MATH. SCAND. 39 (1976), 343–358

ELEMENTARY SPHERICAL FUNCTIONS ON SYMMETRIC SPACES

LARS VRETARE

1. Introduction.

One of the main results in this paper states that the elementary spherical functions on a symmetric space of compact type, with rank l, may be considered as orthogonal polynomials with respect to a positive weight function defined on a region in R^l. This is a generalization of the well-known rank one case in which the elementary spherical functions are Jacobi polynomials. Our result also settles the conjecture in [4] to the effect that the orthogonal polynomials of two variables considered there are elementary spherical functions on a symmetric space of rank 2.

Let φ_Λ be the elementary spherical function corresponding to the highest weight Λ. The second main result of this paper is a recurrence formula for φ_Λ, expressing the product $\varphi_{\Lambda_1}\varphi_\Lambda$ (Λ_1 fixed) as a linear combination of other φ_ν in which the number of terms is independent of Λ. This is also a generalization of well-known facts about Jacobi polynomials and of the recurrence formulas proved in [5] in the case of two variables.

We also obtain recurrence formulas for elementary spherical functions on a symmetric space of non-compact type, by analytic continuation. Some of the coefficients are explicitly computed in terms of Harish–Chandras c-function.

2. Preliminaries and recurrence formulas in the compact case.

General references for this section are [1], [3], [7], [9] and [10]. Root systems are especially studied in [9], representations with weights in [10] and the elementary spherical functions on a compact Lie group in [1] and [7].

Let $\mathfrak{g}_0$ be a noncompact semisimple Lie algebra over R, $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$ and $\mathfrak{u}=\mathfrak{k}_0+i\mathfrak{p}_0$ the corresponding compact real form of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Denote by G_c a simply connected Lie group with Lie algebra $\mathfrak{g}$ and by G, U and K the analytic subgroups of G_c generated by $\mathfrak{g}_0$, $\mathfrak{u}$ and $\mathfrak{k}_0$ respectively. Select a maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}_0}$ of $\mathfrak{p}_0$ and a maximal abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ containing $\mathfrak{h}_{\mathfrak{p}_0}$. Then $\mathfrak{h}_0=\mathfrak{h}_{\mathfrak{k}_0}+\mathfrak{h}_{\mathfrak{p}_0}$ where $\mathfrak{h}_{\mathfrak{k}_0}=\mathfrak{h}_0\cap\mathfrak{k}_0$. Also $\mathfrak{t}=\mathfrak{h}_{\mathfrak{k}_0}+i\mathfrak{h}_{\mathfrak{p}_0}$ is maximal abelian in $\mathfrak{u}$. Complexify $\mathfrak{h}_{\mathfrak{k}_0}$, $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{k}_0$, $\mathfrak{p}_0$

Received March 2, 1976.

and $\mathfrak{h}_0$ to $\mathfrak{h}_{\mathfrak{k}}$, $\mathfrak{h}_{\mathfrak{p}}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{h}$ respectively. Let Δ (Δ_0) be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ $((\mathfrak{g}_0, \mathfrak{h}_{\mathfrak{p}_0}))$. Introduce compatible orderings in the dual spaces of $\mathfrak{h}_{\mathfrak{p}_0}$ and it and write Δ^+ (Δ_0^+) for the set of positive roots with respect to these orderings. By means of the Killing form $\langle\cdot,\cdot\rangle$ we identify it with the set of linear forms Λ on $\mathfrak{h}$ which are real-valued on it. Let σ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then $\tilde{\Lambda}=\frac{1}{2}(\Lambda+\sigma\Lambda)$ is the restriction of Λ to $\mathfrak{h}_{\mathfrak{p}_0}$. Note also that σ is an involutory automorphism of it leaving the scalar product $\langle\cdot,\cdot\rangle$ invariant. The fundamental system of roots for $\Delta, \alpha_1, \ldots, \alpha_n$ may be denumerated in such a way that

$$\sigma\alpha_i = \begin{cases} \alpha_{i'} + \sum_{j=m+1}^{n} n_j^i \alpha_j & \text{if } 1 \leqq i \leqq m \\ -\alpha_i & \text{if } m+1 \leqq i \leqq n \end{cases}$$

where

$$i' = \begin{cases} i & \text{if } 1 \leqq i \leqq l_1 \\ i+l_2 & \text{if } l_1+1 \leqq i \leqq l_1+l_2 = l \\ i-l_2 & \text{if } l_1+l_2+1 \leqq i \leqq l_1+2l_2 = m \end{cases}$$

Moreover $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$ is a fundamental system of roots for Δ_0.

By our assumption G and U are subgroups of the simply connected group G_c. This permits us to identify the irreducible finite dimensional representations of G_c with those of G and U. The highest weights of these representations are precisely the dominant integral linear forms on $\mathfrak{h}$, that is, the linear forms Λ for which $2\langle\Lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$ is a non-negative integer for all $\alpha \in \Delta^+$. The highest weights can also be characterized by means of the fundamental weights π_i, $i=1 \ldots n$ defined by

$$\frac{2\langle\pi_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then Λ is a highest weight if and only if $\Lambda=\sum_{i=1}^{n} m_i\pi_i$ with non-negative integers m_i.

An irreducible finite dimensional representation R is said to be of class one (with respect to K) if there is a vector in the representation space left fixed by all $R(k)$, $k \in K$. For such representations we have the following characterization. A highest weight Λ belongs to a class one representation if and only if $\tilde{\Lambda} = \Lambda$ and $\langle\Lambda,\lambda\rangle/\langle\lambda,\lambda\rangle$ is a non-negative integer for all $\lambda \in \Delta_0^+$. (See [9, chapter 3.3]). To obtain a characterization in terms of the fundamental weights put

$$\mu_i = \begin{cases} \pi_i & \text{if } 1 \leqq i \leqq l_1 \text{ and } \sigma\alpha_i \neq \alpha_i \\ 2\pi_i & \text{if } 1 \leqq i \leqq l_1 \text{ and } \sigma\alpha_i = \alpha_i \\ \pi_i + \pi_{i+l_2} & \text{if } l_1+1 \leqq i \leqq l_1+l_2 = l \end{cases}$$

THEOREM 2.1. *A highest weight Λ belongs to a class one representation if and only if $\Lambda=\sum_{i=1}^{l} m_i\mu_i$ with non-negative integers m_i.*

This theorem is stated without proof in Sugiura [6]. A proof is however obtained from the next two lemmas concerning μ_i.

