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ADDENDUM TO: A FOLIATION OF TEICHMÜLLER SPACE BY TWIST INVARIANT DISKS

ALBERT MARDEN and HOWARD MASUR*

The purpose of this note is to fill a lacuna in our development of the foliation [2] and, at the same time, to exhibit some additional structure of the Teichmüller space T_g with respect to the boundary space $\partial_\gamma T_g$. We will show (Theorem 3.1) that T_g is a trivial principal fibre bundle with base space $\partial_\gamma T_g$ and fibre $D=\{z : |z|<1\}$ which will also be interpreted as a nonabelian group. We will rely extensively on the results and notation of [2].

1. The stabilizer of the boundary space.

1.1. Let $(X,g) \in \partial_\gamma T_g$. The surface X has one or two components and two punctures on X are distinguished, these resulting from the pinching of γ. Associated with these two punctures is a uniquely determined (up to positive multiple) degenerate differential $J_c(X)dq^2$ on X which can be used to represent X as two once punctured disks

$$D(1) = \{0<|\zeta|<1\} \quad \text{and} \quad D(1)' = \{1<|\zeta|<\infty\}$$

with certain identifications involving the unit circle. From this representation of X a Teichmüller disk $D[J]$ can be constructed in T_g, uniquely determined by (X,g), which is "tangent" to $\partial_\gamma T_g$ at (X,g). We will recall further details of this construction in section 1.3 below.

1.2. There is a subgroup $\operatorname{Stab} \partial_\gamma T_g$ of the Teichmüller modular group of T_g that fixes the boundary space $\partial_\gamma T_g$. Let $\operatorname{Fix} \partial_\gamma T_g$ denote the normal subgroup of $\operatorname{Stab} \partial_\gamma T_g$ that fixes $\partial_\gamma T_g$ pointwise. The infinite cyclic group $\{T[\gamma]\}$ generated by the Dehn twist $T[\gamma]$ about γ is a normal subgroup of finite index in $\operatorname{Fix} \partial_\gamma T_g$. In only a few situations does $\operatorname{Fix} \partial_\gamma T_g$ differ from $\{T[\gamma]\}$. One such example is the case that one of the components of X is a once punctured torus. For details concerning these matters we refer to [1].

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Given $(X,g) \in \partial_\gamma T_g$ let Γ^* denote the subgroup of $\operatorname{Stab} \partial_\gamma T_g$ that fixes (X,g). Each $\tau^* \in \Gamma^*$ corresponds to an element $m(\tau^*)$ of the mapping class group of S_0, $(S_0, \mathrm{id.})$ here taken as the origin of T_g. The element $m(\tau^*)$ induces an automorphism (modulo inner automorphisms) of $\pi_1(X)$ which in turn uniquely determines $m(\tau^*)$ up to Dehn twists about $\gamma \subset S_0$. In particular an induced automorphism of $\pi_1(X)$ is inner if and only if $m(\tau^*)$ is a Dehn twist about γ. That τ^* keeps (X,g) fixed means that the action of $m(\tau^*)$ on $\pi_1(X)$ is induced by a conformal automorphism of X. Thus each $\tau^* \in \Gamma^*$ determines (i) a conformal automorphism τ of X which differs from the identity if and only if $\tau^* \notin \{T[\gamma]\}$, and (ii) an automorphism τ of the boundary space $\partial_\gamma T_g$ which differs from the identity if and only if $\tau^* \notin \operatorname{Fix} \partial_\gamma T_g$.

Because of the uniqueness of J_c the action of τ on X induces a conformal automorphism of the pair $(D(1), D(1)')$. Either τ produces a rotation of each factor or it first interchanges and then rotates the factors.

The action of Γ^* on the boundary space $\partial_\gamma T_g$ generates a finite group Γ isomorphic to $\Gamma^*/\operatorname{Fix} \partial_\gamma T_g$.

Let Γ_0^* denote the (finite) subgroup of Γ^* that keeps $D[J]$ pointwise fixed.

