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ON REPRESENTATION FORMULAS FOR INTERMEDIATE DERIVATIVES

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1. Introduction.

In a paper of Domar [2], he studies complex-valued functions φ on R, $n-1$ times absolutely continuous and satisfying $|\varphi^{(n)}(x)| \leqq A$, $|\varphi(x)| \leqq B$, $x \in \mathsf{R}$. For $0<m<n$, a representation formula

$$\text{(i)} \qquad \varphi^{(m)}(0) = \int_{\mathsf{R}} \varphi^{(n)}(-x)\, d\nu_1(x) + \int_{\mathsf{R}} \varphi(-x)\, d\nu_2(x) ,$$

is deduced, where ν_1 and ν_2 are bounded regular Borel measures with ν_1 absolutely continuous, such that equality can be obtained in the resulting inequality

$$\text{(ii)} \qquad |\varphi^{(m)}(0)| \leqq A \int_{\mathsf{R}} |d\nu_1| + B \int_{\mathsf{R}} |d\nu_2| .$$

In this paper, we study real-valued functions φ on an interval I on R, finite or infinite, and assume for a given quadruple of positive numbers (A,B,C,D) that

$$-B \leqq \varphi^{(n)}(x) \leqq A, \qquad -D \leqq \varphi(x) \leqq C, \qquad x \in I .$$

Then we can deduce a representation formula (i) such that equality can be obtained in the inequality

$$\text{(iii)} \quad \varphi^{(m)}(0) \leqq A \int_I d\nu_1(x)^+ + B \int_I d\nu_1(x)^- + C \int_I d\nu_2(x)^+ + D \int_I d\nu_2(x)^- .$$

In fact we obtain this as a special case of a more general representation formula where derivation is exchanged to certain operations of convolution type. This formula is proved by methods adopted from Domar [2].

For $I=\mathsf{R}$ we give explicit expressions for ν_1 and ν_2, and the optimal φ_1 and φ_2, and discuss the properties of these functions. In particular we prove that the measures have compact support if and only if $n \leqq 3$.

This paper is an abbreviated version of some non-published results in [4]. The remaining results in [4] are contained in [5].

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2. A general approach.

In this section we shall give a generalization of a theorem by Domar [2]. We are going to use methods and notations adopted from [2]. A difference is that we are only dealing with real spaces.

We denote by $M(\mathsf{R})$ the Banach space of real bounded Borel measures on R and by $\mathrm{AC}(\mathsf{R})$ the subspace of $M(\mathsf{R})$ consisting of all measures which are absolutely continuous with respect to the Lebesgue measure.

The Fourier–Stieltjes transform $\hat{\mu}$ of a measure μ in $M(\mathsf{R})$ is defined by the relation

$$\hat{\mu}(t) = \int_{\mathsf{R}} e^{-itx}\, d\mu(x)$$

for every t on the dual R. Convolution of elements in $M(\mathsf{R})$ is defined in the usual way so that it corresponds to pointwise multiplication of the Fourier–Stieltjes transforms.

Let μ_1 and μ_2 be given elements in $M(\mathsf{R})$ and μ_0 a third given element with the property that there exist elements ν_1 and ν_2 in $M(\mathsf{R})$ so that

$$\mu_0 = \mu_1 * \nu_1 + \mu_2 * \nu_2 . \tag{1}$$

We assume that there exist a real number α and measures σ_0 and σ_2 in $\mathrm{AC}(\mathsf{R})$ such that the three relations

$$\hat{\mu}_1(t) \neq 0 \tag{2}$$

$$\hat{\mu}_2(t) = \hat{\mu}_1(t)\hat{\sigma}_2(t) \tag{3}$$

$$\hat{\mu}_0(t) = \hat{\mu}_1(t)\hat{\sigma}_0(t) \tag{4}$$

all hold if $|t| \geqq \alpha$.

H denotes the set of pairs of bounded Borel measures (ν_1, ν_2) which satisfy (1). L denotes the set of all pairs of bounded Borel measures (ν_1, ν_2) such that

$$\mu_1 * \nu_1 + \mu_2 * \nu_2 = 0 . \tag{5}$$

We finally form the class K of all pairs of real functions (φ_1, φ_2) in $L^\infty(\mathsf{R})$ such that

$$\varphi_1 * \nu_1 + \varphi_2 * \nu_2 = 0 \tag{6}$$

holds for every (ν_1, ν_2) in L with ν_i in $\mathrm{AC}(\mathsf{R})$, $i = 1, 2$.