LEMMA 2.2. *Let μ_i, $i=1 \dots l$ be defined as above. Then*

1) $\tilde{\mu}_i = \mu_i$

$$2)\ \frac{\langle \mu_i, \tilde{\alpha}_j \rangle}{\langle \tilde{\alpha}_j, \tilde{\alpha}_j \rangle} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i=j \text{ and } 2\tilde{\alpha}_j \notin \Delta_0 \\ 2 & \text{if } i=j \text{ and } 2\tilde{\alpha}_j \in \Delta_0 \end{cases}$$

PROOF. The first statement is equivalent to $\sigma\pi_i=\pi_{i'}$ $1 \leqq i \leqq l_1+2l_2$. To prove this we use the definition of $\pi_{i'}$ and compute the numbers

$$x_{ij} = \frac{2\langle \sigma\pi_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \quad 1 \leqq i \leqq l_1+2l_2,\ 1 \leqq j \leqq n\,.$$

Let us first note that $\sigma\Delta=\Delta$ and that

$$\langle \pi_i, \sigma\alpha_j \rangle = \begin{cases} \langle \pi_i, \alpha_i \rangle & \text{if } j=i' \\ 0 & \text{if } j \neq i' \end{cases}$$

Thus on one hand

$$x_{ii'} = \frac{2\langle \pi_i, \sigma\alpha_{i'} \rangle}{\langle \sigma\alpha_{i'}, \sigma\alpha_{i'} \rangle}$$

that is, an integer and on the other

$$x_{ii'} = \frac{2\langle \pi_i, \sigma\alpha_{i'} \rangle}{\langle \alpha_{i'}, \alpha_{i'} \rangle} = \frac{2\langle \pi_i, \alpha_i \rangle}{\langle \alpha_{i'}, \alpha_{i'} \rangle} = \frac{\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_{i'}, \alpha_{i'} \rangle}\,.$$

By interchanging the role of i and i' we find that this integer has to be 1. Moreover $j \neq i'$ implies that

$$x_{ij} = \frac{2\langle \pi_i, \sigma\alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = 0$$

and we conclude that $\sigma\pi_i=\pi_{i'}$ as desired.

To prove the second statement put

$$y_{ij} = \frac{\langle \mu_i, \tilde{\alpha}_j \rangle}{\langle \tilde{\alpha}_j, \tilde{\alpha}_j \rangle} \quad 1 \leqq i,\ j \leqq l\,.$$

Clearly

$$\langle \tilde{\pi}_i, \tilde{\alpha}_j \rangle = \langle \tilde{\pi}_i, \alpha_j \rangle = \tfrac{1}{2}(\langle \pi_i, \alpha_j \rangle + \langle \pi_{i'}, \alpha_j \rangle)$$

whence $y_{ij}=0$ if $j \neq i$. The values of y_{ii} are obtained in the same way by use of the fact that for any $\alpha \in \Delta$ holds

$$\langle \alpha, \alpha \rangle = n_\alpha \langle \tilde{\alpha}, \tilde{\alpha} \rangle$$

with

$$n_\alpha = \begin{cases} 1 & \text{if } \sigma\alpha = \alpha \\ 2 & \text{if } \sigma\alpha \neq \alpha \text{ and } 2\tilde{\alpha} \notin \Delta_0 \\ 4 & \text{if } \sigma\alpha \neq \alpha \text{ and } 2\tilde{\alpha} \in \Delta_0 \end{cases}$$

Let Λ be the highest weight of a class one representation. In view of the linear independence of the μ_i's we may write $\Lambda = \sum_{i=1}^{l} c_i \mu_i$ with

$$c_i = \frac{\langle \Lambda, \tilde{\alpha}_i \rangle}{\langle \mu_i \tilde{\alpha}_i \rangle} = \begin{cases} \dfrac{\langle \Lambda, \tilde{\alpha}_i \rangle}{\langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle} & \text{if } 2\tilde{\alpha}_i \notin \Delta_0 \\[2ex] \dfrac{\langle \Lambda, 2\tilde{\alpha}_i \rangle}{\langle 2\tilde{\alpha}_i, 2\tilde{\alpha}_i \rangle} & \text{if } 2\tilde{\alpha}_i \in \Delta_0 \end{cases}$$

This proves the only if part of the theorem. The converse is an immediate consequence of

LEMMA 2.3. *Each μ_i belongs to a class one representation.*

PROOF. We have to prove that $\langle \mu_i, \lambda \rangle / \langle \lambda, \lambda \rangle$ is a non-negative integer for all $\lambda \in \Delta_0^+$.

Let W be the Weyl group pf Δ_0. Choose a $S \in W$ and a simple root $\tilde{\alpha}_j \in \Delta_0^+$ such that either $\lambda = S\tilde{\alpha}_j$ or $\lambda = 2S\tilde{\alpha}_j$. In the latter case $\frac{1}{2}\lambda = S\tilde{\alpha}_j$ is also a root, the restriction to $\mathfrak{h}_{p0}$ of some $\beta \in \Delta^+$. Since $\tilde{\beta}$ as well as $2\tilde{\beta}$ belongs to Δ_0^+ we know that $\langle \beta, \beta \rangle = 4\langle \tilde{\beta}, \tilde{\beta} \rangle$. Consequently

$$\frac{\langle \mu_i, \lambda \rangle}{\langle \lambda, \lambda \rangle} = \frac{1}{2} \frac{\langle \mu_i, \tilde{\beta} \rangle}{\langle \tilde{\beta}, \tilde{\beta} \rangle} = \frac{2\langle \mu_i, \beta \rangle}{\langle \beta, \beta \rangle}$$

which is a non-negative integer.

