 0$. Then the natural map*

$$\iota\colon A[[\sigma_1\varphi_1,\ldots,\sigma_n\varphi_n]]\to A[[t_1,\ldots,t_n]]$$

is an injection.

PROOF. We do the proof by induction on n. When $n=1$, $\sigma_1\varphi_1=\varphi_1$ and the lemma is easily seen to be true. Let $N>1$ and suppose (8.1) has been proved for all $n>N$. Let $f\in A[[\sigma_1\varphi_1,\ldots,\sigma_N\varphi_N]]$ and suppose that $\iota f=0$, in $A[[t_1,\ldots,t_N]]$. Write f as:

$$f(\sigma_1\varphi_1,\ldots,\sigma_N\varphi_N) = \sum_{i=0}^{\infty} g_i(\sigma_1\varphi_1,\ldots,\sigma_{N-1}\varphi_{N-1})\cdot(\sigma_N\varphi_N)^i;$$

now set $t_N=0$. Then $\sigma_N\varphi_N=0$ and so

$$0 = \iota f(\sigma'_1\varphi_1,\ldots,\sigma'_{N-1}\varphi_{N-1},0) = \iota g_0(\sigma'_1\varphi_1,\ldots,\sigma'_{N-1}\varphi_{N-1})\,,$$

where $\sigma'_i\varphi_i$ denotes the ith elementary symmetric function in $\varphi_i(t_1),\ldots,\varphi_i(t_{N-1})$. By the inductive hypothesis, $g_0=0$. Assume now that $g_0,g_1,\ldots,g_{s-1}=0$, where $s>0$. We show that then $g_s=0$ and hence (again by induction) $f=0$, which will complete the original inductive argument.

Since $s>0$ we may write

$$f = \sum_{i=s}^{\infty}(g_i)(\sigma_N\varphi_N)^i = \sum_{i=0}^{\infty}((g_{s+i})(\sigma_N\varphi_N)^i)(\sigma_N\varphi_N)^s\,,$$

where $g_i=g_i(\sigma_1\varphi_1,\ldots,\sigma_{N-1}\varphi_{N-1})$. But

$$0 = \iota f = \left[\iota\sum_{i=0}^{\infty}(g_{s+i})(\sigma_N\varphi_N)^i\right][\iota(\sigma_N\varphi_N)^s]\,.$$

Since $\iota(\sigma_N\varphi_N)^s\neq 0$, this implies that $\iota\sum_{i=0}^{\infty}(g_{s+i})(\sigma_N\varphi_N)^i=0$. (Compare [23, p. 79]). By what we have already proved, this shows that $g_s(\sigma_1\varphi_1,\ldots,\sigma_{N-1}\varphi_{N-1})=0$, which completes the proof.

From now on we identify $A[[\sigma_1\varphi_1,\ldots,\sigma_n\varphi_n]]$ with its image by ι.

Given $\varphi \in A[[t]]$, we define degree φ to be the exponent of the *lowest* power of t occurring in φ with a non-zero coefficient. We say that φ is *monic* if the lowest power of t occurring in φ has as coefficient a unit in A. We say that φ is *monic* rel. S if this coefficient is a unit in the subring S.

(8.2) LEMMA. *Let ψ be a monic series of degree one in $A[[t]]$. Then,*

$$A[[\sigma_1,\ldots,\sigma_n]] = A[[\sigma_1\psi,\ldots,\sigma_n\psi]],$$

as subrings of $A[[t_1,\ldots,t_n]]$.

PROOF. It suffices to show that for $1 \leqq i \leqq n$, $\sigma_i \in A[[\sigma_1\psi,\ldots,\sigma_n\psi]]$. Since ψ is monic of degree one there is a series $P \in \tilde{A}[[t]]$ such that $t = P(\psi)$, and so $\sigma_i = \sigma_i(P(\psi))$. But clearly $\sigma_i(P(\psi)) \in A[[\sigma_1\psi,\ldots,\sigma_n\psi]]$, and so the proof is complete.

For $1 \leqq i \leqq n$, let $\hat{\sigma}_i = i$th elementary symmetric function in $t_1^2,\ldots,t_n^2$.

