MATH. SCAND. 39 (1976), 267–270

ON A FOUR-COLOUR THEOREM

HELGE TVERBERG

In 1970 Barnette and Stein [1] proved the following weaker form of the four-colour conjecture.

THEOREM. *Any map on the sphere can be coloured with four colours in such a way that each country has the same colour as at most one of its neighbours.*

The purpose of this note is to give an easy proof of this interesting theorem (while it lasts). The original proof is not very long but it employs a quite difficult theorem by Tutte [3] (also proved in [2, pp. 68–74]). Our proof is very similar to the standard proof of the 5-colour theorem (cf. [2, pp. 84–87]) and the exposition is completely self-contained, modulo general knowledge.

In section 1 we normalize the problem and prove the well known relation

$$12a_0+4a_2+3a_3+2a_4+a_5 = 12+a_7+2a_8+\ldots, \tag{1}$$

(where a_p is the number of p-gons, i.e. countries having p neighbours) which holds for the normalized maps. From (1) we deduce a result due to Wernicke [4] (cf. [2, p. 120]): if $a_0+a_2+a_3+a_4=0$ then the map has a 5-gon with a 5-gon or a 6-gon among its neighbours. This result is the basis for the proof in section 2.

1. Definitions and preliminaries.

A *map* M is described by a finite graph G, embedded in the sphere. A *country* is a component of the complement of G (in the sphere), and two countries are *neighbours* if they have at least one boundary edge in common. A *normal* map M satisfies the conditions:

(a) G is connected.
(b) G is trivalent.
(c) Every edge of G is a boundary edge of two different countries.
(d) Every pair of neighbours share exactly one boundary edge.

Received November 6, 1974; in revised form January 5, 1976.

It is well known that normal maps (with $a_0 = a_2 = 0$) are the same as the maps obtained from simple polyhedra, countries being facets, but we shall not make any use of this property. Another well known property of such maps, which *will* be made use of, is the following: It is always possible to choose an edge ε so hat the map M_ε obtained by deleting ε is also normal. In fact, each country has at least one such "good" edge ε in its boundary. Let the edges of the country be $\varepsilon_1, \ldots, \varepsilon_s$, in clockwise order, with the corresponding neighbours being $N_1, \ldots, N_s$. If ε_1 is "bad", then N_1 has N_r as a neighbour for some r with $2 < r < s$, so that (d) is violated for M_{ε_1}. We may assume the numbering chosen so that r is as small as possible. Then (by Jordan's curve theorem) none of the edges $\varepsilon_2, \ldots, \varepsilon_{r-1}$ can be "bad".

Consider now *any* map M. We shall transform it into a normal map M', the colourability of which will imply that of M.

If G is disconnected, then M has a country P, with vertices from different components of G in its boundary. Connect two such vertices by a new edge running through P. Clearly this edge does not disconnect P. Iterating the procedure, if necessary, we end up with a map satisfying (a).

Now consider (b). First delete every vertex of degree zero.

Then draw a small circle around every vertex of G, not of degree three, and delete what is "inside" the circles. Now (b) will be satisfied.

Consider now an edge which is unwanted by (c), i.e. it has the same country on both sides. We surround it by a Jordan curve (including only two vertices, of course), and delete what is "inside" the curve. Having eliminated such edges one by one, we may assume also (c).

Any edge unwanted by (d) can now be eliminated by the procedure just described. (The reader will easily see that the new country obtained has four neighbours, and not just three, as one might fear.) Thus also (d) can be assumed after elimination of enough edges.

The proof that any normal map M satisfies (1) is carried out by induction on n, the number of countries of M. If $n \leqq 3$, (1) clearly holds, and if $n \geqq 3$, (1) is equivalent to

$$\sum (p_i - 6) = -12 \tag{2}$$

(the ith country being a p_i-gon).

It now suffices to verify that the left hand side of (2) does not change when a "good" edge is deleted. What happens is that the terms $p_1 - 6, \ldots, p_4 - 6$ are replaced by $p_1 + p_2 - 10, p_3 - 7, p_4 - 7$ (in suitable notation).

Wernicke's result is proved as follows. Assume its falsity for the map M. Then there are $5a_5$ neighbourpairs consisting of a 5-gon and a ($\geqq 7$)-gon. As a

p-gon now has at most $[p/2]$ 5-gons among its neighbours, we thus have

$$5a_5 \leqq 3a_7+4a_8+4a_9+5a_{10}+\dots$$

which contradicts (1), as $a_0=a_2=a_3=a_4=0$.