We are now able to establish our first theorem which is a refinement of a theorem in [2].

THEOREM 2.1. 1°. *If* (v_1, v_2) *belongs to* H *or* L *then* v_1 *belongs to* $\mathrm{AC}(\mathbf{R})$.

2°. *There exists a pair* (v_1, v_2) *in* H *with* v_2 *in* $\mathrm{AC}(\mathbf{R})$.

3°. *If* $(\varphi_1, \varphi_2) \in K$ *then* φ_2 *is continuous after a change in a set of Lebesgue measure* 0.

4°. *We form for any* $(v_1, v_2) \in H$ *and any* $(\varphi_1, \varphi_2) \in K$ *with* φ_2 *continuous the functional*

$$F(\varphi_1, \varphi_2, v_1, v_2) = \int_{\mathbf{R}} \varphi_1(-x) v_1'(x)\, dx + \int_{\mathbf{R}} \varphi_2(-x)\, dv_2(x) . \tag{7}$$

The value of the functional (7) *does not depend on the choice of* (v_1, v_2) *in* H.

5°. *Let* A, B, C *and* D *be fixed positive numbers and let* $K(A, B, C, D)$ *denote the subset of all* $(\varphi_1, \varphi_2) \in K$ *such that* $-B \leqq \varphi_1 \leqq A$ *and* $-D \leqq \varphi_2 \leqq C$ *almost everywhere. Then there exists a pair* (ψ_1, ψ_2) *in* $K(A, B, C, D)$ *such that*

$$F(\varphi_1, \varphi_2, v_1, v_2) \leqq F(\psi_1, \psi_2, v_1, v_2)$$

for every (φ_1, φ_2) *in* $K(A, B, C, D)$.

6°. *There exists a pair* (v_1, v_2) *in* H *such that*

$$F(\psi_1, \psi_2, v_1, v_2) = A \int_{\mathbf{R}} v_1'(x)^+ \, dx + B \int_{\mathbf{R}} v_1'(x)^- \, dx + \tag{8}$$

$$+ C \int_{\mathbf{R}} dv_2(x)^+ + D \int_{\mathbf{R}} dv_2(x)^- ,$$

where

$$v_i^+ = \frac{|v_i| + v_i}{2} \quad \textit{and} \quad v_i^- = \frac{|v_i| - v_i}{2} \quad \textit{for } i = 1, 2 .$$

PROOF. The proofs of 1°, 3° and 4° can be found in [2].

PROOF OF 2°. Let (v_3, v_4) be in H. Then we have by (1) that

$$\hat{\mu}_0 = \hat{v}_3 \hat{\mu}_1 + \hat{v}_4 \hat{\mu}_2 . \tag{9}$$

Choose a measure μ in $\mathrm{AC}(\mathbf{R})$ such that

$$\hat{\mu}(t) = 1 \quad \text{for } |t| \leqq \alpha . \tag{10}$$

Define v_1 and v_2 by the formulas

$$v_1 = \sigma_0 - \sigma_0 * \mu + v_3 * \mu \tag{11}$$

and

$$v_2 = v_4 * \mu . \tag{12}$$

Since $\mu \in AC(\mathbf{R})$ we have $\nu_2 \in AC(\mathbf{R})$. Thus it remains to prove that the pair (ν_1, ν_2) defined by (11) and (12) belongs to H.

By (9) we get

$$\hat{\mu}_1\hat{\nu}_1+\hat{\mu}_2\hat{\nu}_2 = \hat{\mu}_1\hat{\sigma}_0+\hat{\mu}(\hat{\mu}_0-\hat{\mu}_1\hat{\sigma}_0)\,.$$

Hence we get for $|t|\leqq\alpha$ by (10) and for $|t|\geqq\alpha$ by (4) that

$$\hat{\mu}_1\hat{\nu}_1+\hat{\mu}_2\hat{\nu}_2 = \hat{\mu}_0$$

and thus (1) is satisfied and (ν_1, ν_2) belongs to H.

