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WARD'S STAUDT–CLAUSEN PROBLEM

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1. Introduction.

Morgan Ward [10] once posed the problem whether a suitable definition for Bernoulli numbers could be framed so that a generalized Staudt–Clausen theorem existed for them within the framework of Jackson's calculus [6].

Ward himself generalized Jackson's calculus of sequences and it is in terms of the more general Ward–Jackson calculus that we offer a solution to Ward's problem. We define our Bernoulli numbers in equation (3.1) and we enunciate types of Staudt–Clausen theorems for them in sections 5 and 7 by suitable adaptations of methods of Carlitz [1] and Rado [9].

Another approach which provides a unique and generalized form of von Staudt's theorem may be found in Kazandzidis [7].

Gould [2] elegantly extended the work of Ward, whose generalized coefficients were rediscoveries of work of Fontené; (see Gould). Following Gould we define Fontené–Ward binomial coefficients by

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{u_n!}{u_k!\, u_{n-k}!}$$

where $\{u_n\}$ is an arbitrary sequence of real or complex numbers such that $u_n \neq 0$ for $n > 1$, $u_0 = 0$, $u_1 = 1$, and $u_n! = u_n u_{n-1} \ldots u_1$ with $u_0! = 1$.

When $\{u_n\} \equiv \{n\}$, the non-negative integers, we get

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \binom{n}{k},$$

the ordinary binomial coefficient. When $\{u_n\} \equiv \{\underline{q}_n\}$, the Fermatian numbers (Dickson [2]), defined by

$$\underline{q}_n = 1 + q + \ldots + q^{n-1} \qquad (n > 0)$$

$$\underline{q}_0 = 0,$$

where q may be indeterminate, $\underline{q}_1 = 1$, and we get

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix},$$

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the well-known q-binomial coefficient. (Note that $\underline{l}_n = n$.)

Another example is given by the fundamental linear recursive sequence of order r, $\{u_n^{(r)}\}$, defined by

$$\begin{aligned} u_n^{(r)} &= 0 \qquad (n \leqq 0) \\ &= 1 \qquad (n = 1) \\ &= \sum_{j=1}^{r} (-1)^{j+1} P_{rj} u_{n-j}^{(r)} \qquad (n \geqq 2) \end{aligned}$$

(in which the P_{rj} are arbitrary integers). This satisfies the criteria to be a $\{u_n\}$ sequence. Note that when $r=2$ we get the Lucas fundamental numbers [8], and the Fontené–Ward coefficients become the Fibonacci binomial coefficients of Hoggatt [4].

2. Ward–Jackson calculus.

We now set out those salient features of the Ward–Jackson calculus of sequences which we shall need. Jackson developed similar results for $\underline{q}_n$ and Ward extended Jackson's work for the more general u_n. Throughout the rest of this paper we shall consider $\{u_n\}$ to be a sequence of integers for $n = 0, 1, \ldots$, and later we shall impose further restrictions.

We define exponents by means of

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{\Sigma s = n} \frac{u_n!}{u_{s_1}! \ldots u_{s_k}!} x_1^{s_1} \ldots x_k^{s_k} .$$

Note that when $x_1 = x_2 = \ldots = x_k = 1$, we get

$$k^n = \sum_{\Sigma s = n} \frac{u_n!}{u_{s_1}! \ldots u_{s_k}!} ,$$

and when $n = 1$, $k = 2$, we get

$$(x_1 + x_2)^1 = x_1 + x_2 ,$$

so that

$$(x_1 + x_2 + \ldots + x_{k-1} + x_k)^n = (x_1 + x_2 + \ldots + (x_{k-1} + x_k))^n .$$

If $F(x)$ denotes the formal power series

$$F(x) = \sum_{n=0}^{\infty} c_n x^n ,$$

we define $F(x+y)$ to mean the series

$$\sum_{n=0}^{\infty} c_n(x+y)^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^{n-m} y^m .$$