Assume now that $\lambda = S\tilde{\alpha}_j$ and write S^{-1} as a product $S_1 \ldots S_p$ of Weyl reflections corresponding to the simple roots. Then

$$\mu_i - S^{-1}\mu_i = (\mu_i - S_1\mu_i) + S_1(\mu_i - S_2\mu_i) + \ldots + S_1 \ldots S_{p-1}(\mu_i - S_p\mu_i) .$$

Each of the terms to the right is of the form $(2\langle \mu_i, \tilde{\alpha}_k \rangle / \langle \tilde{\alpha}_k, \tilde{\alpha}_k \rangle) S' \tilde{\alpha}_k$ for some

$S' \in W$ and some $k \leq l$. But this is non-zero only if $k=i$ so we get

$$\mu_i - S^{-1}\mu_i = 2\frac{\langle \mu_i, \tilde{\alpha}_i \rangle}{\langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle} \sum_{S' \in W'} S'\tilde{\alpha}_i, \qquad W' \subset W.$$

By use of this expression for $S^{-1}\mu_i$ we obtain

$$\frac{\langle \mu_i, \lambda \rangle}{\langle \lambda, \lambda \rangle} = \frac{\langle S^{-1}\mu_i, \tilde{\alpha}_j \rangle}{\langle \tilde{\alpha}_j, \tilde{\alpha}_j \rangle} = \frac{\langle \mu_i, \tilde{\alpha}_j \rangle}{\langle \tilde{\alpha}_j, \tilde{\alpha}_j \rangle} - \frac{\langle \mu_i, \tilde{\alpha}_i \rangle}{\langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle} \sum_{S' \in W'} \frac{2\langle S'\tilde{\alpha}_i, \tilde{\alpha}_j \rangle}{\langle \tilde{\alpha}_j, \tilde{\alpha}_j \rangle}$$

which obviously is an integer. This integer is of course non-negative since λ is a linear combination of the simple roots $\tilde{\alpha}_1 \ldots \tilde{\alpha}_l$ with non-negative coefficients.

Let R_Λ be a class one representation with highest weight Λ. Choose a unit vector e in the representation space V_Λ such that $R(k)e=e$ for all $k \in K$. The elementary spherical function corresponding to Λ is defined by

$$\varphi_\Lambda(g) = (e \mid R_\Lambda(g)e) \qquad g \in G_c\,.$$

Here $(\cdot|\cdot)$ denotes the scalar product in V_Λ. Let $f_0, \ldots, f_q$ be an orthonormal basis of V_Λ such that f_i belongs to the weight $\Lambda_i, \Lambda_0 = \Lambda$. Then

$$R_\Lambda(\exp H)f_i = e^{\Lambda_i(H)}f_i, \qquad H \in \mathfrak{h}$$

and

$$\overline{\varphi_\Lambda}(\exp H) = \sum_{i=0}^{q} |(e \mid f_i)|^2 e^{\Lambda_i(H)}, \qquad H \in \mathfrak{h}\,.$$

As we have seen the highest weight Λ is a linear combination of $\mu_1, \ldots \mu_l$ with integral coefficients. This is in general not true for the other weights Λ_i but we have the following result.

THEOREM 2.4. *Suppose that*

$$\overline{\varphi_\Lambda}(\exp H) = \sum_{i=0}^{q} c_i e^{\Lambda_i(H)}, \qquad H \in \mathfrak{h}\,.$$

Then $c_i \neq 0$ *implies that* $\Lambda_i = \sum_{k=1}^{l} n_k \mu_k$, $n_k \in \mathbf{Z}$.

PROOF. Trying to prove that $\tilde{\Lambda}_i = \Lambda_i$ and that $\langle \Lambda_i, \lambda \rangle / \langle \lambda, \lambda \rangle$ is an integer for all $\lambda \in \Delta_0$ we follow the corresponding proof for the highest weight Λ in [9 p. 210]. $P = \int_K R_\Lambda(k)\,dk$ is a projection of V_Λ onto the one-dimensional subspace spanned by e. If $Pf_i \neq 0$ the proof works and we make the desired conclusion about Λ_i. If however $Pf_i = 0$ we find directly that

$$c_i = |(e \mid f_i)|^2 = |(e \mid Pf_i)|^2 = 0\,.$$

A weight ν for which e^ν appears with non-zero coefficient in some φ_Λ will be called an appearing weight.

From now on we identify the two points $(m_1, \dots m_l) \in \mathbf{Z}^l$ and $\sum_{i=1}^{l} m_i \mu_i \in \mathfrak{h}_{p0}$. We also introduce a partial ordering of $\mathbf{Z}^l$ by putting

$$\Lambda_1 \preceq \Lambda_2 \quad \text{if} \quad \langle \Lambda_1, \mu_i \rangle \leqq \langle \Lambda_2, \mu_i \rangle$$

for all $i = 1 \dots l$. The set of points in $\mathbf{Z}^l$ for which all coordinates are non-negative will be denoted by $\mathbf{Z}^l_+$.

Let us now collect some properties of this ordering of $\mathbf{Z}^l$.

LEMMA 2.5. i) *Let Λ_i be the weights in theorem* 2.4. *Then $\Lambda_i \preceq \Lambda$.*

ii) $\{v \in \mathbf{Z}^l_+, v \preceq \Lambda\}$ *is a finite set for any $\Lambda \in \mathbf{Z}^l_+$.*

iii) *For any $v \in \mathbf{Z}^l$ there is a $S \in W$ such that $Sv \in \mathbf{Z}^l_+$.*

iv) $\mathbf{Z}^l_+$ *is the set of highest weights of class one.*

v) $\mathbf{Z}^l$ *is the set of appearing weights.*

PROOF. i) Follows from the fact that

$$\Lambda_i = \Lambda - \alpha_1 - \alpha_2 \dots \alpha_k \in \Delta_0^+ .$$

ii) Obvious.

iii) Consequence of general weight theory and the observation that $\mathbf{Z}^l$ is invariant under the Weyl group.

iv) Reformulation of theorem 2.1.

v) Follows from iii) and theorem 2.4.

