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ON SPECTRAL SYNTHESIS FOR CURVES IN R^3

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1.

$A(\mathsf{R}^n)$ and $PM(\mathsf{R}^n)$ denote the Banach spaces of Fourier transforms of functions in $L^1(\mathsf{R}^n)$ and $L^\infty(\mathsf{R}^n)$, respectively. Thus $PM(\mathsf{R}^n)$ is the dual of $A(\mathsf{R}^n)$. The elements in $PM(\mathsf{R}^n)$ are defined in distribution sense and are called pseudomeasures. For every closed $E \subset \mathsf{R}^n$, $PM(E)$ denotes the (weakly* closed) subspace of $PM(\mathsf{R}^n)$ consisting of all pseudomeasures with support in E, and $M(E)$ is the subspace of $PM(E)$ consisting of all bounded regular Borel measures with support in E. E is said to be of *sequential spectral synthesis* if $PM(E)$ is the sequential weak* closure of $M(E)$.

The image of an injective C^k mapping $\gamma : [a,b] \to \mathsf{R}^n$, where $k \geqq 1$ and $-\infty < a < b < \infty$, with γ' non-vanishing, is called a *simple C^k curve* in R^n. The results in [3] imply that a simple C^2 curve in R^2, with non-vanishing curvature, is a set of sequential spectral synthesis. We shall now prove:

THEOREM. *A simple C^3 curve in R^3, with non-vanishing torsion, is a set of sequential spectral synthesis.*

It should be pointed out that the method in [3] can be adopted to give rather general results on an extended notion of sequential spectral synthesis for $(n-1)$-dimensional smooth manifolds in R^n, $n \geqq 3$ (see [4]), but that the corresponding method fails in general if the codimension of the manifold is $\geqq 2$ (R. Gustavsson [5] has shown this for curves in R^3). Thus a new idea has been necessary in the proof of our theorem. It is to be expected that the same approach can be used to prove corresponding results for curves in R^n, $n \geqq 4$.

2.

$B(\mathsf{R}^n)$ denotes the Banach space of Fourier–Stieltjes transforms of elements in $M(\mathsf{R}^n)$. Let $E \subset \mathsf{R}^n$ be closed and $f \in C(E)$. We put

$$\|f\|^1_{B(E)} = \inf\{\|F\|_{B(\mathsf{R}^n)} : \ F \in B(\mathsf{R}^n) \cap C^1(\mathsf{R}^n), F|_E = f\} .$$

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If the set to the right is empty, we put $\|f\|^1_{B(E)}=\infty$. The novelty in the new method, in comparison with the method in [3], consists in a refined technique to estimate $\|f\|^1_{B(E)}$, for a special family of functions f on a family of curves $E \subset \mathbf{R}^2$. The main tools in the estimating are the following two elementary lemmas.

LEMMA 1. *There is a constant C such that every $f \in C^1(I)$, where $I \subset \mathbf{R}$ is a compact interval of length $|I|$, satisfies*

$$\|f\|^1_{B(I)} \leqq C(\|f\|_{L^\infty(I)} + (|I|\,\|f\|_{L^\infty(I)}\|f'\|_{L^\infty(I)})^{\frac{1}{2}})\,. \tag{2.1}$$

PROOF OF LEMMA 1. Since the norm in $B(\mathbf{R})$ is invariant under affine mappings of $\mathbf{R}$, we can assume $I=[0,1]$. Then we can extend f to a function $F \in B(\mathbf{R}) \cap C^1(\mathbf{R})$, periodic with period 2π, and such that

$$\|F\|_{L^\infty(\mathbf{R})} \leqq 2\|f\|_{L^\infty(I)}, \qquad \|F'\|_{L^\infty(\mathbf{R})} \leqq \|f'\|_{L^\infty(I)}\,.$$

Then we obtain (2.1) by estimating $\|F\|_{B(\mathbf{R})}$, using for instance Carlson's lemma (Carlson [2]).

