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INFINITESIMAL CHARACTERIZATION OF ANALYTIC VECTORS FOR REPRESENTATIONS OF REAL LIE GROUPS ON LOCALLY CONVEX SPACES

BJARNE SELAND

Introduction.

Let G be a real and connected Lie group and let $g \to \pi(g)$ be a locally equicontinuous representation of G on a complete Hausdorff locally convex vector space E over $\mathbf{C}$. We give an infinitesimal characterization of the analytic vectors for π. In the last part of section 2 we define the notion of entire vectors for π. To illustrate the theory we construct representations π_λ of the Heisenberg group of dimension $2d+1$ on the space of distributions on $\mathbf{R}^d$. For $\lambda \in \mathbf{C}\setminus\{0\}$ π_λ has a dense subspace of entire vectors.

1. Notations.

Let M be a differentiable manifold of dimension d and E a complete Hausdorff locally convex vector space over the field $\mathbf{C}$. We denote the space of smooth functions from M to E by $C^\infty(M,E)$. $C_c^\infty(M,E)$ is the subspace of $C^\infty(M,E)$ consisting of the functions with compact support.

Let $\mathbf{N}^d$ be the set of all d-tuples $\alpha=\{\alpha_1,\ldots,\alpha_d\}$ of non-negative integers. For all $\alpha \in \mathbf{N}^d$, we set $|\alpha|=\alpha_1+\ldots+\alpha_d$ and $\alpha!=\alpha_1!\ldots\alpha_d!$. If $\alpha,\beta \in \mathbf{N}^d$, we define $\alpha+\beta=(\alpha_1+\beta_1,\ldots,\alpha_d+\beta_d)$ and $\binom{\alpha}{\beta}=\alpha!/\beta!(\alpha-\beta)!$. The notation $\beta\leqq\alpha$ means $\beta_j\leqq\alpha_j$ for all $j=1,\ldots,d$.

We put

$$D_j = \frac{\partial}{\partial x_j}, \quad j=1,\ldots,d ,$$

and $D^\alpha=D_1^{\alpha_1}\ldots D_d^{\alpha_d}$. If $x=(x_1,\ldots,x_d) \in \mathbf{R}^d$, we define

$$|x| = \max\{|x_i| : i=1,\ldots,d\} .$$

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2. Characterization of analytic vectors.

Let G be a real connected Lie group with Lie algebra $\mathfrak{g}$ and let $g \to \pi(g)$ be a locally equicontinuous representation of G on a complete Hausdorff locally convex vector space E over $\mathbf{C}$. A vector $v \in E$ is called a C^∞-vector for the representation π if the mapping $g \to \tilde{v}(g) \equiv \pi(g)v$ is smooth from G to E. The subspace of C^∞-vectors for π will be denoted by $E^\infty(\pi)$. We have a representation $\partial\pi$ of $\mathfrak{g}$ on $E^\infty(\pi)$ given by

$$\partial\pi(X)v = \frac{d}{dt}\pi(\exp tX)v\bigg|_{t=0}, \qquad X \in \mathfrak{g}, \qquad v \in E^\infty(\pi). \tag{1}$$

The representation $\partial\pi$ extends uniquely to a representation of the universal enveloping algebra $U(\mathfrak{g}(\mathbf{C}))$ of the complexification $\mathfrak{g}(\mathbf{C})$ of $\mathfrak{g}$, which we also denote by $\partial\pi$.