Proof of 5°. We form the space X of all pairs (ν_1, ν_2) where $\nu_1 \in AC(\mathbf{R})$ and $\nu_2 \in M(\mathbf{R})$. Define

$$p(\nu_1,\nu_2) = A\int_{\mathbf{R}} \nu_1'(x)^+\,dx+B\int_{\mathbf{R}} \nu_1'(x)^-\,dx+C\int_{\mathbf{R}} d\nu_2(x)^+ +D\int_{\mathbf{R}} d\nu_2(x)^-\,.$$

Then $p(\nu_1, \nu_2)$ is a non-negative, positively homogeneous subadditive function on the space X. L is a subspace of X and H is a hyperplane parallel to L. Let d denote the "distance" between H and L, i.e.,

$$d = \inf_{(\nu_1,\nu_2)\in H} p(\nu_1,\nu_2)\,. \tag{13}$$

Obviously we have for every $(\varphi_1, \varphi_2) \in K(A,B,C,D)$ and for every $(\nu_1, \nu_2) \in H$ that

$$F(\varphi_1,\varphi_2,\nu_1,\nu_2) \leqq p(\nu_1,\nu_2)\,. \tag{14}$$

The left-hand side in (14) is according to 4° independent of the choice of (ν_1, ν_2) in H. Thus we have

$$F(\varphi_1,\varphi_2,\nu_1,\nu_2) \leqq d$$

for $(\varphi_1, \varphi_2) \in K(A,B,C,D)$ and $(\nu_1, \nu_2) \in H$.

From the Hahn–Banach theorem (see [6, pp. 135, 143 and 149–150]) we get that there exists a linear functional G on X, vanishing on L and taking the value d on H, such that for every (ν_1, ν_2) in X we have

$$-p(-\nu_1,-\nu_2) \leqq G(\nu_1,\nu_2) \leqq p(\nu_1,\nu_2)\,. \tag{15}$$

The inequality (15) holds in particular on the closed subspace of X consisting of the pairs (ν_1, ν_2) where both ν_1 and ν_2 belong to $AC(\mathbf{R})$. The dual of this space is well known and we obtain from this that there exist bounded measurable functions ψ_1 and ψ_2 such that

$$G(v_1, v_2) = \int_{\mathbf{R}} \psi_1(-x)v_1'(x)\,dx + \int_{\mathbf{R}} \psi_2(-x)v_2'(x)\,dx$$

for $v_i \in \mathrm{AC}(\mathbf{R})$, $i=1,2$. Obviously $-B \leqq \psi_1 \leqq A$ and $-D \leqq \psi_2 \leqq C$ a.e.

Since G vanishes on L the definition of K shows that $(\psi_1, \psi_2) \in K(A, B, C, D)$. In particular we can assume ψ_2 to be continuous.

Let us now for any (v_1, v_2) in H form the functional

$$G^*(v_1, v_2) = \int_{\mathbf{R}} \psi_1(-x)v_1'(x)\,dx + \int_{\mathbf{R}} \psi_2(-x)\,dv_2(x) = F(\psi_1, \psi_2, v_1, v_2)\,.$$

According to 1° and 2° there exists a pair (v_1, v_2) in H such that $v_i \in \mathrm{AC}(\mathbf{R})$, $i=1,2$. For that special pair we have $G^*(v_1, v_2) = G(v_1, v_2) = d$. By 4° G^* is constant on H. Hence $G^*(v_1, v_2) = d$ for every (v_1, v_2) in H. Thus the inequality

$$F(\varphi_1, \varphi_2, v_1, v_2) \leqq d = F(\psi_1, \psi_2, v_1, v_2)$$

holds for every (v_1, v_2) in H and every (φ_1, φ_2) in $K(A, B, C, D)$. Hence 5° is proved.

The proof of 6° can be made analogously to the proof of Theorem 1, statement 5° in [2] and is thus omitted.

3. Intermediate derivatives.

From now on we restrict ourselves to the case where the measures μ_0, μ_1 and μ_2 are given explicitly by the relations

$$\hat{\mu}_1(t) = e^{-t^2}(it)^n \tag{16}$$

$$\hat{\mu}_2(t) = e^{-t^2} \tag{17}$$

$$\hat{\mu}_0(t) = e^{-t^2}(it)^m\,, \tag{18}$$

where m and n are integers $0<m<n$.