LEMMA 2. *Let $E \subset \mathbf{R}^2$ be closed, and let (x_n), $n \in \mathbf{Z}$, be a strictly increasing sequence in $\mathbf{R}$, satisfying*

$$\delta \leqq (x_{n+1}-x_n)(x_n-x_{n-1})^{-1} \leqq \delta^{-1}, \qquad n \in \mathbf{Z}\,,$$

for some $\delta>0$. Put

$$E_m = E \cap \{(x,y) \in \mathbf{R}^2 :\ x_{2m-2} \leqq x \leqq x_{2m+1}\}, \qquad m \in \mathbf{Z}\,.$$

Then there exists a constant C^0, only depending on δ, such that, for every $f \in C(E)$,

$$\|f\|^1_{B(E)} \leqq C^0 \sum_{-\infty}^{\infty} \|f\|^1_{B(E_m)}\,.$$

PROOF OF LEMMA 2. It is possible to define functions $\varphi_m \in C^1(\mathbf{R})$, $m \in \mathbf{Z}$, satisfying

$$\begin{aligned}
&\operatorname{supp}(\varphi_m) \subset [x_{2m-2}, x_{2m+1}]\,, && \\
&0 \leqq \varphi_m(x) \leqq 1, && x \in \mathbf{R}\,, \\
&\varphi_m(x) = 1, && x \in [x_{2m-1}, x_{2m}]\,, \\
&0 \leqq \varphi'_m(x) \leqq 2(x_{2m-1}-x_{2m-2})^{-1}, && x \in [x_{2m-2}, x_{2m-1}]\,, \\
&\varphi_{m+1}(x) = 1-\varphi_m(x), && x \in [x_{2m}, x_{2m+1}]\,.
\end{aligned}$$

Then

$$(2.2) \qquad \sum_{-\infty}^{\infty} \varphi_m(x) = 1, \quad x \in \mathbf{R}.$$

We observe that $\operatorname{supp}(\varphi_m)$ is included in an interval of length less than $(x_{2m}-x_{2m-1})\cdot(1+2\delta^{-1})$, and that

$$\|\varphi_m\|_{L^\infty(\mathbf{R})} = 1, \quad \|\varphi_m'\|_{L^\infty(\mathbf{R})} \leqq (x_{2m}-x_{2m-1})^{-1}2\delta^{-1}.$$

It follows from this, using for instance a variant of Carlson's lemma (Beurling [1, p. 349]), that there exists a constant C^0 such that

$$\|\varphi_m\|_{A(\mathbf{R})} \leqq C^0, \quad m \in \mathbf{Z}.$$

For every m, Φ_m denotes the function on $\mathbf{R}^2$ defined by $\Phi_m(x,y)=\varphi_m(x)$, $(x,y) \in \mathbf{R}^2$. Then $\Phi_m \in B(\mathbf{R}^2)$ and

$$(2.3) \qquad \|\Phi_m\|_{B(\mathbf{R}^2)} = \|\varphi_m\|_{A(\mathbf{R})} \leqq C^0.$$

Choosing $f_m \in B(\mathbf{R}^2)\cap C^1(\mathbf{R}^2)$ as extensions of $f|_{E_m}$, we obtain $\sum f_m\Phi_m \in B(\mathbf{R}^2)\cap C^1(\mathbf{R}^2)$, and by (2.2)

$$\sum f_m(x,y)\Phi_m(x,y) = \sum f(x,y)\Phi_m(x,y) = f(x,y), \quad (x,y) \in \mathbf{R}^2.$$

Thus $\sum f_m\Phi_m$ is an extension of f, and since by (2.3)

$$\|\sum f_m\Phi_m\|_{B(\mathbf{R}^2)} \leqq \sum \|f_m\|_{B(\mathbf{R}^2)}\|\Phi_m\|_{B(\mathbf{R}^2)} \leqq C^0 \sum \|f_m\|_{B(\mathbf{R}^2)},$$

a minimization of $\|f_m\|_{B(\mathbf{R}^2)}$ yields the lemma.

In Section 4 we shall prove the theorem using the following lemma. The proof of the lemma is given in Section 3.