Let p be any continuous seminorm on E and n a non-negative integer. For $v \in E^\infty(\pi)$ we define

$$p_n(v) = \sum_{1 \leq j_1,\ldots,j_n \leq d} p(\partial\pi(X_{j_1}\ldots X_{j_n})v), \qquad n = 1,2,\ldots, \tag{2}$$

$$p_0(v) = p(v)$$

where $\{X_1,\ldots,X_d\}$ is a fixed basis for $\mathfrak{g}$. The vectors $1\otimes X_1,\ldots,1\otimes X_d$ form a basis for $\mathfrak{g}(\mathbf{C})$, and if $X = \xi_1(1\otimes X_1)+\ldots+\xi_d(1\otimes X_d)$ we define

$$|X| = \max\{|\xi_k| : k = 1,\ldots,d\}.$$

Then for $v \in E^\infty(\pi)$ we have

$$p_n(\partial\pi(X^m)v) \leq |X|^m p_{n+m}(v). \tag{3}$$

Let $\tilde{X}$ denote the left invariant differential operator corresponding to $X \in \mathfrak{g}$. For any $v \in E^\infty(\pi)$ we have

$$\tilde{X}_{j_1}\ldots\tilde{X}_{j_n}\tilde{v}(g) = \pi(g)\partial\pi(X_{j_1}\ldots X_{j_n})v, \qquad g \in G \tag{4}$$

for all $n \in \mathbf{N}$ and $1 \leq j_1,\ldots,j_n \leq d$. [6, page 258].

Since π is a locally equicontinuous representation we obtain from (4) that the topology on $E^\infty(\pi)$ described by the family of seminorms $\{p_n\}$ coinsides with the standard topology induced from $C^\infty(G,E)$, which is complete [6, page 253].

A vector $v \in E$ is an analytic vector for π if the function $\tilde{v}(g) \equiv \pi(g)v$ is a real analytic function from G to E. Denoting the subspace of analytic vectors for π by $E^\omega(\pi)$, we have the inclusion $E^\omega(\pi) \subset E^\infty(\pi)$.

THEOREM 1. *A vector $v \in E^\infty(\pi)$ is analytic if and only if there exists a positive constant t such that*

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} p_n(v) < \infty$$

for all $0<s<t$ *and all continuous seminorms* p *on* E.

To prove Theorem 1 we need some results. For each $0<p\leqq 1$, let φ_p denote the map defined by

$$\varphi_\varrho(x_1,\ldots,x_d) = (\varrho - x_1 - \ldots - x_d)^{-1}.$$

One calculates easily that

$$D^\alpha \varphi_\varrho(x) = |\alpha|!\,(\varphi_\varrho(x))^{|\alpha|+1} \tag{5}$$

$$D^\alpha(\varphi_\varrho^{n+1})(x) \geqq D^\alpha(\varphi_\varrho^n)(x), \qquad n=0,1,2,\ldots \tag{6}$$

for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ with $|x_1|+\ldots+|x_d|\leqq 1-\varrho$.

Let (ψ, U) be an analytic chart at the identity e of G with $\psi(e)=(0,\ldots,0)$. Then

$$\bar{X}_i = \sum_{j=1}^{d} a_{ij}(x) D_j, \qquad i=1,\ldots,d$$

where $a_{ij}(x)$ are analytic maps defined in $\Omega=\psi(U)$. Using (5) we can find $\varrho=\varrho_0$ so small that

$$|D^\alpha a_{ij}(0)| \leqq D^\alpha \varphi_{\varrho_0}(0), \qquad 1\leqq i,j\leqq d \tag{7}$$

for all $\alpha \in \mathbb{N}^d$. We set $\varphi=\varphi_{\varrho_0}$.

Let $\boldsymbol{D}(\Omega,E)$ denote the algebra generated by all differential operators in $C^\infty(\Omega,E)$. Following Nelson [5, page 573] we define a semialgebra of *absolute operators*. If A is an element of $\boldsymbol{D}(\Omega,E)$ we define the absolute operator $|A|$ of A to be the set consisting of A alone. Let $|\boldsymbol{D}(\Omega,E)|$ be the free abelian semigroup with the set of all $|A|$ as generators, $A \in \boldsymbol{D}(\Omega,E)$. That is, a typical element of $|\boldsymbol{D}(\Omega,E)|$ is a finite formal sum. If $\boldsymbol{a} \in C^\infty(\Omega,\mathbb{R})$ is a positive function we identify $\boldsymbol{a}$ with $|\boldsymbol{a}I|$, where I is the identify operator in $C^\infty(\Omega,E)$. The product of $v=|A_1|+\ldots+|A_m|$ and $\tau=|B_1|+\ldots+|B_n|$ is given by