(8.3) LEMMA. *Let $\varphi, \psi \in A[[t]]$ with φ an even monic series of degree two and ψ an odd monic series of degree one. Then,*

(i) $A[[\hat{\sigma}_1,\ldots,\hat{\sigma}_{n-1},\sigma_n]] = A[[\sigma_1\varphi,\ldots,\sigma_{n-1}\varphi,\sigma_n\psi]]$,
(ii) $A[[\hat{\sigma}_1,\ldots,\hat{\sigma}_n]] = A[[\sigma_1\varphi,\ldots,\sigma_n\varphi]]$.

PROOF. Let $f \in A[[t_1,\ldots,t_n]]$, and let s be a positive integer. We say that $f \equiv 0 \pmod s$ if f has no term of degree $\leqq s$ in $t_1,\ldots,t_n$ (note that degree $t_i = 1$). We write $f \equiv g \pmod s$ if $f - g \equiv 0 \pmod s$. The following facts are obvious:

(i) If $f \equiv 0$, $g \equiv 0 \pmod s$, then $a_1 f + a_2 g \equiv 0 \pmod s$, $a_1, a_2 \in A$.

(ii) Suppose that $\alpha, \beta \in A[[t]]$ with $\alpha(t) = t + \ldots$, $\beta(t) = t^2 + \ldots$. For any $h \in A[[t_1,\ldots,t_n]]$, if

$$h(\hat{\sigma}_1,\hat{\sigma}_2,\ldots,\hat{\sigma}_{n-1},\sigma_n) \equiv 0 \pmod s,$$

then

$$h(\hat{\sigma}_1,\ldots,\hat{\sigma}_{n-1},\sigma_n) \equiv h(\sigma_1\alpha,\ldots,\sigma_{n-1}\alpha,\sigma_n\beta) \pmod{s+1}.$$

We now can prove (8.3). Suppose that $\varphi(t) = \varepsilon t^2 + \ldots$, $\psi(t) = \delta t + \ldots$, where ε, δ are units in A. Set

$$\bar{\varphi} = \varepsilon^{-1}\varphi, \quad \bar{\psi} = \delta^{-1}\psi.$$

Since φ is even and ψ is odd, we have

$$A[[\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}]] \subset A[[\hat{\sigma}_1,\ldots,\hat{\sigma}_{n-1},\sigma_n]].$$

Thus to prove (i) in the lemma it suffices to show that

$$\sigma_n, \hat{\sigma}_i \in A[[\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}]], \qquad 1 \leqq i \leqq n-1 .$$

Consider first $\sigma_n = t_1 \ldots t_n$. Then, $\sigma_n \equiv \sigma_n\bar{\psi}$ (mod n). Suppose, inductively, we have found a series f_s, for $n \leqq s < N$, such that

$$\sigma_n \equiv f_s(\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}) \quad (\mathrm{mod}\, s) .$$

We show that we then can find f_N, thus completing the inductive step.

Set $g_s = \sigma_n - f_s$, regarded as an element of $A[[\hat{\sigma}_1,\ldots,\hat{\sigma}_{n-1},\sigma_n]]$. Note that $g_s \equiv 0$ (mod s) and so by (ii) above, if we set

$$\hat{g} = g_{N-1}(\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}) ,$$

we have $\hat{g} \equiv g_{N-1}$ (mod N). Set $f_N = f_{N-1} + \hat{g}$. Since

$$\sigma_n - f_{N-1} = g_{N-1} \equiv \hat{g} \ (\mathrm{mod}\, N) ,$$

it follows that $\sigma_n - (f_{N-1} + \hat{g}) \equiv 0$ (mod N) and so

$$\sigma_n \equiv f_{N-1} + \hat{g} = f_N \quad (\mathrm{mod}\, N) ,$$

which completes the inductive step. Thus

$$\sigma_n \in A[[\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}]];$$

in a similar way one shows that $\hat{\sigma}_i \in A[[\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}]]$. Since

$$A[[\sigma_1\bar{\varphi},\ldots,\sigma_{n-1}\bar{\varphi},\sigma_n\bar{\psi}]] = A[[\sigma_1\varphi,\ldots,\sigma_{n-1}\varphi,\sigma_n\psi]] ,$$

Lemma (8.3) (i) is proved. Similarly one shows (8.3) (ii); we omit the details.