In view of this lemma the following corollary is obvious.

COROLLARY 2.6. *Given any two elementary spherical functions φ_{Λ_1} and φ_{Λ_2}. Then there are complex numbers $c_v(\Lambda_1, \Lambda_2)$ such that*

$$\varphi_{\Lambda_1} \varphi_{\Lambda_2} = \sum_{v \preceq \Lambda_1 + \Lambda_2} c_v(\Lambda_1, \Lambda_2) \varphi_v .$$

The restriction to U of the functions φ_Λ are called elementary spherical functions on U (or U/K). They satisfy the functional equation

$$\int_K \varphi_\Lambda(u_1 k u_2)\, dk = \varphi_\Lambda(u_1) \varphi_\Lambda(u_2)$$

which can also be used to define them. By taking complex conjugate we see that $\bar{\varphi}_\Lambda$ is an elementary spherical function too. It will be denoted by $\varphi_{\Lambda'}$. We also have the Schur orthogonality relations

$$(\varphi_{\Lambda_1}, \varphi_{\Lambda_2}) = \int \varphi_{\Lambda_1}(u) \bar{\varphi}_{\Lambda_2}(u)\, du = 0 \quad \text{if } \Lambda_1 \neq \Lambda_2 .$$

Specializing Λ_1 to be one of the μ_i's and dropping the indices corollary 2.6 may be restated as follows:

$$(\varphi_\mu \varphi_\Lambda, \varphi_\nu) \neq 0 \quad \text{only if} \quad \nu \preceq \Lambda + \mu .$$

But

$$(\varphi_\mu \varphi_\Lambda, \varphi_\nu) = \overline{(\varphi_{\mu'} \varphi_\nu, \varphi_\Lambda)}$$

which by the same corollary is non-zero only if $\Lambda \preceq \mu' + \nu$. Thus, the coefficient $c_\nu(\mu, \Lambda)$ in the expansion of $\varphi_\mu \varphi_\Lambda$ is non-zero only if $-\mu' \leqq \nu - \Lambda \leqq \mu$. Note that $\{x \in \mathbf{Z}^l;\ -\mu' \preceq x \preceq \mu\}$ is a finite set contained in the parallel-epiped

$$\{H \in \mathfrak{h}_{\mathfrak{p}0}\ ;\ -\langle \mu', \mu_i \rangle \leqq \langle H, \mu_i \rangle \leqq \langle \mu, \mu_i \rangle,\ i = 1 \dots l\} .$$

We have proved

THEOREM 2.7. *Let μ be a fixed unit vector in $\mathbf{Z}^l_+$. There exist complex-valued functions $c_x(\Lambda)$ defined on $\mathbf{Z}^l_+$ such that*

$$\varphi_\mu(g)\varphi_\Lambda(g) = \sum_{\substack{-\mu' \preceq x \preceq \mu \\ \Lambda + x \in \mathbf{Z}^l_+}} c_x(\Lambda)\varphi_{\Lambda+x}(g), \qquad g \in G_c .$$

The number of terms in the sum is independent of Λ.

3. Spherical functions as orthogonal polynomials.

In this section we consider polynomials of l variables $X_1, \dots X_l$. In view of the identification made in section 2 of an appearing weight $\Lambda = \sum_{i=1}^l m_i \mu_i$ and the point $(m_1, \dots m_l) \in \mathbf{Z}^l$ let us denote the monomial $X_1^{m_1} \dots X_l^{m_l}$ by X^Λ. A polynomial $P(X)$ is said to have degree Λ if

$$P(X) = \sum_{\nu \preceq \Lambda} a_\nu X^\nu, \quad a_\Lambda \neq 0 .$$

It follows from corollary 2.6 that

$$P(\varphi) = P(\varphi_{\mu_1}, \dots, \varphi_{\mu_l}) = \sum_{\nu \preceq \Lambda} b_\nu \varphi_\nu, \quad b_\Lambda \neq 0 .$$

Hence if P is a polynomial such that $P(\varphi(u)) = 0$ for all $u \in U$, then P is the zero-polynomial. This allows us to talk about polynomials in the variables $\varphi_{\mu_1} \dots \varphi_{\mu_l}$.

Besides the partial ordering $\prec$ we will use a total ordering $<$ of $\mathbf{Z}^l_+$ having the two properties

$$\Lambda_1 < \Lambda_2 \quad \text{if} \quad \Lambda_1 \prec \Lambda_2$$

and $\{\nu \in \mathbf{Z}^l_+\ ;\ \nu \leq \Lambda\}$ is finite for any $\Lambda \in \mathbf{Z}^l_+$.

An example of such an ordering is the lexicographic ordering with respect to an orthogonal basis in $\mathfrak{h}_{p0}$ with a first element v satisfying

$$\langle v, \mu_i \rangle > 0, \quad \langle v, \tilde{\alpha}_i \rangle > 0, \quad i = 1, \ldots l .$$

We may e.g. take

$$v = \tilde{\varrho} = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \tilde{\alpha} .$$

By use of induction with respect to this ordering we can now prove.

THEOREM 3.1. *φ_Λ is a polynomial of degree Λ in the variables $\varphi_{\mu_1} \ldots \varphi_{\mu_l}$.*

PROOF. This is obvious if $\Lambda = 0$. Suppose it is true for all $\nu < \Lambda$. Choose an index i such that $\Lambda - \mu_i \in \mathbf{Z}^l_+$. By corollary 2.6

$$\varphi_\Lambda = c\varphi_{\mu_i}\varphi_{\Lambda - \mu_i} + \sum_{\nu < \Lambda} c_\nu \varphi_\nu, \qquad c \neq 0$$

which clearly is a polynomial of degree Λ.