$$v\tau = \sum_{i=1}^{m}\sum_{j=1}^{n} |A_i B_j|.$$

Next, we define a preordering $\ll$ in $|\boldsymbol{D}(\Omega,E)|$ by putting $v\ll\tau$ in case

$$p(A_1 f(0))+\ldots+p(A_m f(0)) \leqq p(B_1 f(0))+\ldots+p(B_n f(0))$$

for all $f \in C^\infty(\Omega,E)$ and all continuous seminorms p on E. With each $\alpha \in \mathbb{N}^d$ we associate a linear operator D^α in $|\boldsymbol{D}(\Omega,E)|$ defined by

$$D^{\alpha}\left(\left|\sum_{\gamma\in\mathbf{N}^d} a_\gamma(x)D^\gamma\right|\right) = \sum_{\gamma\in\mathbf{N}^d}\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}|D^\beta a_\gamma(x)D^{\alpha-\beta}D^\gamma|$$

where $\sum_{\gamma\in\mathbf{N}^d} a_\gamma(x)D^\gamma \in \boldsymbol{D}(\Omega, E)$.

LEMMA 2. *For $n=1,2,\ldots$ we have*

$$|D^\alpha \bar{X}_{j_1}\ldots\bar{X}_{j_n}| \ll D^\alpha\left(n!\,(4d\varphi^2)^n \sum_{1\leqq|\gamma|\leqq n}\frac{1}{\gamma!}|D^\gamma|\right)$$

for all $\alpha\in\mathbf{N}^d$ and $1\leqq j_1,\ldots,j_n\leqq d$.

PROOF. We prove the claim by induction on n. For $\alpha\in\mathbf{N}^d$ and $i=1,\ldots,d$ we have

$$|D^\alpha\bar{X}_i| = \left|D^\alpha\left(\sum_{k=1}^d a_{ik}D_k\right)\right| = \left|\sum_{k=1}^d\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta a_{ik}D^{\alpha-\beta}D_k\right|.$$

Because of (7) and (6) $|D^\beta a_{ik}(0)|\leqq D^\beta\varphi^2(0)$ for all $\beta\in\mathbf{N}^d$ and $1\leqq i,k\leqq d$. Hence we get

$$|D^\alpha\bar{X}_i| \ll \sum_{k=1}^d\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}|D^\beta\varphi^2D^{\alpha-\beta}D_k|$$

$$= D^\alpha(|\varphi^2D_1|+\ldots+|\varphi^2D_d|) \ll D^\alpha\left(4d\varphi^2\sum_{|\gamma|=1}\frac{1}{\gamma!}|D^\gamma|\right).$$

Next suppose the claim is true for some $n\geqq1$. For $\alpha\in\mathbf{N}^d$ and $i=1,\ldots,d$ we have

$$|D^\alpha\bar{X}_i\bar{X}_{j_1}\ldots\bar{X}_{j_n}| = \left|D^\alpha\left(\sum_{k=1}^d a_{ik}D_k\right)\bar{X}_{j_1}\ldots\bar{X}_{j_n}\right|$$

$$= \left|\sum_{k=1}^d\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta a_{ik}D^{\alpha-\beta}D_k\bar{X}_{j_1}\cdots\bar{X}_{j_n}\right|$$

$$\ll \sum_{k=1}^d\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta\varphi|D^{\alpha-\beta}D_k\bar{X}_{j_1}\ldots\bar{X}_{j_n}|$$

$$\ll \sum_{k=1}^d\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta\varphi D^{\alpha-\beta}D_k\left(n!\,(4d\varphi^2)^n\sum_{1\leqq|\gamma|\leqq n}\frac{1}{\gamma!}|D^\gamma|\right)$$

$$= \sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta\varphi D^{\alpha-\beta}\left(n!\,n(4d\varphi^2)^{n-1}8d^2\varphi^3\sum_{1\leqq|\gamma|\leqq n}\frac{1}{\gamma!}|D^\gamma|+\right.$$

$$\left.+n!\,(4d\varphi^2)^n\sum_{1\leqq|\gamma|\leqq n}\frac{1}{\gamma!}(|D_1D^\gamma|+\ldots+|D_dD^\gamma|)\right).$$