(8.4) Lemma. *Let $\varphi_1,\ldots,\varphi_n \in A[[t]]$ with each φ_i monic rel. S. Then*

$$A[[\sigma_1\varphi_1,\ldots,\sigma_n\varphi_n]] \cap S[[t_1,\ldots,t_n]] = S[[\sigma_1\varphi_1,\ldots,\sigma_n\varphi_n]] .$$

We precede the proof by some remarks. Recall the lexicographic ordering of monomials in $A[[t_1,\ldots,t_n]]$. Given monomials $\alpha = t_1^{a_1} \ldots t_n^{a_n}$ and $\beta = t_1^{b_1} \ldots t_n^{b_n}$, we say that $\alpha < \beta$ if either $\deg \alpha < \deg \beta$ (recall that $\deg t_i = 1$) or $\deg \alpha = \deg \beta$ and for some j, $1 \leqq j \leqq n$, $a_i = b_i$ for $i > j$ and $a_j < b_j$.

Note the following simple fact:

(8.5) If α, β, γ are monomials in $A[[t_1,\ldots,t_n]]$ with $\beta < \gamma$, then $\alpha\beta < \alpha\gamma$.

Let $\{\lambda_i\}$, $i \geqq 0$, denote the monomials in $A[[t_1,\ldots,t_n]]$ with the above ordering: so, $\lambda_0 = 1$, $\lambda_1 = t_1,\ldots$, $\lambda_{n+1} = t_1 t_n,\ldots$. Thus, any series φ in $A[[t_1,\ldots,t_n]]$ has a unique expression

$$\varphi = \sum_{i=0}^{\infty} a_i \lambda_i, \quad a_i \in A .$$

We define the *leading coefficient* of φ to be the first non-zero a_i. The main fact needed to prove (8.4) is the following:

(8.6) LEMMA. *Let ξ and η be series in $A[[t_1,\ldots,t_n]]$ such that the leading coefficient of η is a unit in S. If η and $\xi\eta \in S[[t_1,\ldots,t_n]]$, then $\xi \in S[[t_1,\ldots,t_n]]$.*

PROOF. Let $\xi=\sum_{i=0}^{\infty} s_i\lambda_i$, $s_i \in A$. Suppose, by induction, that $s_0,\ldots,s_{N-1} \in S$, for some $N \geqq 0$. We show that then $s_N \in S$, which will complete the inductive step and prove the lemma. If $s_N=0$, then trivially $s_N \in S$, so we assume $s_N \neq 0$. Set

$$\xi' = \xi - \sum_{0}^{N-1} s_i\lambda_i = \sum_{N}^{\infty} s_i\lambda_i;$$

clearly $\xi'\eta \in S[[t_1,\ldots,t_n]]$. Suppose that $\eta=\sum_{i=M}^{\infty} r_i\lambda_i$, where $r_M \neq 0$. By hypothesis, $r_M=\varepsilon$, a unit in S. Write $\xi'\eta=\sum_K^{\infty} q_i\lambda_i$, where $q_K \neq 0$. By (8.5), $\lambda_K=\lambda_N\cdot\lambda_M$ and hence $q_K=s_N\cdot r_M=s_N\cdot\varepsilon$. Since $\xi'\eta \in S[[t_1,\ldots,t_n]]$ we have $q_K \in S$. Thus, $s_N=q_K\varepsilon^{-1} \in S$, which completes the proof.

PROOF OF (8.4). Using (8.6) the proof of (8.4) is very similar to that given for (8.1); we leave the details to the reader.

We turn now to the proof of Lemma (7.4) — recall we are given a series $\varphi \in \tilde{Q}[[t]]$. For $r \geqq 1$ we write

$$\varphi^r = \sum_{0\leqq i} \alpha_i(r)t^{i+r} = \sum_{0\leqq i} \frac{\beta_i(r)}{\gamma_i(r)} t^{i+r},$$

where $\beta_i(r), \gamma_i(r) \in \mathbf{Z}$ and $(\beta_i(r),\gamma_i(r))=1$. We set $\beta_i=\beta_i(1)$, $\gamma_i=\gamma_i(1)$.

(8.7) LEMMA. *Given $r \geqq 1$, for each $k \geqq 0$ there are integers $d_k(r)$ and $e_k(r)$ so that*

(i) $$\alpha_k(r) = \frac{r\beta_k d_k(r)+\gamma_k e_k(r)}{\gamma_k\cdot d_k(r)},$$

(ii) $$d_k(1) = 1, \quad e_k(1) = 0,$$

(iii) $$d_k(r) = d_k(r-1)\cdot\prod_{i=1}^{k-1} \gamma_i\cdot\gamma_{k-i}(r-1), \qquad r \geqq 2.$$

(We interpret the product over the empty set as 1.)