We next assume that the sequence $\{u_n\}$ is chosen in such a way that

$$E(x) = \sum_{n=0}^{\infty} x^n/u_n!$$

is convergent in the neighbourhood of $x=0$. It accordingly is an element of an analytic function of x which Ward called the basic exponential. There exists then a positive number ϱ such that the basic exponential series converges absolutely within the circle $|x|=\varrho$. The basic exponential has the following most important property for sufficiently small absolute values of its arguments x_i:

$$E(x_1+x_2+\ldots+x_k) = E(x_1)E(x_2)\ldots E(x_k) .$$

We also define an operator D_x which transforms the power series $F(x)$ into

$$D_x F(x) = \sum_{n=1}^{\infty} u_n c_n x^{n-1} ,$$

with function of function rules

$$D_x F^m(x) = u_m F^{m-1}(x) D_x F(x)$$

and

$$D_x y D_y x = 1 .$$

In particular then

$$D_x x^n = u_n x^{n-1} .$$

The operator D_x is easily shown to be linear and distributive, and it converts a polynomial of degree n in x into one of degree $n-1$. Associated with D_x we define an inverse operator I_x in a form convenient for this paper:

$$f(t) = |I_x D_x f(x)|_0^t .$$

When $n \neq -1$,

$$I_x x^n = x^{n+1}/u_{n+1} + C ,$$

where C is independent of x.

3. Divisibility sequences.

A sequence of integers $\{u_n\}$, $n=1,2,\ldots$, is said to be a divisibility sequence if $u_s \mid u_t$ whenever $s \mid t$. The properties of divisibility sequences have been examined by Ward [10] and others (mentioned by Williams [12]). For example, the sequence of Fermatian numbers $\{q_n\}$ is a divisibility sequence since if $s|t$ then $q_s|q_t$. Another example of a divisibility sequence is the sequence of Fibonacci numbers $\{F_n\}$ defined by the recurrence relation $F_n = F_{n-1}+F_{n-2}$, $n>2$, with initial terms $F_1=F_2=1$.

Specifically, Ward has proved two theorems related to what he calls Properties A and B. A divisibility sequence is said to have property A provided that:

A: if $C=(a,b)$, then $u_c=(u_a, u_b)$, for every pair of terms u_a, u_b of $\{u_n\}$.

It is said to have property B if:

B: for every prime divisor p and every positive integer a, $u_m \equiv 0 \pmod{p^a}$ when and only $m\equiv 0 \pmod{z}$, where z is the rank of apparition of p^a in $\{u_n\}$.

The two theorems of Ward referred to above are:

THEOREM A: *Property* A *and Property* B *are equivalent to one another.*

THEOREM B: *The binomial coefficients belonging to every divisibility sequence having Property* A *or Property* B *are all integers.*

We now define generalized Bernoulli numbers, B_n, by

$$\frac{t}{E(t)-1} = \sum_{n=0}^{\infty} B_n t^n/u_n! \tag{3.1}$$

where $\{u_n\}$ is a divisibility sequence with Property A. (This condition on the $\{u_n\}$ will apply in the rest of this paper.)

4. Generalized Hurwitz series.

We now define some generalized Hurwitz series. Ordinary Hurwitz series have the form $\sum_{n=0}^{\infty} a_n t^n/n!$ where the a_n are integers; the ordinary exponential series is an example of a Hurwitz series. We shall call a series of the form

$$\sum_{n=0}^{\infty} a_n t^n/u_n! \tag{4.1}$$

where the a_n are arbitrary integers, a generalized Hurwitz series (GH-series). The Cauchy product of (4.1) and another GH-series $\sum_{n=0}^{\infty} b_n t^n/u_n!$ is also a GH-series

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left\{{n \atop m}\right\}a_m b_{n-m}\right)t^n/u_n!$$

since the $\{{n \atop m}\}$ are integral for divisibility sequences with Property A. (This is, of course, a special use of the term "product" and is discussed by Ward [10].)

GH-series are changed into other GH-series by the operators D_x and I_x:

$$D_x \sum_{n=0}^{\infty} a_n x^n/u_n! = \sum_{n=0}^{\infty} a_{n+1} x^n/u_n!\ ,$$

$$I_x \left| \sum_{n=0}^{\infty} a_n x^n/u_n! \right|_0^t = \sum_{n=1}^{\infty} a_{n-1} t^n/u_n!\ .$$

For a series without constant term

$$H_1(t) = \sum_{n=1}^{\infty} a_n t^n/u_n!\ ,$$

it follows from the function of a function rule that

$$D_x H_1^k(x) = u_k H_1^{k-1}(x) D_x H_1(x)\ .$$

Then

$$H_1^k(t) = |I_x D_x H_1^k(x)|_0^t$$
$$= |I_x u_k H_1^{k-1}(x) D_x H_1(x)|_0^t$$

and

$$\frac{1}{u_k!} H_1^k(t) = \left| I_x \frac{H_1^{k-1}(x)}{u_{k-1}!} D_x H_1(x) \right|_0^t .$$

So, by induction on k, we can prove that

$$\frac{1}{u_k!} H_1^k(t) \tag{4.2}$$

is a GH-series for all $k \geqq 1$.