LEMMA 3. *Let* $\varepsilon>0$, $0<h<1$, $t \in \mathbf{R}$. φ, f *and* g *are real-valued functions on* $\mathbf{R}$ *with the following properties*:

φ *is a non-negative function in* $\mathscr{D}(\mathbf{R})$ *with* $\operatorname{supp}(\varphi)\subset[0,1]$ *and satisfying* $\int_{\mathbf{R}} \varphi(\sigma)\,d\sigma=1$.

$$(2.4) \qquad f \in C^1(-1,1], \quad \varepsilon < f'(s) < \varepsilon^{-1}, \quad s \in [-1,1],$$

and either

$$(2.5) \qquad g \in C^2[-1,1], \quad \varepsilon < |g''(s)| < \varepsilon^{-1}, \quad s \in [-1,1],$$

or

$$(2.6) \qquad g \in C^3[-1,1], \quad \varepsilon < |g'''(s)| < \varepsilon^{-1}, \quad s \in [-1,1].$$

Then the function G on the curve $\Gamma_0: \{(f(s), g(s)) : s \in [0,1]\}$ *with values*

$$G(f(s), g(s)) = K(s) = \int_0^1 \exp\{itg(s-\sigma h)\}\varphi(\sigma)\,d\sigma\,, \tag{2.7}$$

satisfies

$$\|G\|^1_{B(\Gamma_0)} \leqq C_0\,, \tag{2.8}$$

for some constant C_0, *depending on* ε *and* φ, *but not depending on* h, t, f *and* g.

3.

PROOF OF LEMMA 3. If $t=0$, (2.7) takes the value 1, i.e. the left-hand member of (2.8) is 1. Thus we can in the following disregard that case and assume that $t \neq 0$.

It is of course no restriction to assume that f and g are defined on R, and that the conditions (2.4) and (2.5) or (2.6) hold with $[-1,1]$ exchanged to R. Then we can enlarge the set Γ_0 to the set

$$\Gamma: \{(f(s), g(s)) : \ s \in \mathsf{R}\}\,,$$

and G and K can be extended to Γ and R, respectively, by the formula (2.7). Obviously (2.8) holds if we can prove

$$\|G\|^1_{B(\Gamma)} \leqq C_0\,.$$

For every compact interval $I \subset \mathsf{R}$, we define

$$\Gamma(I) = \{(f(s), g(s)) : \ s \in I\}\,.$$

We shall base the proof of Lemma 3 on three different upper estimates of $\|G\|^1_{B(\Gamma(I))}$. Using Lemma 2 we shall then obtain an estimate of $\|G\|^1_{B(\Gamma)}$ by means of a suitably chosen covering of Γ by curves $\Gamma(I)$.

In the following we shall use the letter C to denote any finite positive constant, which may depend on ε and φ, but not on h, t, f and g, and not on I.

1°. The first estimate is obtained by considering extensions of G on $\Gamma(I)$ such that they only depend on the first variable. Since the norm in $B(\mathsf{R}^2)$ of such a function equals the norm in $B(\mathsf{R})$ of the corresponding function of the first variable, we have by Lemma 1 an upper estimate

$$C(\|H\|_{L^\infty(f(I))} + (|f(I)|\,\|H\|_{L^\infty(f(I))}\|H'\|_{L^\infty(f(I))})^{\frac{1}{2}})\,,$$

where $H \circ f = K$, $s \in \mathsf{R}$. The inequalities (2.4) imply that we obtain the estimate

$$\|G\|^1_{B(\Gamma(I))} \leqq C(\|K\|_{L^\infty(I)} + (|I|\,\|K\|_{L^\infty(I)}\|K'\|_{L^\infty(I)})^{\frac{1}{2}})\,. \tag{3.1}$$

By (2.7) and the assumptions on φ we have

$$\|K\|_{L^\infty(\mathbf{R})} \leqq 1 . \tag{3.2}$$

Futhermore, using a change of variable, we obtain from (2.7)

$$K'(s) = h^{-1} \int_0^1 \exp\{itg(s-\sigma h)\}\varphi'(\sigma)\,d\sigma, \qquad s \in \mathbf{R} ,$$

and hence

$$\|K'\|_{L^\infty(\mathbf{R})} \leqq Ch^{-1} .$$

Thus (3.1) gives the estimate

$$\|G\|^1_{B(\Gamma(I))} \leqq C(1+(|I|h^{-1})^{\frac{1}{2}}) . \tag{3.3}$$

2°. The second and third estimates depend on the fact that

$$\|G\|^1_{B(\Gamma(I))} = \|G\chi\|^1_{B(\Gamma(I))} ,$$

for every bounded continuous character χ on $\mathbf{R}^2$. Taking

$$\chi(x,y) = \exp\{-ity\}, \qquad (x,y) \in \mathbf{R}^2 ,$$

we can thus as well estimate the norm of extensions to $B(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ of the function on $\Gamma(I)$ which for the point corresponding to the parameter value s takes the value

$$\begin{aligned} G(f(s), g(s))\exp\{-itg(s)\} &= L(s) \\ &= \int_0^1 \exp\{it(g(s-\sigma h)-g(s))\}\varphi(\sigma)\,d\sigma . \end{aligned} \tag{3.4}$$