The proof is by induction on r, using the fact that $\varphi^r = \varphi \cdot \varphi^{r-1}$. We omit the details.

As a consequence of (8.7) we have

(8.8) LEMMA. *Let r and k be positive integers. Then,*

$$d_k(r) \text{ is a product of powers of } \gamma_1, \ldots, \gamma_{k-1}\,,$$
$$\gamma_k(r) \text{ is a product of powers of } \gamma_1, \ldots, \gamma_k\,.$$

Note that by (8.7) (i), $\gamma_k(r) \mid \gamma_k \cdot d_k(r)$. Thus (8.8) follows from (8.7) (iii) by induction on r; we omit the details.

PROOF OF LEMMA (7.4). The proof is by induction on r; by hypothesis, the lemma is true for $r=1$. Suppose then that $r>1$ and that φ^i has infinitely many primes for $1 \leq i < r$. We show that φ^r does also.

Let N be a positive integer. Since φ has infinitely many primes, there is a prime p and an integer s such that

$$p > N, \quad p > r, \quad p \mid \gamma_s\,.$$

For a given prime p, let s be the least integer with this property. Thus, $p \nmid \gamma_1, \ldots, \gamma_{s-1}$ and so by (8.8), $p \nmid d_s(r)$. To prove (7.4) we show: $p \mid \gamma_s(r)$.

For by (8.7) (i), $\beta_s(r) \mid r\beta_s d_s(r) + \gamma_s e_s(r)$, and $\gamma_s(r) \mid \gamma_s \cdot d_s(r)$. Since $p \mid \gamma_s$ and $p \nmid r\beta_s d_s(r)$, it follows that $p \nmid r\beta_s d_s(r) + \gamma_s e_s(r)$. Thus the factor p in $\gamma_s d_s(r)$ persists when we write $\alpha_s(r)$ as $\beta_s(r)/\gamma_s(r)$, and so $p \mid \gamma_s(r)$ as claimed. This proves lemma (7.4).

9. Appendix II: The m-sequence $\hat{A}$.

An important role in topology is played by the m-sequence $\hat{A}$, with characteristic series $t/2\sinh(t/2)$. Since $\hat{A}$ is $Z[\frac{1}{2}]$-integral one may apply the theorems of sections 2–7, taking coefficient domain $= Q$, $S = Z[\frac{1}{2}]$. However, by proceeding in a slightly different way, we obtain more precise results.

Define $\mathscr{M}$ to be the set of all triples (M, ξ, d), where M is a smooth manifold, ξ is a vector bundle over M and d is a class in $H^2(M; Z)$ such that $d \bmod 2 = W_2\xi + W_2 M$. Let csh denote the real m-sequence with characteristic series $\cosh(t/2)$. For any space Y and any subring $S \subset Q$, define

(9.1) $S_m(Y, \hat{A}) =$ all classes $\theta \in H^{**}(Y, Q)$ such that for all triples in $\mathscr{M}$ and all maps $f\colon M \to Y$,

$$2^s\{f^*\theta \cdot e^{d/2} \cdot \mathrm{csh}(\xi) \cdot \hat{A}(M)\}[M] \in S\,,$$

where ξ is a $(2s+\varepsilon)$-bundle, $\varepsilon = 0$ or 1.

By the Atiyah–Singer theorem (e.g., see [11]), $1 \in S_m(Y, \hat{A})$.

Suppose that ω is a complex bundle over a smooth manifold M, with $c_1\omega = d \in H^2(M; \mathbf{Z})$. Then (see [11]),

$$\operatorname{td}(\omega) = e^{d/2} \cdot \hat{A}(\omega),$$

and so

$$S_m(Y, \hat{A}) \subset S(Y, \operatorname{td}).$$

Our result is:

(9.2) THEOREM. *For any compact connected Lie group G and any subring $S \subset \mathbf{Q}$,*

$$S_m(\mathrm{B}G, \hat{A}) = S(\mathrm{B}G, \operatorname{td}).$$

PROOF. We need only show:

$$(*) \qquad S(\mathrm{B}G, \operatorname{td}) \subset S_m(\mathrm{B}G, \hat{A}).$$

Recall that $S(\mathrm{B}T(n), \operatorname{td}) = \mathbf{Z}[[E_1, \ldots, E_n]]$, where $E_i = 1 - e^{-t_i}$. Since $\mathbf{Z}[[E]] = \mathbf{Z}[[e^{-t}]]$ and since for any class $u \in H^2(M; \mathbf{Z})$, $2nu + d \equiv d \bmod 2$, (*) follows for $G = T(n)$, $n \geq 1$.