Since $D^\beta\varphi^m(0) \leqq D^\beta\varphi^{m+1}(0)$ for all $\beta \in \mathbf{N}^d$ and $m=1,2,\ldots$, we obtain

$$|D^\alpha \bar{X}_i \bar{X}_{j_1} \ldots \bar{X}_{j_n}| \ll \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta}\left((n+1)!\,(4d)^{n+1}\varphi^{2n+1} \sum_{1 \leqq |\gamma| \leqq n+1} \frac{1}{\gamma!}|D^\gamma|\right)$$
$$= D^\alpha\left((n+1)!\,(4d\varphi^2)^{n+1} \sum_{1 \leqq |\gamma| \leqq n+1} \frac{1}{\gamma!}|D^\gamma|\right).$$

Proof of Theorem 1. Suppose $v \in E^\omega(\pi)$, and let p be any continuous seminorm on E. Since $f = \tilde{v} \circ \psi^{-1}$ is analytic at the origin, there exists a constant $t_0 > 0$ independent of p and a constant $M > 0$ such that

$$\frac{1}{\gamma!} p(D^\gamma f(0)) t_0^{|\gamma|} \leqq M$$

for all $\gamma \in \mathbf{N}^d$. Using the identity (4) and setting $\alpha = (0,\ldots,0)$ in Lemma 2 we get

$$p_n(v) = \sum_{1 \leqq j_1,\ldots,j_n \leqq d} p(\bar{X}_{j_1} \ldots \bar{X}_{j_n}\tilde{v}(e))$$
$$\leqq d^n n!\,(4d\varphi^2(0))^n \sum_{1 \leqq |\gamma| \leqq n} \frac{1}{\gamma!} p(D^\gamma f(0))$$
$$\leqq n!\,(4d^2\varrho_0^{-2})^n \sum_{1 \leqq |\gamma| \leqq n} M t_0^{-|\gamma|}$$

for $n=1,2,\ldots$. Then it follows that

$$\sum_{n=0}^{\infty} \frac{1}{n!} p_n(v) s^n < \infty$$

for all $0 < s < \frac{1}{4} d^{-2} \varrho_0^2 \cdot \min\{1, t_0\}$.

Conversely, suppose there exists a constant $t > 0$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} p_n(v) s^n < \infty \tag{8}$$

for all $0 < s < t$ and all continuous seminorms p on E. Since the mapping

$$x \to e(x) = \exp(x_1 X_1) \ldots \exp(x_d X_d)$$

is an analytic diffeomorphism from an open neighbourhood Ω of 0 in $\mathbf{R}^d$ to an open neighbourhood of e in G, it is sufficient to prove that $F(x) = \pi(e(x))v$ is analytic at the origin.

For any $x \in \Omega$ and $\alpha \in \mathbf{N}^d$ one has the formula [1, page 62]

$$D^\alpha F(x) = \pi(e(x))\partial\pi(Z_1(x))^{\alpha_1} \ldots \partial\pi(Z_{d-1}(x))^{\alpha_{d-1}}\partial\pi(X_d)^{\alpha_d} v \tag{9}$$

where

$$Z_j(x) = \mathrm{Ad}\,(\exp(x_{j+1}X_{j+1})\ldots\exp(x_dX_d))^{-1}X_j, \quad j=1,\ldots,d-1 .$$