By the same arguments as those leading to (3.1) we now get

$$\|G\|^1_{B(\Gamma(I))} \leqq C(\|L\|_{L^\infty(I)} + (|I|\,\|L\|_{L^\infty(I)}\|L'\|_{L^\infty(I)})^{\frac{1}{2}}) . \tag{3.5}$$

Differentiating (3.4) we obtain

$$L'(s) = ith \int_0^1 \frac{g'(s-\sigma h)-g'(s)}{h} \exp\{it(g(s-\sigma h)-g(s))\}\varphi(\sigma)\,d\sigma , \tag{3.6}$$

which gives

$$|L'(s)| \leqq |th|\,\|g''\|_{L^\infty(s+[-h,0])} .$$

Since (3.2) and (3.4) give

$$\|L\|_{L^\infty(\mathbf{R})} \leqq 1 , \tag{3.7}$$

we obtain from (3.5)

$$\|G\|^1_{B(\Gamma(I))} \leqq C(1+(|I|\,|th|\,\|g''\|_{L^\infty(I+[-h,0])})^{\frac{1}{2}}) . \tag{3.8}$$

3°. In the third estimate we have to assume that g' does not vanish in $I+[-h,0]$. Then, for $s \in I$, we can integrate (3.4) partially, obtaining

$$L(s) = \frac{1}{ith}\int_0^1 \left(\frac{\varphi'(\sigma)}{g'(s-\sigma h)} + \frac{h\varphi(\sigma)g''(s-\sigma h)}{(g'(s-\sigma h))^2}\right)\exp\{it(g(s-\sigma h)-g(s))\}\,d\sigma\,,$$

which gives

$$\|L\|_{L^\infty(I)} \leqq C|th|^{-1}\left\|\left|\frac{1}{g'}\right| + \left|\frac{hg''}{g'^2}\right|\right\|_{L^\infty(I+[-h,0])}. \tag{3.9}$$

In the same way we change (3.6), obtaining

$$L'(s) = \int_0^1 \left(\varphi'(\sigma)\frac{g'(s-\sigma h)-g'(s)}{hg'(s-\sigma h)} - \frac{\varphi(\sigma)g'(s)g''(s-\sigma h)}{(g'(s-\sigma h))^2}\right)\cdot \exp\{it(g(s-\sigma h)-g(s))\}\,d\sigma\,,$$

which gives

$$\|L'\|_{L^\infty(I)} \leqq C \sup_{u,v \in I+[-h,0]}\left(\left|\frac{g''(u)}{g'(v)}\right| + \left|\frac{g'(u)g''(v)}{(g'(v))^2}\right|\right). \tag{3.10}$$

(3.5) and (3.7) give

$$\|G\|_{B(\Gamma(I))}^{1} \leqq C\|L\|_{L^\infty(I)}^{\frac{1}{2}}(1+(|I|\,\|L'\|_{L^\infty(I)})^{\frac{1}{2}})\,.$$

Assuming that

$$\|g'\|_{L^\infty(I+[-h,0])}\left\|\frac{1}{g'}\right\|_{L^\infty(I+[-h,0])} \leqq C\,, \tag{3.11}$$

for some constant C, we obtain thus from (3.9) and (3.10)

$$\|G\|_{B(\Gamma(I))}^{1} \leqq C\left[|th|^{-1}\left\|\frac{1}{g'}\right\|_{L^\infty(I+[-h,0])}\left(1+\left\|\frac{hg''}{g'}\right\|_{L^\infty(I+[-h,0])}\right)\right]^{\frac{1}{2}}\cdot \left[1+\left(|I|\left\|\frac{g''}{g'}\right\|_{L^\infty(I+[-h,0])}\right)^{\frac{1}{2}}\right]. \tag{3.12}$$

This is our third estimate. We stress that we have to assume that (3.11) holds, for some constant C.