COROLLARY 3.2. *Complex conjugation permutes $\varphi_{\mu_1}, \ldots, \varphi_{\mu_l}$, that is, for any μ_k there is a μ_j such that $\overline{\varphi_{\mu_k}} = \varphi_{\mu_j}$.*

PROOF. Recall the notation $\varphi_{\Lambda'} = \overline{\varphi_\Lambda}$. Let $P(\varphi)$ be the polynomial $\varphi_{\mu_k'}$. Then

$$\varphi_{\mu_k} = \overline{P(\varphi)} = \bar{P}(\varphi_{\mu_1'}, \ldots, \varphi_{\mu_l'}) .$$

If the degree of P is $\sum_{j=1}^l n_j \mu_j$ we see that φ_{μ_k} is a polynomial of degree $\sum_{j=1}^l n_j \mu_j'$. This is possible only if $\mu_k = \mu_j'$ for some j.

Let Ω be the image in $\mathbf{C}^l$ of $i\mathfrak{h}_{p0}$ under the mapping

$$F \colon iH \curvearrowright (\varphi_{\mu_1}(\exp iH), \ldots \varphi_{\mu_l}(\exp iH)), \qquad H \in \mathfrak{h}_{p0} .$$

By transformation of the Schur orthogonality relations for the elementary spherical functions on U/K we will prove that these functions are orthogonal polynomials on Ω with respect to a positive weight function.

Let us first compute $\det F$, the Jacobian of the mapping F.

LEMMA 3.3. *Put*

$$\Sigma = \{\alpha \in \Delta_0^+ \; ; \; 2\alpha \notin \Delta_0^+\} .$$

Then

$$\det F = c \prod_{\alpha \in \Sigma} \sin \alpha .$$

PROOF. By theorem 2.4 $\det F$ is a linear combination of exponentials e^{ν}, $\nu \in \mathbf{Z}^l$, $\nu \preceq \Lambda_0$ where

$$\Lambda_0 = \sum_{i=1}^{l} \mu_i .$$

Moreover since F is W-invariant, $\det F$ is skew, that is,

$$\det F(SH) = \det S \det F(H) .$$

It follows that

$$\det F = c \sum \det S e^{S\Lambda_0} .$$

Put

$$\Lambda_1 = \sum_{\alpha \in \Sigma} \alpha$$

and let S_i be the Weyl reflection corresponding to the simple root $\tilde{\alpha}_i$ $i=1,\ldots,l$. S_i permutes the roots of Σ except for $\tilde{\alpha}_i$ if $2\tilde{\alpha}_i \notin \Delta_0^+$ and $2\tilde{\alpha}_i$ if $2\tilde{\alpha}_i \in \Delta_0^+$. Hence $S_i\Lambda_1 = \Lambda_1 - n\cdot\tilde{\alpha}_i$, $n=2$ or 4. But

$$S_i\Lambda_0 = \Lambda_0 - \frac{2\langle \mu_i, \tilde{\alpha}_i\rangle}{\langle \tilde{\alpha}_i \tilde{\alpha}_i\rangle}\tilde{\alpha}_i .$$

This shows that $\Lambda_1 - \Lambda_0$ is invariant under S_i, $i=1,\ldots,l$ and we conclude that $\Lambda_1 = \Lambda_0$. Since $\prod_{\alpha\in\Sigma} \sin\alpha$ is also a skew linear combination of e^{ν}, $\nu \in \mathbf{Z}^l$, $\nu \preceq \Lambda_0$ the lemma follows.

Put

$$D = \{H \in i\mathfrak{h}_{\mathfrak{p}0} \ ; \ \alpha(H) \in \pi i \mathbf{Z} \text{ for some } \alpha \in \Delta_0\}$$

$$E = \{H \in i\mathfrak{h}_{\mathfrak{p}0} \ ; \ \alpha(SH-H) \in \pi i \mathbf{Z} \text{ for all } \alpha \in \Delta_0 \text{ and some } S \in W\}$$

$$A = \exp i\mathfrak{h}_{\mathfrak{p}0}$$

and let A' be the complement of $\exp D \cup \exp E$ in A. Consider the mapping

$$F'\colon a \curvearrowright (\varphi_{\mu_1}(a)), \ldots, \varphi_{\mu_l}(a)), \qquad a \in A' .$$

LEMMA 3.4. *Let p and q be the number of elements of W and $J = K \cap A$ respectively. F' is a regular pq-to-one mapping of A' onto a subset Ω' of Ω.*

PROOF. The regularity follows from lemma 3.3. Let us determine the inverse image of $F'(a)$, $a \in A'$. It follows from theorem 3.1 that

$$\text{if } F'(b) = F'(a), \text{ then } \varphi_\Lambda(b) = \varphi_\Lambda(a) \text{ for all } \Lambda \in \mathbf{Z}^l_+ .$$

Owing to the completeness of the elementary spherical functions on U/K we conclude that $b=k_1ak_2$ for some k_1 and k_2 in K. Then we can find $S \in W$ and $j \in J$ such that $b=a^Sj$ (see [3, p. 384]).

It remains to prove that the elements a^Sj are different and belong to A'. A' is obviously W-invariant. The invariance under multiplication by elements $j \in J$ follows from the fact that $j^2=e$, or in other words, $\alpha(H) \in \pi i\mathbf{Z}$ for all $\alpha \in \Delta_0$ and all $H \in i\mathfrak{h}_{\mathfrak{p}_0}$ such that $\exp H \in J$. Suppose now that $a^{S_1}j_1=a^{S_2}j_2$. Then we cand find $S \in W$ such that $a^Sa^{-1} \in J$ that is, $a \in \exp E$. This contradicts the assumption that $a \in A'$ and the proof is finished.

Let us assume in view of corollary 3.2 that

$$\mu_j' = \begin{cases} \mu_{j+k} & \text{if } j=1,\ldots,k \\ \mu_{j-k} & \text{if } j=k+1,\ldots,2k \\ \mu_j & \text{if } j=2k+1,\ldots,l \end{cases}$$

The mapping

$$\psi\colon\ (z_1,\ldots,z_l) \curvearrowright (x_1,\ldots,x_l)$$

where

$$x_j = \begin{cases} \frac{1}{2}(z_{j+k}+z_j) & \text{if } j=1,\ldots,k \\ (2i)^{-1}(z_{j-k}-z_j) & \text{if } j=k+1,\ldots,2k \\ z_j & \text{if } j=2k+1,\ldots,l \end{cases}$$

maps Ω into $\mathbf{R}^l$.