1.3. Lemma 1.1. *The subgroup Γ^* of* $\operatorname{Stab} \partial_\gamma T_g$ *preserves $D[J]$ and its restriction there generates an infinite cyclic group isomorphic to Γ^*/Γ_0^*. Its pull-back to D generates a cyclic group of parabolic transformations with fixed point $z=1$.*

Proof. Represent the unit disk D as the right half plane $\{w : \operatorname{Re} w > -1\}$ by means of $w = 2z/(1-z)$. For each $w = u + iv$ the point of $D[J]$ corresponding to w or z is obtained as follows (see [2, § 5.3]). The image of the annulus

$$\{(R^{1+u})^{-\frac{1}{2}} < |\zeta| < 1\} \subset D(1)$$

under the map

$$H_z \colon \zeta \mapsto e^{i\varphi} R^{1+u} \zeta, \qquad \varphi = v \log R$$

is glued to the outer contour of the annulus

$$\{1 < |\zeta| < (R^{1+u})^{\frac{1}{2}}\} \subset D(1)'$$

without further rotation to obtain the annulus

$$A_z = \{1 < |\zeta| < R^{1+u}\}.$$

Thus via $J_c(X)$, for each w, a specific presentation of $\pi_1(X)$ determines a closed Riemann surface together with a presentation of its fundamental group which is uniquely determined up to a power of a Dehn twist about the loop corresponding to the loop in A_z separating ∂A_z. The twist is effected through the construction by the parabolic transformation $w \mapsto w + 2\pi i/\log R$.

Given an element of Γ^* assume first the conformal automorphism τ of X that it determines fixes each distinguished puncture. (The group of such τ is either finite cyclic or a direct product of two such, depending upon whether X has one component or two.) Denote by Ψ, Ψ', $-\pi < \Psi, \Psi' \leqq \pi$, the angles of rotation determined by τ acting in $D(1), D(1)'$ respectively (in each case taken about the origin $\zeta = 0$). Since τ has finite order, these angles are rational multiples of 2π. If $\tau \neq \text{id.}$, it determines a new presentation of $\pi_1(X)$. The effect on the resulting closed surface of first applying τ and then the construction for given w is the same as first performing the construction for w and then twisting A_z by the angle $\Psi - \Psi'$. Thus τ determines an automorphism of $D[J]$ which in view of its particular action on the fundamental group of the surfaces is the restriction of an element of Γ^*. Because every element $\tau^* \in \Gamma^*$ which determines τ corresponds to the same element of the mapping class group of S_0 up to Dehn twists about γ, every such τ^* preserves $D[J]$. The pull-back to $\{\operatorname{Re} w > -1\}$ of such a τ^* is given by a translation

$$w \mapsto w + 2\pi i m/n \log R + 2\pi i k/\log R$$

where n is the order of τ and $\Psi - \Psi' = 2\pi m/n$.

Now suppose τ interchanges the two distinguished punctures of X so that τ interchanges $D(1)$ and $D(1)'$ followed by rotations of angles Ψ, Ψ'. The effect in A_z is a conformal automorphism $\zeta \mapsto R^{1+u}/\zeta$ followed by a twist of angle $\Psi - \Psi'$. It is only the twist that has an effect on the points of $D[J]$.

We conclude that the pull-back to $\{\operatorname{Re} w > -1\}$ of the restriction of Γ^* to $D[J]$ is the group generated by a parabolic transformation

$$w \mapsto w + 2\pi i/n \log R$$

where $2\pi/n$ is the smallest positive value of $\Psi - \Psi'$ coming from the automorphisms τ of X determined by elements of Γ^*.

REMARK. We have shown that the elements of $\operatorname{Fix} \partial_\gamma T_g$ preserve every disk $D[J]$ arising from $\partial_\gamma T_g$.

1.4. Note that if $\Psi' \equiv \Psi \pmod{2\pi}$ there is an extension $\tau^* \in \Gamma^*$ of τ which keeps $D[J]$ pointwise fixed. Such will be the case if, for example, τ has order two in each component of X.

For instance consider the case that $g = 2$ and that γ is a dividing cycle. Then X has two components, each a once-punctured torus. Let τ_i, $i = 1, 2$, be the conformal automorphism of X that keeps one of the components of X pointwise fixed, and on the other is the involution that keeps the puncture fixed. Each τ_i has order two and so does $\sigma = \tau_1\tau_2$. The surfaces corresponding to each point on $\partial_\gamma T_g$ have such conformal automorphisms.

Lemma 1.1 shows that corresponding to each τ_i is an element $\tau_i^* \in \operatorname{Fix} \partial_\gamma T_g$ which fixes every disk $D[J]$ and satisfies $(\tau_i^*)^2 = T[\gamma]$. On the other hand σ has no such property. This is due to the fact that what should be an extension of σ to T_g (from the point of view of its action on the fundamental group) is actually the hyperelliptic involution which is possessed by all surfaces of genus 2. This does not act effectively on T_g.