We are now prepared to prove the lemma. We shall first consider the case when (2.5) holds. In this case, the method in [3] can be applied, but we prefer to use an approach analogous to the one needed in the second case.

By (2.5), which we assume true for $s \in \mathbf{R}$, g' has a zero a. The representation

$$g'(s) = (s-a)\int_0^1 g''(a+(s-a)t)\,dt, \qquad s \in \mathbf{R}\,,$$

shows that

$$C|s-a| \leqq |g'(s)| \leqq C^{-1}|s-a|, \qquad s \in \mathbf{R}, \tag{3.13}$$

for some C.

We introduce, for a number $d>0$ to be fixed later, the set of points $a \pm d2^n$, $n \in \mathbf{N}$, on $\mathbf{R}$. This set satisfies the conditions of Lemma 2 with $\delta = \frac{1}{2}$. Thus, for some absolute constant C,

$$\|G\|_{B(\Gamma)}^1 \leqq C \sum_{-\infty}^{\infty} \|G\|_{B(\Gamma(I_n))}^1,$$

where

$$\begin{aligned} I_0 &= [a-2d, a+2d], \\ I_n &= [a+2^{2n-2}d, a+2^{2n+1}d], \qquad n \in \mathbf{Z}_+, \\ I_{-n} &= 2a - I_n, \qquad n \in \mathbf{Z}_+. \end{aligned}$$

Thus it suffices to prove that d can be chosen so that for some C

$$\sum_{-\infty}^{\infty} \|G\|_{B(\Gamma(I_n))}^1 < C. \tag{3.14}$$

For the term with index 0 we use the estimates 1° and 2°. (3.3) gives

$$\|G\|_{B(\Gamma(I_0))}^1 < C(1+(dh^{-1})^{\frac{1}{2}}), \tag{3.15}$$

and (3.8) and (2.5) give

$$\|G\|_{B(\Gamma(I_0))}^1 < C(1+(d|th|)^{\frac{1}{2}}). \tag{3.16}$$

Hence

$$\|G\|_{B(\Gamma(I_0))}^1 < C, \tag{3.17}$$

if we choose, say,

$$d = \operatorname{Max}\left(3h, \frac{1}{|th|}\right).$$

For the remaining intervals this choice guarantees, by (3.13), that (3.11) holds, for some C, independent of n, for all the intervals I_n, $n \neq 0$. Thus we can use (3.12). Observing that

$$|I_n| \cdot \left\| \frac{g''}{g'} \right\|_{L^\infty(I_n + [-h,0])} < C,$$

we obtain

$$\|G\|_{B(\Gamma(I_n))}^{\frac{1}{2}} < C(|th|\cdot d2^{2n})^{-\frac{1}{2}} < C2^{-n}, \tag{3.18}$$

for $n \neq 0$, and then (3.14) follows from (3.17) amd (3.18).

Next we consider the case when (2.6) holds and g' has at least one zero on $\mathbf{R}$. Since g''' has a constant sign, we have exactly two zeros which may coincide. We denote them a and b, where $a \leqq b$. The formulas

$$g'(s) = (s-a)(s-b)\int_0^1\int_0^1 g'''(a+(b-a)t+(s-b)tu)t\,dt\,du$$

and

$$g''(s) = (s-a)^{-1}g'(s)+(s-a)\int_0^1 g'''(a+(s-a)t)t\,dt$$

show that

$$C|(s-a)(s-b)| \leqq |g'(s)| \leqq C^{-1}|(s-a)(s-b)| \tag{3.19}$$

and

$$|g''(s)| \leqq C^{-1}(|s-a|+|s-b|)\,. \tag{3.20}$$

Also in this case we introduce points on $\mathbf{R}$, starting with a basic length d. When $d<b-a$ it is convenient to assume that

$$d = \frac{(b-a)}{12}\cdot 2^{-2M}, \tag{3.21}$$

for some $M \in \mathbf{N}$.