According to [2], K then consists of all pairs of bounded functions of the form $(\varphi^{(n)}, \varphi)$ where φ is absolutely continuous together with its $n-1$ first derivatives and the functional $F(\varphi^{(n)}, \varphi, v_1, v_2) = \varphi^{(m)}(0)$. Hence we get by Theorem 2.1 that for every quadruple (A, B, C, D) of positive numbers there is a representation formula

$$\varphi^{(m)}(0) = \int_{\mathbf{R}} \varphi^{(n)}(-x)v_1'(x)\,dx + \int_{\mathbf{R}} \varphi(-x)\,dv_2(x)$$

such that equality can be attained in the resulting inequality

$$\varphi^{(m)}(0) \leqq A \int_{\mathbf{R}} v_1'(x)^+ \, dx + B \int_{\mathbf{R}} v_1'(x)^- \, dx + C \int_{\mathbf{R}} dv_2(x)^+ + D \int_{\mathbf{R}} dv_2(x)^- \tag{19}$$

where $-B \leqq \varphi^{(n)} \leqq A$ and $-D \leqq \varphi \leqq C$ a.e.

In this particular case the value of (8) and the extremal functions are known. According to Hörmander [3, p. 45] the extremal functions can be found among the pairs (ψ_1, ψ_2) with $\psi_1 = \psi_2^{(n)}$ and

$$\psi_2(x) = \alpha^{-n} h_n(\alpha(x+\gamma); B, A) + \beta \, , \tag{20}$$

where

$$h_n(x; B, A) = \frac{B+A}{(n+1)!}\left(\bar{B}_{n+1}\left(x + \frac{A}{2(A+B)}\right) - \bar{B}_{n+1}\left(x - \frac{A}{2(A+B)}\right)\right).$$

Here $\bar{B}_n$ is the periodically continued restriction to $[0,1]$ of the Bernoulli polynomial B_n of degree n. The numbers α and β are so defined that the infimum and supremum of ψ_2 are $-D$ and C respectively. This extremal character is not proved in [3] but as Hörmander indicates, the proof given by Bang [1] of Kolmogorov's inequality [7] can easily be applied in this more general case too.

The pair (v_1, v_2) in H giving equality in (19) is uniquely determined. More precisely we have the following theorem.

THEOREM 3.1. *There exists in H a pair (v_a, v_2) such that* $\operatorname{sign} v_1'(x) = \operatorname{sign} \psi_1(-x)$ *while v_2 is a discrete measure composed of non-negative point masses at the points where $\psi_2(-x)$ has its local maxima and non-positive point masses at the points where $\psi_2(-x)$ has its local minima.*

The pair (v_1, v_2) is uniquely determined by the condition that $v_1'(x)$ and $\psi_1(-x)$ changes sign at the same points and the condition that $v_2(x)$ has its support at the points where $\psi_2(-x)$ has its local extrema.

PROOF. Let us study the function

$$g_n(x) = \alpha^{-n} h_n(\alpha x; B, A) + \beta \, ,$$

where h_n, α and β are defined by (20). Let γ be the real number, closest to zero, for which $g_n^{(m)}$ takes its maximum.

Then the functions

$$\psi_1(x) = \alpha^{-1} h_0(\alpha(x+\gamma); B, A)$$

and

$$\psi_2(x) = \alpha^{-n}h_n(\alpha(x+\gamma); B, A)+\beta$$

will constitute an extremal pair in the sense of Theorem 2.1, statement 5°. The function ψ_1 has jumps at the points $\pm b+k\alpha^{-1}-\gamma$ where $b=A(2(A+B)\alpha)^{-1}$ and $k \in \mathbf{Z}$. (Cf. [3].)

Let (v_1, v_2) be an optimal pair in the sense of Theorem 2.1,6°. Obviously it has to satisfy the conditions on the signs of v_1' and v_2. Thus it remains to prove the uniqueness.

From our formulas (1) and (16)–(18) we get

$$\hat{v}_1(t) = (it)^{-n}((it)^m - \hat{v}_2(t)) . \tag{21}$$

Let us first consider the case when $n-m>1$. Then we have $\hat{v}_1 \in L^1(\mathbf{R})$ and we can thus suppose v_1' to be continuous. The sign variations of v_1' imply

$$v_1'(\pm b+\gamma+k\alpha^{-1}) = 0, \quad k \in \mathbf{Z} .$$

Define

$$\hat{f}_j(t) = \hat{v}_1(t)e^{it((-1)^j b+\gamma)}, \quad j=1,2 .$$

Then $\hat{f}_j \in L^1(\mathbf{R})$ and is thus the Fourier transform, in the L^1-sense, of the function $v_1'(x-((-1)^j b+\gamma))$. The periodic function