We choose $\varepsilon>0$ such that $\{x \in \mathbf{R}^d : |x|\leqq\varepsilon\}\subset\Omega$, and let φ be any continuous linear functional on E. Since π is locally equicontinuous there exists a continuous seminorm p on E such that

$$\text{(10)} \qquad |\langle D^\alpha F(x),\varphi\rangle| \leqq p\big(\partial\pi(Z_1(x))^{\alpha_1}\ldots\partial\pi(Z_{d-1}(x))^{\alpha_{d-1}}\partial\pi(X_d^{\alpha_d})v\big)$$

for all $|x|\leqq\varepsilon$ and $\alpha \in \mathbf{N}^d$. The mappings $x \to Z_j(x)$ are continuous from Ω to $\mathfrak{g}$. Hence there exists a constant M_ε such that $|Z_j(x)|\leqq M_\varepsilon$ for all $|x|\leqq\varepsilon$, $j=1,\ldots,d-1$. Combining (10) and (3) we obtain that

$$|\langle D^\alpha F(x),\varphi\rangle| \leqq M_\varepsilon^{|\alpha|}p_{|\alpha|}(v)$$

for all $|x|\leqq\varepsilon$ and $\alpha \in \mathbf{N}^d$. From (8) it follows that F is $\sigma(E,E')$-analytic at the origin. Hence analytic by Lemma 3 of [4, Chapter 6].

A vector $v \in E^\infty(\pi)$ is called an entire vector for the representation π if

$$\sum_{n=0}^{\infty}\frac{1}{n!}p_n(v)s^n < \infty$$

for all $s>0$ and all continuous seminorms p on E. Let $E_\infty^\omega(\pi)$ denote the subspace of E consisting of entire vectors for π. We give $E_\infty^\omega(\pi)$ the topology described by the family of seminorms

$$P_{s,p}(v) = \sum_{n=0}^{\infty}\frac{s^n}{n!}p_n(v), \qquad s>0,\ p \in \Lambda$$

where Λ is the set of all continuous seminorms on E. The inclusion i: $E_\infty^\omega(\pi) \to E^\infty(\pi)$ is continuous, and it is easy to show that $E_\infty^\omega(\pi)$ is complete

Let $v \in E_\infty^\omega(\pi)$ and $X \in \mathfrak{g}(\mathbf{C})$. Then the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\partial\pi(X^n)v$$

converges absolutely in $E^\infty(\pi)$ and we define

$$\mathrm{Exp}\,\partial\pi(X)v = \sum_{n=0}^{\infty}\frac{1}{n!}\partial\pi(X^n)v .$$

Because of (3) $E_\infty^\omega(\pi)$ is invariant under $\mathrm{Exp}\,\partial\pi(X)$ and $\mathrm{Exp}\,\partial\pi(X)$ is a continuous linear map. Furthermore, for each $v \in E_\infty^\omega(\pi)$ the function $X \to \mathrm{Exp}\,\partial\pi(X)v$, $X \in \mathfrak{g}(\mathbf{C})$, is continuous.

Let G^c be the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}(\mathbf{C})$. Fix $\varepsilon>0$ such that $X \to \exp X$ is bijective from $\{X \in \mathfrak{g}(\mathbf{C}) : |X|<\varepsilon\}$ to

$U \subseteq G^c$. Then for $g = \exp X$ with $|X| < \varepsilon$ we define

$$\pi^\omega(g)v = \operatorname{Exp} \partial\pi(X)v, \quad v \in E_\infty^\omega(\pi)$$

π^ω is a local representation of G^c on $E_\infty^\omega(\pi)$ [1, Proposition 2.3].

There exists a neighbourhood V of the origin in $\mathfrak{g}$ [2, page 95] such that

$$\pi(\exp X)v = \operatorname{Exp} \partial\pi(X)v$$

for all $X \in V$ and $v \in E_\infty^\omega(\pi)$. Hence the closure of $E_\infty^\omega(\pi)$ in E is invariant under π, so that if $E_\infty^\omega(\pi) \neq (0)$ and π is topologically irreducible, $E_\infty^\omega(\pi)$ must be dense in E.