In what follows we will use the notation Γ^*/Γ_0^* to indicate the effective action of Γ^* on $D[J]$.

2. Local cross sections.

2.1. Given $(X, g) \in \partial_\gamma T_g$ apply the construction of section 1.3 at the point $w = 0$. Especially because of the action on X determined by Γ^* there is certain ambiguity in determining a corresponding point of $D[J]$. What is unambiguously determined by (X, g) is the *origin-orbit* (Γ^*/Γ_0^*) (0^*) where 0^* denotes a specific point in $D[J] \subset T_g$ corresponding to $w = 0$.

2.2. Let U be a neighborhood of 0^* in T_g so small that (i) $A(U) \cap U = \varnothing$ for all elements A of $\operatorname{Stab} \partial_\gamma T_g$ with $A \notin \Gamma_0^*$. We may also assume (ii) that $A(U) = U$ for all $A \in \Gamma_0^*$.

Index the components of $\{\Gamma^*(U)\}$ as $U, U_{-1}, U_1, \ldots, U_{-k}, U_k, \ldots$ where U_k has "center" $A_k(0^*)$ and A_k is the element of Γ^*/Γ_0^* which when pulled back to $\{\operatorname{Re}\omega > -1\}$ is $w \mapsto w + 2\pi k i/n \log R$, n being the order of the finite cyclic group $(\Gamma^*/\Gamma_0^*)/\{T[\gamma]\}$.

2.3. Now choose a neighborhood V of (X, g) in $\partial_\gamma T_g$ so small that (i) $\sigma(V) \cap V = \varnothing$ for the restriction σ to $\partial_\gamma T_g$ of any element of $\operatorname{Stab} \partial_\gamma T_g$ not in Γ. We may also assume (ii) that $\sigma(V) = V$ for all $\sigma \in \Gamma$. To each $(Y, h) \in V$ corresponds a Teichmüller disk D_1 "tangent" at (Y, h): The subgroup of $\operatorname{Stab} \partial_\gamma T_g$ which preserves D_1 is a subgroup of Γ^* which is different from $\operatorname{Fix} \partial_\gamma T_g$ only if (Y, h) is fixed under a non-trivial element of Γ. In any case (Y, h) uniquely determines an *origin-orbit* in D_1. We require (iii) that V be so small that the origin-orbit of (Y, h) lies in $\bigcup U_k$ for all $(Y, h) \in V$.

In each U_k there is at most one point of the origin-orbit of a point in V.

2.4. We want to ensure that every origin-orbit has exactly one point in U. If this is not the case for a point (Y, h) of V we must modify the construction of the origin-orbit for this point. This will be done by performing a shift as follows.

We know that a point of the origin-orbit of (Y, h) lies in U_k for some $0<k<n$. Recall that a point of the origin-orbit of (Y, h) is determined by attaching $\{R^{-\frac{1}{2}}<|\zeta|<1\}$ in $D(1)$ under the map $H_0\colon \zeta \mapsto R\zeta$ to $\{1<|\zeta|<R^{\frac{1}{2}}\}$ in $D(1)'$ and the origin-orbit is obtained by letting the appropriate subgroup of Γ^* act.

Form instead a *new* origin-orbit as follows. Attach $\{R^{-\frac{1}{2}}<|\zeta|<1\}$ to $\{1<|\zeta| <R^{\frac{1}{2}}\}$ but under the map $\zeta \mapsto e^{i\Psi}R\zeta$ where $\Psi=-2\pi k/n$. Take the orbit of this new point in D_1 under the same subgroup of Γ^*.

If applied to (X, g) the new origin-orbit would be the same as the old, but as applied to (Y, h) we obtain the required shift.

2.5. With this shift carried out wherever necessary in V define the function $\mathscr{C}\colon V \mapsto U$ by setting $\mathscr{C}((Y, h))$ equal to that point in the origin-orbit of (Y, h) that lies in U.

LEMMA 2.1. *$\mathscr{C}$ is continuous in terms of the Teichmüller metrics on $\partial_\gamma T_g$ and T_g.*

PROOF. Assume $(Y_m h_m) \to (Y, h)$ in V. We may assume that for some $0 \leqq k<n$, a point of the origin-orbit for (Y_m, h_m) lies in U_k for all m. By continuity of J_c on $\partial_\gamma T_g$, a point of the origin-orbit of (Y, h) must also lie in U_k and be the limit of these other points. But now the shift simultaneously moves all these points from U_k up to U.