$$\sum_{k=-\infty}^{\infty} \hat{f}_j(t+k2\pi\alpha), \quad j=1,2 ,$$

which locally belongs to L^1 has its Fourier coefficients determined by the values $v_1'((-1)^j b+\gamma+k\alpha^{-1})$ in such a way that they must vanish too. Hence we have

$$\sum_{k=-\infty}^{\infty} \hat{f}_j(t+kh) = 0 \text{ a.e.}, \quad j=1,2 , \tag{22}$$

where $h=2\pi\alpha$.

Define α_1 and α_2 as the real numbers closest to zero for which g_n has maximum and minimum, respectively. Then ψ_2 has extreme values at the points $\alpha_j-\gamma+k\alpha^{-1}$ for $j=1,2$ and $k \in \mathbf{Z}$.

Suppose now that v_2 has the mass a_{kj} at the point $-(\alpha_j-\gamma+k\alpha^{-1}), j=1,2$. Then

$$\hat{v}_2(t) = \sum_{k=-\infty}^{\infty} a_{k1}e^{it(\alpha_1-\gamma+k\alpha^{-1})}+\sum_{k=-\infty}^{\infty} a_{k2}e^{it(\alpha_2-\gamma+k\alpha^{-1})} .$$

For every integer q we have then

$$\hat{v}_2(t+qh) = \hat{v}_2(t)e^{iqh(\alpha_1-\gamma)}+A(t)e^{-iqh\gamma}(e^{iqh\alpha_2}-e^{iqh\alpha_1}) \tag{23}$$

where

$$A(t) = \sum_{k=-\infty}^{\infty} a_{k2} e^{it(\alpha_2 - \gamma + k\alpha^{-1})} .$$

If we insert (21) and (23) into (22) for $j=1,2$ we get two equations from which we can eliminate $A(t)$ and then get $\hat{v}_2(t)$ uniquely. By our formula (21) we then get $\hat{v}_1$ uniquely.

The case when $n-m=1$ must be treated differently. Suppose there are two optimal pairs (v_1, v_2) and (v_3, v_4). Then both v_1' and v_3' are continuous on $\mathbf{R}$ except for jumps of size 1 at zero. Hence their difference $v_1'-v_3'$ is continuous on $\mathbf{R}$. We can then apply the above method to conclude that $v_1'-v_3'=0$. Thus we have uniqueness in this case too.

4. On the supports of the optimal measures (v_1, v_2).

In this section we shall study the optimal pair (v_1, v_2) more closely. We begin by formulating our main result.

THEOREM 4.1. *The optimal pair of measures* (v_1, v_2) *in the sense of Theorem* 2.1.6° *have compact support if and only if* $n \leqq 3$.

PROOF. Our method to prove the uniqueness of the optimal pair (v_1, v_2) in section 3 gave explicit expressions for $\hat{v}_1$ and $\hat{v}_2$. Since the formulas are not very convenient in the general case we will restrict ourselves to the case when $A=B$, that is the case studied by Domar [2]. This restriction is not essential in what concernes the compactness of the supports. The details in [2] are not sufficient for our purpose. Let us return to our formulas in section 3 and insert $A=B$. We then get $b=(4\alpha)^{-1}$. If we let $p=4\pi\alpha$ we get that v_1' has zeros at the points $(2k+1)\pi p^{-1}+\gamma$, $k \in \mathbf{Z}$.

Define

$$\hat{f}(t) = \hat{v}_1(t) e^{it(\pi/p+\gamma)} .$$

If $n-m>1$ we have

$$\sum_{k=-\infty}^{\infty} \hat{f}(t+kp) = 0 . \tag{24}$$

The function ψ_2 has extreme values at the points $\alpha_1+2q\pi p^{-1}-\gamma$, $q \in \mathbf{Z}$. Hence we can write

$$\hat{v}_2(t) = \sum_{q=-\infty}^{\infty} a_q e^{it(\alpha_1 - \gamma + 2q\pi p^{-1})} .$$