EXAMPLE. Let $\mathfrak{g}$ be the real Heisenberg algebra of dimension $2d+1$. We choose a basis $\{X_1, \ldots, X_{2d+1}\}$ for $\mathfrak{g}$ with commutation relations $[X_i, X_{d+i}] = X_{2d+1}$, $i = 1, \ldots d$, $[X_i, X_j] = 0$ for $1 \leqq i, j \leqq d$, $[X_i, X_j] = 0$ for $d+1 \leqq i, j \leqq 2d$ and $[X_i, X_{2d+1}] = 0$ for $i = 1, \ldots, 2d$. The Heisenberg group G of dimension $2d+1$ is the connected and simply connected Lie Group which corresponds to $\mathfrak{g}$. For each $\lambda \in \mathbf{C}$ we may realize a representation V_λ of G on $C_c(\mathbf{R}^d, \mathbf{C})$ with

$$[V_\lambda(\exp(\xi_1 X_1 + \ldots + \xi_d X_d))f](x) = f(x_1 - \xi_1, \ldots, x_d - \xi_d)$$

$$[V_\lambda(\exp(\xi_1 X_{d+1} + \ldots + \xi_d X_{2d}))f](x) = \exp(\lambda\xi_1 x_1 + \ldots + \lambda\xi_d x_d)f(x)$$

$$[V_\lambda(\exp(\xi X_{2d+1}))f](x) = \exp(\lambda\xi)f(x)\,.$$

We denote the space pf distributions on $\mathbf{R}^d$ by E. For each $\lambda \in \mathbf{C}$ we have a representation π_λ of G on E defined by

$$\pi_\lambda(g)T(\varphi) = \langle T, V_\lambda(g^{-1})\varphi\rangle, \quad g \in G,\ T \in E \text{ and } \varphi \in C_c^\infty(\mathbf{R}^d, \mathbf{C})$$

π_λ is locally equicontinuous. Evidently, $\partial\pi_\lambda(X_i) = D_i$, $\partial\pi_\lambda(X_{d+i}) =$ multiplication by λx_i, $i = 1, \ldots, d$, and $\partial\pi_\lambda(X_{2d+1}) =$ multiplication by λ. We claim that $f(x) \equiv 1$ is an entire vector for π_λ.

Let B be any bounded subset of $C_c^\infty(\mathbf{R}^d, \mathbf{C})$. We choose a constant $K > 0$ such that the support of φ is contained in $\{x \in \mathbf{R}^d : |x| \leqq K\}$ for all $\varphi \in B$. Setting $M = \max\{1, |\lambda|, K\}$ we get

$$(11) \qquad |\partial\pi(X_{j_1} \ldots X_{j_{2m}})f(x)| \leqq (2m-1)(2m-3) \ldots 3M^{2m}$$

$$(12) \qquad |\partial\pi(X_{j_1} \ldots X_{j_{2m+1}})f(x)| \leqq (2m)(2m-2) \ldots 2M^{2m+1}$$

for all x with $|x| \leqq K$ and all $1 \leqq j_1, \ldots, j_{2m+1} \leqq 2d+1$, $m = 1, 2, \ldots$. Using the estimates (11) and (12) we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqq j_1, \ldots, j_n \leqq 2d+1} \sup\{|\langle\partial\pi_\lambda(X_{j_1} \ldots X_{j_n})f, \varphi\rangle| : \varphi \in B\} s^n < \infty$$

for all $s > 0$.

Lemma 6 and Theorem 4 of [3] imply that π_λ is topologically irreducible if $\lambda \neq 0$. Hence $E^{\omega}_{\infty}(\pi_\lambda)$ must be dense in E when $\lambda \neq 0$.

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UNIVERSITY OF TRONDHEIM
NORWAY