It is important to note that by the statement

$$\sum_{n=0}^{\infty} a_n t^n/u_n! \equiv \sum_{n=0}^{\infty} b_n t^n/u_n!\ (\operatorname{mod} m)$$

we mean that the system of congruences

$$a_n \equiv b_n \ (\operatorname{mod} m) \qquad n=0,1,\dots\ ,$$

is satisfied. This is equivalent to the assertion

$$\sum_{n=0}^{\infty} a_n t^n/u_n! = \sum_{n=0}^{\infty} b_n t^n/u_n! + mH(t)$$

where $H(t)$ is some GH-series. Thus the result concerning (4.2) can be stated in the form

(4.3) $$H_1^k(t) \equiv 0 \pmod{u_k!}$$

provided $H_1(0)=0$.

Accordingly, we now define the GH-series

(4.4) $$f(t) = \sum_{n=1}^{\infty} t^n/u_n!$$

so that

$$f(t) = E(t)-1 \quad \text{with } f(0) = 0\,.$$

If we consider the formal inverse function $f^{-1}(t)$, then

$$\begin{aligned} E(f^{-1}(t)) &= 1+f(f^{-1}(t)) \\ &= 1+t; \end{aligned}$$

so

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} f^n(t)/u_n\,.$$

5. Staudt–Clausen Theorem.

LEMMA.

$$f^{p-1}(t) \equiv -\sum_{n=1}^{\infty} t^{n(p-1)}/u_{n(p-1)}! \pmod p \qquad \text{for prime } p\,.$$

PROOF.

$$\begin{aligned} f(t) &= E(t)-1 \\ f^{p-1}(t) &= \sum_{j=0}^{p-1} (-1)^j \begin{Bmatrix} p-1 \\ j \end{Bmatrix} E((p-j-1)t) \\ &= (-1)^{p-1} + \sum_{j=0}^{p-2} (-1)^j \begin{Bmatrix} p-1 \\ j \end{Bmatrix} E((p-j-1)t) \\ &= 1+\sum_{j=0}^{p-2} (-1)^j \begin{Bmatrix} p-1 \\ j \end{Bmatrix} E(p-j-1)t) \qquad \text{for } p>2\,. \\ D_t^{p-1} f^{p-1}(t) &= \sum_{j=0}^{p-2} (-1)^j u_{p-j-1}^{p-1} \begin{Bmatrix} p-1 \\ j \end{Bmatrix} E((p-j-1)t) \end{aligned}$$

$$\equiv \sum_{j=0}^{p-2} (-1)^j \left\{ \begin{matrix} p-1 \\ j \end{matrix} \right\} E((p-j-1)t) \quad (\operatorname{mod} p)$$

$$\equiv -1 + f^{p-1}(t) \ (\operatorname{mod} p)$$

since $u_{p-j-1}^{p-1} \equiv 1 \ (\operatorname{mod} p)$, $j = 0, 1, \ldots, p-2$ (from Fermat's theorem), $(u_j, p) = 1$.

A solution of this differential congruence is given by

$$f^{p-1}(t) \equiv \sum_{n=1}^{\infty} t^{n(p-1)}/u_{n(p-1)}! \quad (\operatorname{mod} p) .$$

This can be verified as follows:

$$D_t^{p-1}\left(-\sum_{n=1}^{\infty} t^{n(p-1)}/u_{n(p-1)}! \right) = -\sum_{n=1}^{\infty} t^{(n-1)(p-1)}/u_{(n-1)(p-1)}!$$

$$= -\sum_{n=0}^{\infty} t^{n(p-1)}/u_{n(p-1)}!$$

$$= -1 + \left(-\sum_{n=1}^{\infty} t^{n(p-1)}/u_{n(p-1)}! \right),$$

which is what we seek. When $p = 2$,

$$f^{p-1}(t) = \bar{f}(t) = \sum_{n=1}^{\infty} t^n/u_n!$$

$$\equiv -\sum_{n=1}^{\infty} t^n/u_n! \quad (\operatorname{mod} 2) .$$

We now introduce the arithmetical function $\delta(m, s)$ defined by

$$\delta(m, s) = \begin{cases} 1 & \text{if } m \mid s , \\ 0 & \text{if } m \nmid s . \end{cases}$$