For every integer k we then get

$$\hat{v}_2(t+kp) = \hat{v}_2(t)e^{ikp(\alpha_1-\gamma)} . \tag{25}$$

From (21), (24) and (25) we get

$$\hat{v}_2(t) = \frac{\sum_{k=-\infty}^{\infty}(-1)^k(i(t+kp))^{m-n}e^{ikp\gamma}}{\sum_{k=-\infty}^{\infty}(-1)^k(i(t+kp))^{-n}e^{ikp\alpha_1}} .$$

If we put

$$A_s(t) = \lim_{N\to\infty}\sum_{k=-N}^{N}(-1)^k\frac{e^{ikps}}{t-kp} ,$$

we can write

$$\hat{v}_2(t) = \frac{(-i)^m(n-1)!\,A_\gamma(t)^{(n-m-1)}}{(n-m-1)!\,A_{\alpha_1}(t)^{(n-1)}} . \tag{26}$$

We have (cf. [7])

$$\gamma = \begin{cases}0 & \text{if } n-m \text{ is even}\\ \pi/p & \text{if } n-m \text{ is odd}\end{cases} \tag{27}$$

and

$$\alpha_1 = \begin{cases}0 & \text{if } n \text{ is even}\\ \pi/p & \text{if } n \text{ is odd} .\end{cases} \tag{28}$$

Moreover we have the well known relations

$$A_0(t) = \pi/p \operatorname{cosec} \pi t/p \tag{29}$$

and

$$A_{\pi/p}(t) = \pi/p \cot \pi t/p . \tag{30}$$

Our formulas (27)–(30) show that the right-hand side in (26) contains derivatives of cot of even order and derivatives of cosec of odd order. By induction on k we can prove the formulas

$$\operatorname{cosec}^{(2k-1)}(x) = \cos x\frac{P_{k-1}(\sin^2 x)}{(\sin x)^{2k}} \tag{31}$$

and

$$\cot^{(2k)}(x) = \cos x\frac{Q_{k-1}(\sin^2 x)}{(\sin x)^{2k+1}} , \tag{32}$$

where P_{k-1} and Q_{k-1} are polynomials of degree $k-1$.

It can be proved that the formulas (26)–(30) will determine the optimal measure v_2 also in the case when $n-m=1$.

With the aid of our formulas (26)–(32) we can now see that if $n>3$ there exists at least one complex zero of the denominator in the right-hand side of (26) which is not a zero of the numerator. Hence $\hat{v}_2$ is not an entire function and thus v_2 cannot have compact support if $n>3$.

If $n\leqq 3$ we have the following three cases.

$$n=2,\ m=1: \quad \hat{v}_2(t) = ip/\pi \sin \pi t/p$$

$$n=3,\ m=1: \quad \hat{v}_2(t) = ip/\pi \sin \pi t/p$$

$$n=3,\ m=2: \quad \hat{v}_2(t) = -(p/\pi)^2 \sin^2 \pi t/p\ ,$$

where p is appropriately defined. Hence v_2 has compact support if and only if $n\leqq 3$. By (21) the same holds for v_1.

5. Generalizations to compact intervals.

From Section 4 we get that there are pairs (v_1, v_2) in H such that v_1 and v_2 have compact support. At least this is the case if $n\leqq 3$.

More generally, we can choose v_2 as a discrete measure with support on the arbitrary interval I and then choose the masses so that $\hat{v}_2(t)=(it)^m+O(t^n)$ in a neighbourhood of zero. If we define v_1 by (21) we get that v_1 has its support on I. Hence to every interval I there is a pair $(v_1, v_2)\in H$ with $\operatorname{supp} v_i \subseteqq I$, $i=1,2$.

Let I be an arbitrary interval on R. Let $H(I)$ denote the subset of H consisting of the pairs (v_1, v_2) in H with $\operatorname{supp} v_i \subseteqq I$, $i=1,2$. Moreover let $-I$ denote the set $(x \mid -x \in I)$.

Let us once again regard our representation formula

$$(33) \qquad \varphi^{(m)}(0) = \int_{\mathsf{R}} \varphi^{(n)}(-x) v_1'(x)\, dx + \int_{\mathsf{R}} \varphi(-x)\, dv_2(x)\ .$$

If we have $|\varphi^{(n)}(x)|\leqq M_n$ and $|\varphi(x)|\leqq M_0$ a.e. on I we get

$$(34) \qquad |\varphi^{(m)}(0)| \leqq M_n \int_{-I} |v_1'(x)|\, dx + M_0 \int_{-I} |dv_2(x)|$$

for every (v_1, v_2) in $H(-I)$.