In the case when $d \geqq b-a$ we take the set of points of the form $b+d2^n$ or $a-d2^n$, $n \in \mathbf{N}$. In the case when $d<b-a$ we add to this set all points $a+d2^n$ and $b-d2^n$, where $n \in \mathbf{N}$, $n \leqq 2M+2$, M defined by (3.21). In both cases we obtain sets which fulfill the assumptions of Lemma 2 with $\delta=\frac{1}{4}$. Thus it suffices to prove that, for some C,

$$\sum_{I \subset Q} \|G\|_{B(\Gamma(I))}^{\frac{1}{2}} < C, \tag{3.22}$$

where Q *in case* $d \geqq b-a$ is the family of intervals $[a-2d, b+2d]$, $[b+2^{2n-2}d, b+2^{2n+1}d]$, $[a-2^{2n+1}d, a-2^{2n-2}d]$, $n \in \mathbf{Z}_+$, *in case* $d<b-a$ is the family of intervals $[b-2d, b+2d]$, $[b+2^{2n-2}d, b+2^{2n+1}d]$, $n \in \mathbf{Z}_+$, $[b-2^{2n+1}d, b-2^{2n-2}d]$, $1 \leqq n \leqq M+1$, and their images when reflecting in $s=\frac{1}{2}(b+a)$.

When estimating, we have now to single out the interval $[a-2d, b+2d]$ in the first case and the intervals $[b-2d, b+2d]$ and $[a-2d, a+2d]$ in the second case. Exactly as before we can apply (3.3) which gives, with I as any of these intervals,

$$(3.23) \qquad \|G\|^{1}_{B(\Gamma(I))} \leqq C(1+(dh^{-1})^{\frac{1}{2}})$$

(cf. 3.15) and (3.8) which, using (3.20), gives

$$(3.24) \qquad \|G\|^{1}_{B(\Gamma(I))} \leqq C\big(1+\big(|th|d(d+h+(b-a))\big)^{\frac{1}{2}}\big)$$

(cf. 3.16).

Choosing $d_0 = \text{Max}\,(3h, \delta)$, where $\delta > 0$ satisfies

$$\delta(\delta+(b-a)) = |th|^{-1},$$

we can easily find a constant C such that the construction, including the condition (3.21), can be performed for some d with

$$Cd_0 < d < C^{-1}d_0 .$$

By (3.23) and (3.24) we find that the contribution to (3.22) of the considered terms is then dominated by some constant C.

For the remaining intervals (3.11) is evident by (3.19). Thus (3.12) can be applied, and using (3.19) and (3.20) we obtain for those intervals which are situated at distance $2^{2n-2}d$ from the nearest of the points a and b

$$\|G\|^{1}_{B(\Gamma(I))} \leqq C\left[|th|^{-1}\frac{1}{2^{2n}d(b-a+2^{2n}d)}\left(1+\frac{h}{d2^{2n}}\right)\right]^{\frac{1}{2}} \cdot$$

$$\cdot\left(1+\left(d2^{2n}\cdot\frac{1}{d2^{2n}}\right)^{\frac{1}{2}}\right) \leqq C2^{-n} .$$

All these estimates prove (3.22).

It remains to discuss the case when (2.6) holds but g' does not vanish on R. We assume that $|g'|$ attains its minimum for $s=a$, and define the real-valued function g_0 such that the relation

$$g_0'(s) = g'(s)-g'(a) ,$$

holds for $s \in$ R. Then g_0 is a function for which the previous discussion is applicable. Following the estimating procedure for g_0 but inserting g instead, we find that the obtained estimates hold as well. Hence the lemma holds in this case, too.

4.

PROOF OF THE THEOREM. Let γ be an injective C^3 mapping from $[a,b]$ to R^3. We put $\gamma(s) = (x(s), y(s), z(s))$, $s \in [a,b]$ and assume that

$$\begin{vmatrix} x'(s) & x''(s) & s'''(s) \\ y'(s) & y''(s) & y'''(s) \\ z'(s) & z''(s) & z'''(s) \end{vmatrix} \neq 0, \tag{4.1}$$

$s \in [a, b]$. We have to prove that $\Gamma = \{(x(s), y(s), z(s)) : s \in [a, b]\}$ is a set of sequential spectral synthesis.