2.6. As in [2, § 5.3] it follows easily from Lemma 2.1 that there is a natural homeomorphism of $V \times D$ onto the neighborhood of 0^* in T_g consisting of the union of all those Teichmüller disks $D[J]$ which are "tangent" to a point of V.

2.7. Consider the situation now that cross sections $\mathscr{C}_1, \mathscr{C}_2$ have been constructed as above for neighborhoods V_1, V_2 in $\partial_\gamma T_g$ with $V_1 \cap V_2 \neq \varnothing$.

LEMMA 2.2. *For each point $P=(Y, h) \in V_1 \cap V_2$ there exists a biholomorphic automorphism A_P of the Teichmüller disk corresponding to P which varies continuously with P and satisfies*

$$\mathscr{C}_2(P) = A_P \circ \mathscr{C}_1(P).$$

The pull-back to $\{\operatorname{Re} w>-1\}$ of A_P is a parabolic transformation $w \mapsto w+a(P)2\pi i$ where $a(P) \in \mathbf{R}$ depends continuously on P.

PROOF. Given a component of $V_1 \cap V_2$ and a point P in it our construction of $\mathscr{C}_1$ and $\mathscr{C}_2$ shows that $\mathscr{C}_1(P)$ and $\mathscr{C}_2(P)$ are related by such an A_P. That A_P depends continuously on P follows from the fact that $\mathscr{C}_1$ and $\mathscr{C}_2$ do.

3. Construction of the principal bundle.

3.1. It is convenient to represent the open unit disc D as the right half plane $\{\operatorname{Re} w > -1\}$ by means of the transformation $w = 2z/(1-z)$ which sends $z=1$ to $w = +\infty$. Via this transformation we will interpret D as a topological group as follows.

For $w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$ in this half plane define

$$w_1 \cdot w_2 = u_1 + u_2 + u_1 u_2 + i(v_1 + v_2 + u_1 v_2) .$$

With this operation, D is a non-abelian group with $w=0$ serving as the identity element and the inverse $w^{-1} = -w/(1+u)$.

This group operation arises from the formula for composition of maps

$$f_1(z) = z|z|^{w_1}, \quad f_2(z) = z|z|^{w_2}$$

yielding

$$f_2 \circ f_1(z) = z|z|^{w_1 \cdot w_2} .$$

The numbers w with $\operatorname{Re} w = 0$ form an abelian subgroup G_0 isomorphic to the additive group of real numbers.

The translation $w \mapsto w + ia$, $a \in \mathbb{R}$, appears in G as left multiplication $w \mapsto ia \cdot w$.

3.2. THEOREM 3.1. *T_g is a principal $G \equiv D$ bundle over $\partial_\gamma T_g$ with projection determined by the decomposition of T_g by Teichmüller disks "tangent" to $\partial_\gamma T_g$. It is equivalent to the principal product G-bundle $\partial_\gamma T_g \times D$.*

PROOF. The group $G \equiv D$ acts on T_g as a group of homeomorphisms preserving the Teichmüller disk corresponding to each point of $\partial_\gamma T_g$. This action is as follows. To each point $(S, f) \in T_g$ corresponds a Jenkins differential φ on S determined by $f(\gamma)$. Let $z \in D$. Define the action

$$z \cdot (S, f) = (f_z(S), f_z \circ f)$$

where f_z is the extremal Teichmüller map of S with complex dilation $-z\bar{\varphi}/|\varphi|$. G acts freely on T_g. Together with Lemmas 2.1 and 2.2 this proves the first statement of the Theorem.

Note that T_g is in fact a fibre bundle with respect to the subgroup G_0.

The second statement follows either from the fact that $\partial_\gamma T_g$ is contractible [3, Corollary 11.6] or from the fact that G is contractible [3, Corollary 12.3].

3.3. A consequence [3, Theorem 12.2] is that any cross section into T_g defined on a closed subset of $\partial_\gamma T_g$ can be extended to a global cross-section. In particular, the locally defined section $\mathscr{C}$ in section 2.5 can be so extended. This is what was needed in [2] and completes the proof of that Theorem. The proof given there is incomplete because of our failure to take account of the full group Γ^*; only $\{T[\gamma]\}$ was considered.

The argument presented above works just as well for the more general Teichmüller space $T(g,n)$. As a consequence, T_g can be parameterized as described in [2, § 5.4].

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