LEMMA 3.5. *Let w be defined on $\psi(\Omega)$ by*

$$w(\psi(F(iH))) = \left|\prod_{\alpha\in\Delta^+} \sin\tilde{\alpha}(H) \prod_{\tilde{\alpha}\in\Sigma} (\sin\tilde{\alpha}(H))^{-1}\right|, \qquad H \in \mathfrak{h}_{\mathfrak{p}_0}.$$

Then

$$\int_{\psi(\Omega)} f(\psi^{-1}(x))w(x)\,dx = c\int_U f(\varphi_{\mu_1}(u),\ldots,\varphi_{\mu_l}(u))\,du$$

for all continuous functions f on Ω.

PROOF. Follows from lemma 3.4 by noting that the complements of A' in A and of $\psi(\Omega')$ in $\psi(\Omega)$ are sets of measure zero. The following well-known integral formula is also used

$$\int_U f(u)\,du = \int_A f(a)\,Da\,da$$

where

$$D(\exp iH) = \left|\prod_{\alpha\in\Delta^+} \sin\tilde{\alpha}(H)\right|, \qquad H \in \mathfrak{h}_{\mathfrak{p}_0}.$$

As a corollary we obtain.

THEOREM 3.6. *The elementary spherical functions on U/K may be considered as orthogonal polynomials with respect to the positive weight function w, see lemma* 3.5, *defined on a region in* $\mathbf{R}^l$.

PROOF. φ_Λ is a polynomial of $\varphi = (\varphi_{\mu_1}, \ldots \varphi_{\mu_l})$, hence also of $\psi(\varphi)$. Denote this polynomial by P_Λ that is, $P_\Lambda(\psi(\varphi(u)) = \varphi_\Lambda(u)$. The orthogonality relations follow from lemma 3.5.

$$\int_{\psi(\Omega)} P_{\Lambda_1}(x)\overline{P_{\Lambda_2}(x)}\, w(x)\, dx = c \int_U \varphi_{\Lambda_1}(\mu)\, \overline{\varphi_{\Lambda_2}(u)}\, du\,.$$

4. Closer study of the recurrence formula and the non-compact case.

General references for this section are [2] and [9].

Let $G = KAN$ be an Iwasawa decomposition of G and define $H(g)$, $g \in G$ by $g = k \exp H(g) n$. The functions

$$\Phi_\lambda(g) = \int_K e^{(i\lambda - \varrho)(H(gk))}\, dk, \qquad \lambda \in \mathfrak{h}_{\mathfrak{p}}$$

where $\varrho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \tilde{\alpha}$ are called elementary spherical functions on G (or G/K). For certain values of λ they coincide with φ_Λ, namely

$$\varphi_\Lambda(g) = \Phi_{-i(\Lambda+\varrho)}(g), \qquad g \in G\,.$$

This fact will be used to get more information about the coefficients $c_x(\Lambda)$ in the recurrence formula, theorem 2.7, and to extend the formula by analytic continuation.

For any $M \geqq 0$ *let* $\mathfrak{h}_{\mathfrak{p}_0}(M)$ be the set of $H \in \mathfrak{h}_{\mathfrak{p}_0}$ such that $\tilde{\alpha}_i(H) > M$ for all $i = 1, \ldots, l$. Denote the points Sx, $S \in W, -\mu' \precsim x \precsim \mu$ by $x_0, \ldots, x_p$ and chose M so large that $\Lambda + x_i \in \mathbf{Z}^l_+$ for all $\Lambda \in \mathfrak{h}_{\mathfrak{p}_0}(M) \cap \mathbf{Z}^l_+$ and all $i = 0, \ldots, p$.

Let D be the Casimir operator of G. Then

$$D\Phi_\lambda = -(\langle\lambda,\lambda\rangle + \langle\varrho,\varrho\rangle)\Phi_\lambda$$

and

$$D\varphi_\Lambda(g) = \chi(\Lambda)\varphi_\Lambda(g), \qquad g \in G,\ \chi(\Lambda) = \langle\Lambda, \Lambda + 2\varrho\rangle\,.$$

It is clear that $\chi(\Lambda+x_i)=\chi(\Lambda+x_j)$, $i\neq j$, if and only if Λ belongs to an affine hyperplane τ_{ij} in $\mathfrak{h}_{p_0}$. Denote the complement in $\mathbf{Z}^l_+$ of the hyperplanes τ_{ij}, $i\neq j$, $i,j=0,\ldots,p$ by $\bar{\mathbf{Z}}^l_+$ and put

$$\bar{\mathbf{Z}}^l_+(M) = \mathfrak{h}_{p_0}(M) \cap \bar{\mathbf{Z}}^l_+ .$$

For $\Lambda \in \bar{\mathbf{Z}}^l_+(M)$ we write the recurrence formula in theorem 2.7 as

$$\varphi_\mu\varphi_\Lambda = \sum_{i=0}^{p} c_{x_i}(\Lambda)\varphi_{\Lambda+x_i}$$

where $c_{x_i}(\Lambda)\neq 0$ only if $-\mu'\preceq x_i\preceq\mu$.

LEMMA 4.1. *There exist rational functions $d_{x_i}(\lambda)$, $\lambda \in \mathfrak{h}_p$, $i=0,\ldots,p$ such that*

$$c_{x_i}(\Lambda) = d_{x_i}(\Lambda+\varrho) \quad \textit{if } \Lambda \in \bar{\mathbf{Z}}^l_+(M) .$$

Moreover $d_{x_i}(S\lambda)=d_{S^{-1}x_i}(\lambda)$, $S \in W$.