We regard $\mathbf{R}^3$ as an euclidean space with the xyz-coordinates as orthonormal coordinates. Take any $s_0 \in [a, b]$, and introduce new orthornormal coordinates ξ, η, ζ such that the ξ-axis has the direction of the tangent vector $(x'(s_0), y'(s_0), z'(s_0))$ at the point $(x(s_0), y(s_0), z(s_0))$. Then the curve gets the equation

$$\begin{aligned} \xi &= \xi(s) = a_1 x(s) + a_2 y(s) + a_3 z(s) + a_0 \\ \eta &= \eta(s) = b_1 x(s) + b_2 y(s) + b_3 z(s) + b_0 \\ \zeta &= \zeta(s) = c_1 x(s) + c_2 y(s) + c_3 z(s) + c_0, \end{aligned} \tag{4.2}$$

where (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) are orthogonal unit vectors. It follows from the construction and (4.1) that $\eta'(s_0) = 0$ and that $|\xi'(s_0)|$ and $|\eta''(s_0)| + |\eta'''(s_0)|$ have a positive lower bound which does not depend on the choice of the η and ζ coordinate axis. Due to the assumed continuity we can conclude the following:

There is an $\varepsilon > 0$ such that for every orthonormal change of coordinates into coordinates such that the ξ-axis forms an angle $< \varepsilon$ with at least one of the vectors

$$(x'(s), y'(s), z'(s)), \qquad s \in [s_0 - \varepsilon, s_0 + \varepsilon],$$

the corresponding functions, defined by (4.2), satisfy

$$\varepsilon < |\xi'(s)| < \varepsilon^{-1}, \qquad s \in [s_0 - \varepsilon, s_0 + \varepsilon], \tag{4.3}$$

and *either*

$$\varepsilon < |\eta''(s)| < \varepsilon^{-1}, \qquad s \in [s_0 - \varepsilon, s_0 + \varepsilon], \tag{4.4}$$

or,

$$\varepsilon < |\eta'''(s)| < \varepsilon^{-1}, \qquad s \in [s_0 - \varepsilon, s_0 + \varepsilon]. \tag{4.5}$$

Furthermore, for every c, $a < c < b$, we can obviously find intervals $[s_n - \varepsilon_n, s_n + \varepsilon_n]$, $n = 1, \ldots, N$, of $[a, b]$ such that

$$[c, b[\subset \bigcup_1^N]s_n, \delta_n + \varepsilon_n[,$$

and such that (4.1), (4.2) and (4.3) hold if s_0 and ε are changed to s_n and ε_n, respectively.

We have to prove that a pseudomeasure on Γ lies in the sequential weak* closure of $M(\Gamma)$. Any pseudomeasure on Γ can be written as the sum of a pseudomeasure with support in $\Gamma_{[a,d]}$ and a pseudomeasure with support in $\Gamma_{[c,b]}$, if $a<c<d<b$. Here

$$\Gamma_I = \{(x(s), y(s), z(s)) : \; s \in I\}$$

for every $I \subset [a,b]$. Of course it is enough to discuss a pseudomeasure with support in some $\Gamma_{[c,b]}$, $a<c<b$. By a further partitioning we find that it is enough to study pseudomeasures with support in one of the sets $\Gamma_{[s_n, s_n+\varepsilon_n]}$, as defined above. Changing affinely the parameter, if necessary, we can describe

$$\Gamma^0 = \Gamma_{[s_n-\varepsilon_n, s_n+\varepsilon_n]}$$

as the set of values of a C^3 function with values $\gamma(s) = (x(s), y(s), z(s))$, $s \in [-1,1]$, where (4.1) holds for $s \in [-1,1]$. Then, for some $\varepsilon>0$, whenever new coordinates (ξ, η, ζ) are introduced by an orthonormal transformation, such that the ξ-axis forms an angle $<\varepsilon$ with at least some of the vectors $(x'(s), y'(s), z'(s))$, $s \in [-1,1]$, then the corresponding functions (4.2) satisfy (4.3) and *either* (4.4) *or* (4.5). We have reduced the problem to prove sequential spectral synthesis for a psedomeasure μ, supported by $\Gamma^0_{[0,1]}$.

We refer to [3] and [4, Theorem 2.9] for the details of the following discussion. The paper [2] concerns just R^2, but many of the arguments hold in any R^n, $n \geqq 2$.