$$\sum_{n=0}^{\infty} B_n f(t) t^n / u_n = t$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} f^n(t)/u_n$$

$$= \sum_{n=0}^{\infty} (-1)^n f^{n+1}(t)/u_{n+1} .$$

Since the coefficients of $f^n(t)$ are multiples of $u_n!$ from result (4.3), if $\delta(u_{n+1}, u_n!) = 1$, then u_{n+1} divides $u_n!$ divides the coefficients of $f^n(t)$. Let $H(t)$ be a GH-series. We then get

THEOREM 1.

$$\sum_{n=0}^{\infty} B_n f(t) \frac{t^n}{u_n!} = H(t) + \sum_{n=1}^{\infty} (1 - \delta(u_n, u_{n+1}!))$$

in which

$$f^{p-1}(t) \equiv -\sum_{n=1}^{\infty} t^{n(p-1)}/u_{n(p-1)}! \pmod p .$$

That the theorem is a generalization of the ordinary Staudt-Clausen theorem can be seen if we let $\{u_n\} = \{n\}$: then since

$$\sum_{n=0}^{\infty} B_n t^n/n! = H(t) + \sum_{p} (-f(t))^{p-1}/p$$

and

$$f^{p-1}(t) \equiv -\sum_{n=1}^{\infty} t^{n(p-1)}/n(p-1)! \pmod p ,$$

we get

$$pB_{n(p-1)} \equiv (-1)^p \pmod p$$

which is a form of the ordinary Staudt–Clausen theorem. The generalization follows since $\delta(n, (n-1)!) = 0$ for all ordinary composite $n > 4$, and $\delta(p, (p-1)!) = 1$ for all ordinary primes p.

6. Examples.

We shall illustrate the result for the Fibonacci numbers.

We set $B_0 = 1$ and $B_{2n+1} = 0$, $n \geqq 1$, as with ordinary Bernoulli numbers and we establish that

$$B_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} B_{n-k} . \tag{6.1}$$

PROOF. Since

$$E(t)E(-t) = E(t-t) = E(0) = 1 ,$$

we have that

$$t(1-E(t)) = (-t)(E(t)-1)$$

and

$$t(1-E(t)) = tE(t)(E(-t)-1)$$

so that

$$\frac{tE(t)}{E(t)-1} = \frac{(-t)}{E(-t)-1}$$
$$= \sum_{n=0}^{\infty} B_n(-t)^n/u_n! \,.$$

But

$$\frac{tE(t)}{E(t)-1} = E(t)\sum_{n=0}^{\infty} B_n t^n/u_n!$$
$$= \sum_{m=0}^{\infty} t^m/u_m! \sum_{n=0}^{\infty} B_n t^n/u_n!$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} B_{n-k} t^n/u_n!$$

and so

$$B_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} B_{n-k}$$

since $B_{2n+1}=0$ for $n \geqq 1$. Thus

$$B_0 = 1,\ B_1 = \frac{-1}{u_2},\ B_2 = \frac{1}{u_2}-\frac{1}{u_3},\ B_4 = \frac{1}{u_2}-\frac{1}{u_5}-\frac{u_4}{u_2}\left(\frac{1}{u_2}-\frac{1}{u_3}\right).$$

When $\{u_n\}=\{n\}$, these give the known results for the ordinary Bernoulli numbers.

Consider $\{u_n\}=\{F_n\}$:

$$B_1 = -1,\ B_2 = 1-\tfrac{1}{2},\ B_4 = 1-\tfrac{1}{5}-3(1-\tfrac{1}{2})\,.$$
$$\delta(F_2, F_1!) = 1,\ \delta(F_3, F_2!) = 0,\ \delta(F_5, F_4!) = 0;$$
$$F_3B_2 \equiv -1 \pmod{F_3};\ F_3B_4 \equiv -1 \pmod{F_3};\ F_5B_4 \equiv -1 \pmod{F_5}\,.$$

7. Generalized Euler–Maclaurin Formula.

Another approach can be made using a generalization of the Euler–Maclaurin sum which is an important use of the ordinary Bernoulli numbers. The generalization in question is

$$\sum_{j=1}^{n-1} j^k = \sum_{j=0}^{k} \frac{n^{k+1-j}}{u_{k+1-j}} \begin{Bmatrix} k \\ j \end{Bmatrix} B_j\,.$$