Now let G be any compact, connected Lie group. By (3.2) there is a maximal torus $T \subset G$ such that td is (G, S)-regular with respect to T. Let $j \colon \mathrm{B}T \to \mathrm{B}G$ be induced by the inclusion. Using (3.3), to prove (*) we need only show: given any class $\theta \in H^{**}(\mathrm{B}G, \mathbf{Q})$ such that $j^*\theta \in S_m(\mathrm{B}T, \hat{A})$ then for any triple (M, ξ, d) and any map $f \colon M \to \mathrm{B}G$,

$$(**) \qquad 2^s\{f^*\theta \cdot e^{d/s} \cdot \operatorname{csh}(\xi) \cdot \hat{A}(M)\}[M] \in \mathbf{Z}.$$

Let

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\iota} & \mathrm{B}T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle j=\pi} \\ M & \xrightarrow{f} & \mathrm{B}G \end{array}$$

denote the diagram given in (3.4). As before, let β_F denote the bundle along the fiber, with complex structure chosen so that $p_*(\operatorname{td}\beta_F) = 1 \in H^0(M)$. Let $\alpha = c_1(\beta_F)$. Then

$$\hat{A}(\beta_F) = e^{-\alpha/2} \cdot \operatorname{td}\beta_F$$

and so

$$\hat{A}(\hat{M}) = p^*\hat{A}(M) \cdot e^{-\alpha/2} \operatorname{td}\beta_F$$

Let (M,ξ,d) be the triple in $\mathscr{M}$ and θ the class in $H^*(\mathrm{B}\,G)$, as above. Set $\hat{d}=p^*d$, $\hat{\xi}=p^*\xi$. Note that

$$\begin{aligned}\{p^*f^*\theta\cdot e^{(\hat{d}+\alpha)/2}\cdot\mathrm{csh}\,(\hat{\xi})\cdot\hat{\mathrm{A}}\,(\hat{M})\}[\hat{M}] &= p_*\{p^*f^*\theta\cdot e^{(\hat{d}+\alpha)/2}\cdot\mathrm{csh}\,(\hat{\xi})\,\hat{\mathrm{A}}\,(\hat{M})\}[M]\\ &= \{f^*\theta\cdot e^{d/2}\cdot\mathrm{csh}\,(\xi)\cdot\hat{\mathrm{A}}\,(M)\}[M]\,.\end{aligned}$$

Let $\bar{\alpha}=\alpha \bmod 2$. Then,

$$\begin{aligned}W_2\hat{M}+W_2\hat{\xi} &= p^*W_2M+\bar{\alpha}+p^*W_2\xi\\ &= (\hat{d}+\alpha)\ \mathrm{mod}\,2\,,\end{aligned}$$

and so, since $j^*\theta\in S_m(\mathrm{B}\,T,\hat{\mathrm{A}})$ and $p^*f^*=l^*j^*$, we have

$$2^s\{p^*f^*\theta\cdot e^{(\hat{d}+\alpha)/2}\cdot\mathrm{csh}\,(\hat{\xi})\cdot\hat{\mathrm{A}}\,(\hat{M})\}[\hat{M}]\in\mathbf{Z}\,,$$

which proves (**). Thus

$$\begin{aligned}j^*S_m(\mathrm{B}\,G,\hat{\mathrm{A}}) &= j^*H^{**}(\mathrm{B}\,G)\cap S_m(\mathrm{B}\,T,\hat{\mathrm{A}})\\ &= j^*H^{**}(\mathrm{B}\,G)\cap S(\mathrm{B}\,T,\mathrm{td})\\ &= j^*S(\mathrm{B}\,G,,\mathrm{td})\,.\end{aligned}$$

Since j^* is injective, this proves (9.2).