μ can be considered as a bounded linear functional on $C^1(\mathsf{R}^3)$, whose value when applied to a function F does only depend on the restriction of F to $\Gamma^0_{[0,1]}$. Given φ and h as in Lemma 3, we define on Γ^0 a measure μ_h with a smooth density function with respect to the parameter s. For the parameter value s_0 the density function is defined as $\langle H_{s_0}, \mu \rangle$, where H_{s_0} is any function in $C^1(\mathsf{R}^3)$, such that

$$H_{s_0}(x(s), y(s)) = \frac{1}{h}\varphi\left(\frac{s_0 - z}{h}\right), \qquad s \in [0,1] \, .$$

We shall prove that $\mu_h \to \mu$ weakly*, as $h \to +0$. The only difficult part is to prove that there exists a constant C such that for every bounded continuous character χ on R^3, and independently of h, the function on $\Gamma^0_{[0,1]}$ with value

$$\int_{\mathsf{R}} \chi(x(s-\sigma h), y(s-\sigma h), z(s-\sigma h)) \varphi(\sigma) \, d\sigma \, ,$$

at $(x(s), y(s), z(s))$, $s \in [0,1]$, can be extended to a function in $B(\mathsf{R}^3) \cap C^1(\mathsf{R}^3)$ with its norm in $B(\mathsf{R}^3)$ bounded above by C.

To prove this, let us first consider a character χ which is constant in a plane which forms an angle $<\varepsilon$ with some tangent of Γ^0. Then we change coordinates orthonormally to (ξ, η, ζ), taking the new ξ and ζ coordinate axis parallell to the plane with the ξ-axis forming an angle $<\varepsilon$ with some tangent to Γ^0. Due to the properties (4.3), (4.4), (4.5) of the corresponding representation (4.2) it is natural to restrict to extensions which are independent of ζ, and use Lemma 3 which gives immediately the desired uniform bound.

It remains to discuss the case when the angle between the tangents of Γ^0 and the planes where χ is constant are all $>\varepsilon$. Then we introduce a new orthonormal system (ξ, η, ζ), with the η-axis normal to these planes, Then, for some $C>0$, independent of χ, $\eta \in C^3[-1,1]$ and

$$C < |\eta'(s)| < C^{-1}, \quad s \in [-1,1] \tag{4.6}$$

$$|\eta''(s)| < C^{-1}, \quad s \in [-1,1] . \tag{4.7}$$

The function to be extended takes the value

$$\int_{\mathbf{R}} \exp\{it\eta(s-\sigma h)\}\varphi(\sigma)\, d\sigma$$

at $(\xi(s), \eta(s), \zeta(s))$, $s \in [0,1]$. As in the deduction of the estimates 2° and 3° in the proof of Lemma 3 we multiply by the conjugate of χ, and are left with the problem of estimating $\|F\|_{B(\Gamma^0_{[0,1]})}^{1}$, where F is defined on $\Gamma^0_{[0,1]}$ by

$$F(\xi(s), \eta(s), \zeta(s)) = L(s) = \int_0^1 \exp\{it(\eta(s-\sigma h)-\eta(s))\}\varphi(\sigma)\, d\sigma , \tag{4.8}$$

if χ takes the values $\exp(it\eta)$ at (ξ, η, ζ). We consider extensions of F which do only depend on the variable η. Due to (4.6) and Lemma 1, it suffices to find a constant C, such that

$$\|L\|_{L^\infty[0,1]} + (\|L\|_{L^\infty[0,1]}\|L'\|_{L^\infty[0,1]})^{\frac{1}{2}} \leqq C . \tag{4.8}$$

A partial integration of (4.8) (cf. (3.9)) shows by (4.6) and (4.7) that

$$\|L\|_{L^\infty[0,1]} \leqq \frac{C}{|th|} ,$$

for some constant C, and obviously

$$\|L\|_{L^\infty[0,1]} \leqq 1 .$$

Differentiation of (4.8) (cf. (3.6)) shows by (4.6) and (4.7) that

$$\|L'\|_{L^\infty[0,1]} \leqq C|th| ,$$

for some constant C. From these relations we obtain (4.8), and the theorem is proved.

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