Our next question will then be to ask whether there can be equality in (34). The answer is affirmative which we formulate in a theorem.

THEOREM 5.1. *Let I be an arbitrary interval on R with $0\in I$ and let (M_0, M_n) be an arbitrary pair of positive numbers. Moreover let φ be a real valued function absolutely continuous together with its $n-1$ first derivatives on I and satisfying, $|\varphi^{(k)}(x)|\leqq M_k$ on I, $k=0,n$. Then for $0<m<n$ there is a representation formula*

$$(35) \qquad \varphi^{(m)}(0) = \int_{-I} \varphi^{(n)}(-x) v_1'(x)\, dx + \int_{-I} \varphi(-x)\, dv_2(x)$$

where ν_1 and ν_2 are bounded Borel measures on I with ν_1 absolutely continuous, such that equality can be obtained in the resulting inequality

$$|\varphi^{(m)}(0)| \leqq M_n \int_{-I} |\nu_1'(x)|\,dx + M_0 \int_{-I} |d\nu_2(x)| \,. \tag{36}$$

PROOF. The ideas and methods used in [2] and in our section 2 are enough also in this more general case. A detailed proof is given in [4] but is omitted in this paper.

The functions giving equality in (36) are not explicitly known. It follows from (35) and (36) that the extremal function φ and the corresponding optimal pair (ν_1, ν_2) have to be connected by the relation

$$\varphi^{(n)}(-x) = M_n \operatorname{sign} \nu_1'(x)$$

and by the condition that the support of ν_2 is a part of the set

$$(x \mid |\varphi(-x)| = M_0) \,.$$

Hence the extremal function has to be a polynomial spline function of degree n, where the number and the location of the knots are given by the changes of sign of the optimal measure ν_1'. See also Tihomirov [8].

Any choice of pair (ν_1, ν_2) in $H(-I)$ in formula (36) gives an upper bound for intermediate derivatives. Hence to get upper bounds we have to look for pairs (ν_1, ν_2) in $H(-I)$ such that the total variations of ν_1 and ν_2 can be calculated. This idea and a more direct approach to a formula similar to (35) (by use of a theorem by Peano) is used in [5] to get upper bounds for intermediate derivatives on finite intervals.

Theorem 5.1 can be generalized in the direction of formula (19).

From Theorem 5.1 the following corollary immediately follows.

COROLLARY. *Let I be an arbitrary interval on* R *and let M_0 and M_n be arbitrarily given positive numbers. Let $C_{n,m,x}(I)$ be the least possible numbers satisfying*

$$|f^{(m)}(x)| \leqq C_{n,m,x}(I) \tag{37}$$

for every function f with $|f^{(k)}(x)| \leqq M_k$ on I, $k=0,n$.

Then for every n, m, x and I there is a function f giving equality in (37).

Let $C_{n,m}(I) = \sup_{x\in I} C_{n,m,x}(I)$. From our theorems 4.1 and 5.1 the following result follows.

THEOREM 5.2. *For every $n>3$ and every interval $I \neq$* R *we have*

$$C_{n,m}(I) > C_{n,m}(\mathbf{R}) \,. \tag{38}$$

PROOF. Suppose on the contrary that there are integers m, n and an interval $I \neq \mathsf{R}$ with

$$C_{n,m}(I) = C_{n,m}(\mathsf{R}) .$$

Let φ be an extremal function corresponding to R. Then φ must be extremal with respect to I. We may suppose that $0 \in I$ and $\varphi^{(m)}(0) = C_{n,m}(I)$. According to Theorem 5.1 there is then a corresponding optimal pair (v_1, v_2) in $H(-I)$. Hence we have

$$\begin{aligned} \varphi^{(m)}(0) &= M_n \int_I |v_1'(x)|\, dx + M_0 \int_I |dv_2(x)| \\ &= M_n \int_{\mathsf{R}} |v_1'(x)|\, dx + M_0 \int_{\mathsf{R}} |dv_2(x)| . \end{aligned}$$

Hence the pair (v_1, v_2) is optimal corresponding to R. But, according to Theorem 3.1 this pair is uniquely determined and according to Theorem 4.1 the measures v_1 and v_2 cannot have compact support if $n > 3$. Since v_2 is either odd or even its support cannot be a half-line. Thus a contradiction is achieved and the theorem is proved.

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