PROOF.

$$u_k!\,x \sum_{j=0}^{n-1} E(jx) = u_k!\,x \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} \frac{j^i x^i}{u_i!}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \frac{u_k!}{u_i!} j^i x^{i+1}$$

and the coefficient of x^{k+1} is $\sum_{j=0}^{n-1} j^k$ on the right hand side;

$$u_k!\,x \sum_{j=0}^{n-1} E(jx) = u_k! \frac{x}{E(x)-1}(E(nx)-1)$$

$$= u_k! \sum_{j=0}^{\infty} B_j \frac{x^j}{u_j!} \sum_{i=0}^{\infty} n^{i+1} \frac{x^{i+1}}{u_{i+1}!}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{n^{i-j+1} u_k!}{u_j!(u_{i-j+1})!} B_j x^{i+1}$$

and the coefficients of x^{k+1} is

$$\sum_{j=0}^{k} \frac{n^{k-j+1}}{u_{k-j+1}} \begin{Bmatrix} k \\ j \end{Bmatrix} B_j$$

as required.

The proof of the next result parallels Rado [9]:

$$\sum_{j=1}^{p-1} j^k \equiv -\delta(p-1,k) \pmod p .$$

PROOF. If $\delta(p-1,k)=1$, by Fermat's theorem then

$$j^k \equiv 1 \pmod p .$$

So

$$\sum_{j=1}^{p-1} j^k \equiv p-1 \pmod p$$

$$\equiv -\delta(p-1,k) \pmod p .$$

If $\delta(p-1,k) \neq 1$, and g is a primitive root of k, then

$$\sum_{j=1}^{p-1} (jg)^k \equiv \sum_{j=1}^{p-1} j^k \pmod p$$

or

$$(g^k-1) \sum_{j=1}^{p-1} j^k \equiv 0 \pmod p .$$

But $g^k \not\equiv 1 \pmod p$.

THEOREM 2. *For any prime p,*

$$pB_{2n} \equiv -\delta(p-1,2n) - \sum_{j=0}^{2n-1} \frac{p^{2n+1-j}}{u_{2n+1-j}} \left\{ {2n \atop j} \right\} B_j \pmod{p}.$$

PROOF. From the previous results in this section we have that

$$\delta(p-1,k) + \sum_{j=0}^{k} \frac{p^{k+1-j}}{u_{k+1-j}} \left\{ {k \atop j} \right\} B_j \equiv 0 \pmod{p}$$

or

$$pB_{2n} + \delta(p-1,2n) + \sum_{j=0}^{2n-1} \frac{p^{2n+1-j}}{u_{2n+1-j}} \left\{ {2n \atop j} \right\} B_j \equiv 0 \pmod{p}$$

since $u_1 = 1$. This gives the required result.

When $\{u_n\} = \{n\}$, this result becomes

$$pB_{2n} \equiv -\delta(p-1,2n) \pmod{p}$$

which is another form of the ordinary Staudt–Clausen theorem. As an example, consider the Fibonacci numbers:

$$n = 1, \quad p = 2, \quad pB_{2n} = 2B_2 = 1 \equiv -1 \pmod{2},$$

and

$$-\delta(1,2) \dot{-} \sum_{j=0}^{1} \frac{2^{3-j}}{F_{3-j}} \left\{ {2 \atop j} \right\} B_j \equiv -1 - \frac{8}{2} B_0 + 4B,$$

$$\equiv -1 \pmod{2};$$

$$n = 2, \quad p = 2, \quad pB_{2n} = 2B_4 \equiv -3 \equiv -1 \pmod{2};$$

and

$$-\delta(1,4) - \sum_{j=0}^{3} \frac{2^{5-j}}{F_{5-j}} \left\{ {4 \atop j} \right\} B_j = -1 - \frac{32}{5} + \frac{16}{3} \times 3 - \frac{8}{2} \times 3 \times \frac{2}{2} - 2 \times \frac{4}{5}$$

$$\equiv -1 \pmod{2}.$$

To specify the result any further, two conditions would be needed:

$$u_n < 2^{n-1}$$

which is satisfied by the Fibonacci numbers and ordinary integers; and

$$2B_n \equiv -1 \pmod{2}$$

which is obtained with the ordinary integers but perhaps not with any other sequence.

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