In a similar fashion one can prove:

(9.3) THEOREM. $S_m(\mathrm{MU}\,(n),\hat{\mathrm{A}})=S(\mathrm{MU}\,(n),\mathrm{td})$, *for* $n\geqq 1$.

We omit the details.

10. Appendix III. PL and topological manifolds.

So far we have assumed that all manifolds are smooth. We note here what can be said about PL and topological manifolds.

A real m-sequence K can be regarded as a class in $H^{**}(\mathrm{BSO})$. (We assume the coefficient domain is $\mathbf{Q}$). Since the natural H-space maps $\mathrm{BSO}\to\mathrm{BSPL}\to\mathrm{BSTOP}$ induce isomorphisms

$$H^*(\mathrm{BSO})\approx H^*(\mathrm{BSPL})\approx H^*(\mathrm{BSTOP})\,,$$

we can regard K as defined on oriented PL and topological manifolds. Given a subring S of $\mathbf{Q}$, and a space Y, we define $S^p(Y,K)$ (respectively $S^t(Y,K)$) to be the subgroup of $H^{**}(Y,\mathbf{Q})$ defined as in (1.4), where we replace smooth manifolds by PL (respectively topological) manifolds. Clearly,

$$S^t(Y,K) \subset S^p(Y,K) \subset S(Y,K). \tag{10.1}$$

We prove:

(10.2) THEOREM. *Let S be a subring of the rationals and let K be a real* m-*sequence that is S-integral. Then, $S^p(Y,K)=S(Y,K)$ for the following spaces Y:* B $T(n)$, BU (n), BSO (n), MU (n), MSO (n), K $(\mathrm{Z}, 2n)$, $n \geqq 1$.

The proof consists in observing that in sections 2–7 we have used only the following properties of smooth manifolds: (i) Thom transversality, (ii) The "pull-back" property given in (3.4), (iii) Poincaré duality and (iv) the Gysin homomorphism. Since these properties also hold for PL manifolds, it is easily checked that for each space Y in (10.2), equality holds in (10.1) between $S^p(Y,K)$ and $S(Y,K)$.

On the other hand, I do not know whether $S^t(Y,K)=S(Y,K)$ for the spaces Y given in (10.2). The difficulty is that one does not always have Thom transversality for topological manifolds, as for example, when one maps an n-manifold into the total space of an $(n-4)$-plane bundle.

11. Appendix IV: Bordism.

We note here a geometric interpretation of the group $S(Y,K)$. Recall [1] an m-sequence K can be regarded as a ring homomorphism $K\colon \Omega_*^G \to A$, where $G = \mathrm{U}$ or SO and Ω_*^G denotes bordism of G-manifolds. Consider now the bordism groups $\Omega_*^G(Y)$, for Y a space [7]. We will say that a homomorphism $\varphi\colon \Omega_*^G(Y) \to A$ is K-linear (K an m-sequence) if

$$\varphi(\alpha \cdot u) = K(\alpha) \cdot \varphi(u),$$

where $\alpha \in \Omega_*^G$, $u \in \Omega_*^G(Y)$ and where $\alpha \cdot u$ denotes the usual module action of Ω_*^G on $\Omega_*^G(Y)$.

Suppose now that $H_*(Y;\mathrm{Z})$ has no torsion. Using the fact ([8])

$$\Omega_*^G(Y) \otimes A \approx H_*(Y;\mathrm{Z}) \otimes \Omega_*^G \otimes A,$$

it follows that any K-linear morphism φ has the form $\varphi = \lambda \otimes K$ where $\lambda \in H^{**}(Y;A)$. Thus we have:

(11.1) THEOREM. *Let $\varphi\colon \Omega_*^G(Y) \to A$ be a K-linear homomorphism, where $H_*(Y;\mathrm{Z})$ is torsion-free, and let S be a subring of A. Then $\varphi(\Omega_*^G(Y)) \subset S$ if, and only if, $\varphi = \lambda \otimes K$, where $\lambda \in S(Y,K) \subset H^{**}(Y;A)$.*

REMARK. The work of Mayer [16] can be interpreted as a study of *group* homomorphisms $\psi\colon \Omega_*^G(Y) \to \mathrm{Z}$, where $Y = \mathrm{BU}\,(n)$ or $\mathrm{BSO}\,(n)$.

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