PROOF. Applying succesively the operators D^j, $j=0,\ldots,p$ to the recurrence formula and putting $g=e$, we obtain since $\Phi_\lambda(e)=1$ an equation system for $c_{x_i}(\Lambda)$ with the non-zero determinant

$$\prod_{0\leqq k<l\leqq p} (\chi(\Lambda+x_l)-\chi(\Lambda+x_k)) .$$

The solution of this system is

$$c_{x_i}(\Lambda) = \sum_{j=0}^{p} a^i_j(\Lambda)(D^j\varphi_\mu\varphi_\Lambda)(e)$$

where $a^i_j(\Lambda)$ is the coefficient of z^j in the polynomial

$$\prod_{\substack{k=0\\k\neq i}}^{p} (z-\chi(\Lambda+x_k))(\chi(\Lambda+x_i)-\chi(\Lambda+x_k))^{-1} .$$

It follows from lemma 46 in [2] that $(D^j\varphi_\mu\Phi_\lambda)(e)$ is analytic on $\mathfrak{h}_p$ and bounded by a polynomial. Hence it must be a polynomial itself. Extending the rational functions a^i_j to $\mathfrak{h}_p$ in the natural way we see that the functions

$$d_{x_i}(\lambda) = \sum_{j=0}^{p} a^i_j(\lambda-\varrho)(D^j\varphi_\mu\Phi_{-i\lambda})(e)$$

have the desired properties. Note that $\Phi_{S\lambda}=\Phi_\lambda$, $S \in W$.

COROLLARY 4.2. *Let W_x be the isotropy group of x in W and let Γ be the set of all $x \in \mathbf{Z}^l_+$ satisfying $-\mu' \preceq x \preceq \mu$. Then*

$$\varphi_\mu \varphi_\Lambda = \sum_{\substack{S \in W/W_x \\ x \in \Gamma}} d_x(S(\Lambda+\varrho))\varphi_{\Lambda+S^{-1}x}$$

for all $\Lambda \in \bar{\mathbf{Z}}^l_+(M)$.

Let us write φ_λ instead of $\Phi_{-i(\lambda+\varrho)}$ for all $\lambda \in \mathfrak{h}_\mathfrak{p}$ and put for a fixed $H \in \mathfrak{h}_{\mathfrak{p}_0}(0)$

$$f'(\lambda) = \varphi_\mu(\exp H)\varphi_\lambda(\exp H) - \sum_{\substack{x \in \Gamma \\ S \in W/W_x}} d_x(S(\lambda+\varrho))\varphi_{\lambda+S^{-1}x}(\exp H)\,.$$

By analytic continuation we will show that $f'(\lambda)=0$ not only on $\bar{\mathbf{Z}}^l_+(M)$ but for all $\lambda \in \mathfrak{h}_\mathfrak{p}$ for which the rational functions $d_x(S(\lambda+\varrho))$ are defined. To avoid singularities we multiply $f'(\lambda)$ by a polynomial $A(\lambda)$ such that $A(\lambda)f'(\lambda)$ is analytic on $\mathfrak{h}_\mathfrak{p}$. Let $B(\lambda)$ be a polynomial which is zero on the hyperplanes τ_{ij}, $i,j=0,\ldots,l$, used in the definition of $\bar{\mathbf{Z}}^l_+$. Then

$$f(\lambda) = e^{-\lambda(H)}B(\lambda)A(\lambda)f'(\lambda)$$

is analytic function on $\mathfrak{h}_\mathfrak{p}$ which is zero on $\mathbf{Z}^l_+(M)=\mathbf{Z}^l_+ \cap \mathfrak{h}_{\mathfrak{p}_0}(M)$. The behaviour of $f(\lambda)$ at infinity follows from.

LEMMA 4.3. *For any $H \in \mathfrak{h}_{\mathfrak{p}_0}(0)$ and any $\eta \in \mathfrak{h}_\mathfrak{p}$ the function $e^{-\lambda(H)}\varphi_{\lambda+\eta}(\exp H)$ is bounded if* $\operatorname{Re}\lambda \in \mathfrak{h}_{\mathfrak{p}_0}(0)$.

PROOF. Immediate consequence of the definition of Φ_λ and the fact that $\cdot(H(\exp Hk)) \leq \nu(H)$ for all $\nu, H \in \mathfrak{h}_{\mathfrak{p}_0}(0)$ and all $k \in K$, (see [2, lemma 35]).

COROLLARY 4.4. *$f(\lambda)$ is bounded by a polynomial if* $\operatorname{Re}\lambda \in \mathfrak{h}_{\mathfrak{p}_0}(0)$.

To see that $f(\lambda)$ is identically zero we use

Carlson's THEOREM (see [8, p. 186]). *Assume that $g(z)$ is an analytic function of one complex variable z such that*

$$g(z) = \begin{cases} O(e^{a|z|}) & 0<a<\pi \quad \text{if } \operatorname{Re} z \geqq 0 \\ 0 & \phantom{0<a<\pi} \quad \text{if } z=0,1,2,\ldots\,. \end{cases}$$

Then $g(z)$ is identically zero.

THEOREM 4.5. *The following recurrence formula holds for all $g \in G$ and all $\lambda \in \mathfrak{h}_\mathfrak{p}$ which are not singularities of the coefficients.*

$$\Phi_{i-(\mu+\varrho)}(g)\cdot\Phi_\lambda(g) = \sum_{\substack{x \in \Gamma \\ S \in W/W_x}} d_x(iS\lambda)\Phi_{\lambda-iS^{-1}x}(g)\,.$$

PROOF. We have to verify that $f(\lambda)$ is identically zero on $\mathfrak{h}_{\mathfrak{p}}$. Put $g_1(z_1) = f(z_1\mu_1 + \Lambda)$. By corollary 4.4, g_1 fulfills the assumptions in Carlson's theorem. Hence $f(z_1\mu_1 + \Lambda) = 0$ for all complex numbers z_1 and all $\Lambda \in \mathbf{Z}^l_+(M)$. Next putting $g_2(z_2) = f(z_1\mu_1 + z_2\mu_2 + \Lambda)$ another application of Carlson's theorem yields that $f(z_1\mu_1 + z_2\mu_2 + \Lambda) = 0$ for all complex numbers z_1 and z_2 and all $\Lambda \in \mathbf{Z}^l_+(M)$. After l steps we arrive to the desired conclusion.