2. Proof of the Theorem.

The proof (for normal maps, which suffices) is carried out by induction on the number of countries. Consider a normal map M with n countries, and assume that the theorem holds for all normal maps with $n-1$ countries. We may assume that $n \geqq 9$. Let X be any country of M. Delete a "good" boundary edge ε of X. The map M_ε can be coloured as desired, by the hypothesis of induction. This gives a partial colouring of M, with X left uncoloured.

If X is a 3-gon, we can obviously colour X, too. If X is a 4-gon, we may assume that its neighbours, Q, R, S, T, in clockwise order are coloured by A, B, C, D (the available colours) respectively. Consider the union of the closures of the countries coloured A or C. One component of this set contains Q, and is called the AC-chain of Q. If we interchange A and C everywhere on the AC-chain of Q, we get a new partial colouring of M, with only X uncoloured. If the chain does not include S, then only B, C, D are used around X, so that X may be coloured A. On the other hand, if the chain includes S, then the BD-chain of R does not include T, and the argument above can be repeated.

We may thus assume that M has no ($\leqq 4$)-gon. Then Wernicke's result applies and we choose X as a 5-gon, having a ($\leqq 6$)-gon among its neighbours. Let P, Q, R, S, T be the neighbours of X, in clockwise order. We may assume, as above, that all countries of M, except X, have been coloured, and also that the colour sequence around X is either A, A, B, C, D or A, B, A, C, D.

In the former case, read once more the proof given in the case when X is a 4-gon, (start with: "If we . . .").

In the latter case, with colour sequence A, B, A, C, D consider the BC-chain of Q. If it does not include S, we are clearly all right. If it includes S then, together with X, it separates R from P and T, and so the colour sequence can be changed to A, B, D, C, D. After a permutation this becomes C, D, A, B, A; thus the colour pattern around X has rotated by two units in clockwise direction. Repeating the argument, we either finish the proof or obtain a further rotation by two units. After at most two more repetitions we have thus obtained that the colour B is assigned to a 5-gon or a 6-gon, i.e. we may assume that P is a 5-gon or a 6-gon, and that the colour sequence is B, A, C, D, A.

Before proceeding further, we transfer B from P to X, so that from now on P is the only uncoloured country.

Assume first P to be a 5-gon. Three of its neighbours are Q, X and T, with colours A, B and A, respectively. Any colour, except A, not used on the two remaining neighbours, can now be assigned to P.

Let now P be a 6-gon, with neighbours U, V, Q, X, T and W, in clockwise order. If B is not used for U, V or W, it can be used for P. This remark applies also to C and D, so we assume that B, C and D are used for U, V, and W in some order. Let E denote the colour used for U, and consider the EA-chain of U.

Assume first that this chain excludes Q and T (and hence R, S and X, too). As it also excludes V and W, we can clearly change the colour of U to A, while conserving the situation otherwise. This case has been dealt with already.

Assume now that the EA-chain of U includes Q, say. Then it separates V from W, (in the complement of P) so that the colour of V can be changed into that of W, without any change at Q, R, S, T, U, W or X. This brings us back to an earlier case, and the proof is finished.

ADDED IN PROOF. I am grateful to the referee for pointing out an error and some obscurities in the original manuscript. He also showed that for any given country X in the map M, there is a colouring of M in which X is coloured differently from all its neighbours. (Distort M so that X covers the northern hemisphere and contains the "equator". For each country $Y \neq X$ introduce a new country Y', the reflection of Y in the "equatorial plane", and apply the theorem to the map thus obtained).

Quite recently Appel and Haken has announced the solution of the four-colour problem itself (cf. Bull. Amer. Math. Soc. 82 (1976), 711–712).

REFERENCES

1. D. V. Barnette and S. K. Stein, *On a theorem of Tutte, a 4-colour theorem and B-sets*, in *Recent Trends in Graph theory* (Lecture Notes in Mathematics 186), Springer-Verlag, Berlin, Heidelberg, New York, 1971.
2. Ø. Ore, *The four-colour problem*, Academic Press, New York and London 1967.
3. W. T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. 82 (1956), 99–116.
4. P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. 58 (1904), 413–426.

UNIVERSITY OF BERGEN
NORWAY