It is possible to express some of the coefficients in the recurrence formula in terms of Harish–Chandra's c-function. More precisely we have

LEMMA 4.6.

$$d_\mu(\lambda) = \frac{c(-i(\mu+\varrho))c(-i\lambda)}{c(-i(\lambda+\mu))}.$$

PROOF. Fix a $H \in \mathfrak{h}_{\mathfrak{p}_0}(0)$. Then

$$\lim_{t\to\infty} e^{(\varrho - i\lambda)(tH)}\Phi_\lambda(\exp tH) = c(\lambda)$$

for all $\lambda \in \mathfrak{h}_{\mathfrak{p}}$ with $-\operatorname{Im}\lambda \in \mathfrak{h}_{\mathfrak{p}_0}(0)$ except for certain affine hyperplanes (see [2, p. 291]). It is however an easy consequence of Vitali's convergence theorem (see [8 p. 168]) and corollary 1 to lemma 28 in [2] that this holds on the hyperplanes as well, that is, for all $\lambda \in \mathfrak{h}_{\mathfrak{p}}$ with $-\operatorname{Im}\lambda \in \mathfrak{h}_{\mathfrak{p}_0}(0)$. Applying this to the recurrence formula multiplied by $e^{-\mu+\varrho-i\lambda}$ the lemma follows.

COROLLARY 4.7. *For any $\alpha \in \Delta_0^+$ let $m(\alpha)$ be the number of roots in Δ^+, whose restriction to $\mathfrak{h}_{\mathfrak{p}_0}$ is α. Then*

$$d_\mu(\lambda) = \frac{b(\lambda)}{b(\varrho)}$$

where

$$b(\lambda) = \prod_{\alpha\in\Delta_0^+} \prod_{0\leq k\leq\langle\mu,\alpha\rangle/\langle\alpha,\alpha\rangle-1} \frac{\frac{1}{2}m(\alpha)+\frac{1}{4}m(\alpha/2)+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle+k}{\frac{1}{4}m(\alpha/2)+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle+k}.$$

PROOF. Follows from the explicit expression for $c(\lambda)$

$$c(\lambda) = \frac{I(i\lambda)}{I(\varrho)}$$

where

$$I(\lambda) = \prod_{\alpha\in\Delta_0^+} \beta(\tfrac{1}{2}m(\alpha), \tfrac{1}{4}m(\alpha/2)+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle),$$

and β denotes the Beta function.

Let $<$ be a total ordering of $\mathfrak{h}_{\mathfrak{p}_0}$ such that $x \prec y$ implies that $x < y$ and let μ_0 be the lowest one of $\mu_1, \ldots, \mu_l$ with respect to this ordering. In this case Γ consists of the two points 0 and μ_0 only, so the only non-zero coefficient in the recurrence formula except for $d_{s\mu_0}$, $s \in w$ is d_0. Moreover d_0 is determined by putting $g = e$.

THEOREM 4.8. *Let* μ_0 *be as above. Then*

$$\Phi_{-i(\mu_0+\varrho)}(g)\Phi_\lambda(g) = \sum_{S \in W/W_{\mu_0}} d_{\mu_0}(iS\lambda)\Phi_{\lambda - iS^{-1}\mu_0}(g) + d_0(i\lambda)\Phi_\lambda$$

where

$$d_{\mu_0}(\lambda) = \frac{c(-i(\mu_0+\varrho))c(-i\lambda)}{c(-i(\lambda+\mu_0))}$$

and

$$d_0(\lambda) = 1 - \sum_{S \in W/W_{\mu_0}} d_{\mu_0}(iS\lambda).$$

REMARK. Let U be a compact connected semisimple Lie group and U' the universal covering group of U. Then $U = U'/T$ for some subgroup T of U' (see [3, p. 274]), and it is easy to see that the elementary spherical functions on U coincide with those on U' for which $\varphi_\Lambda(u't) = \varphi'_\Lambda(u')$, $u' \in U'$, $t \in T$. Assume that φ_{Λ_1} and φ_{Λ_2} are such functions. By corollary 2.6 we have

$$(\varphi_{\Lambda_1}\varphi_{\Lambda_2} * \varphi_\nu)(u') = c_\nu(\Lambda_1, \Lambda_2)\varphi_\nu(u')(\varphi_\nu, \varphi_\nu)$$

from which is seen that $\varphi_\nu(u't) = \varphi_\nu(u')$ if $c_\nu(\Lambda_1, \Lambda_2) \neq 0$. We conclude that the elementary spherical functions on U also satisfies recurrence formulas with uniformly bounded number of terms.

When we replace a non-compact connected semisimple Lie group G with finite center by the universal covering group of G we don't change the set of elementary spherical functions. Hence the recurrence formulas hold also in this case.

REFERENCES

1. E. Cartan, *Sur la détermination d'un système orthogonal complet dans un espace de Riemann symmetrique clos*, Rend. Circ. Mat. Palermo 53 (1929), 217–252.
2. Harish–Chandra, *Spherical functions on a semisimple Lie group* I, Amer. J. Math. 80 (1958), 241–310.
3. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
4. T. H. Koornwinder, *Orthogonal polynomials in two variables which are eigenfunctions of two independent partial differential operators* I–IV, Indag. Math. 36 (1974), 48–66 and 357–381.

5. I. G. Sprinkhuizen, *Orthogonal polynomials in two variables. A further analysis of the polynomials orthogonal on a region bounded by two lines and a parabola*, Math. Centrum Amsterdam Rep TW 144, 1974.
6. M. Sugiura, *Representations of compact groups realized by spherical functions on symmetric spaces*, Proc. Japan Acad. 38 (1962), 111–113.
7. M. Takeuchi, *Polynomial representations associated with symmetric bounded domains*, Osaka J. Math. 10 (1973), 441–475.
8. E. C. Titchmarsh, *The theory of functions*, Oxford University Press, London, second ed., 1939.
9. G. Warner, *Harmonic analysis on semisimple Lie groups* I *and* II (Grundlehren Math. Wissensch. 188 and 189), Springer-Verlag, Berlin · Heidelberg · New York, 1972.
10. H. Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen*, Math. Z. 23 (1925), 271–309; 24 (1926), 328–395, 789–791.

UNIVERSITY OF LUND, SWEDEN