MATH. SCAND. 39 (1976), 367–370

OPERATORS WITH THIN SPECTRAL SETS

RUDOLF A. HIRSCHFELD

1.

Let T be a bounded linear operator in a complex Hilbert space H and $\Lambda \subset \mathbf{C}$ a compact set containing the spectrum $\sigma(T)$ of T. Denote by $R(\Lambda)$ the algebra of those rational functions on $\mathbf{C}$ that have their poles off Λ, endowed with the sup-norm

$$\|u\|_\Lambda = \max\{|u(\lambda)| : \lambda \in \Lambda\} .$$

For every $u \in R(\Lambda)$ the map $u(T)$ is safely defined by a mere substitution and it represents a bounded linear operator in H. Using von Neumann's terminology (see [3, §§ 154, 155]) in a slightly modified way, we shall say that Λ is a *spectral set* for T if

$$M = \sup\{\|u(T)\| : u \in R(\Lambda) \text{ and } \|u\|_\Lambda \leqq 1\}$$

is a finite number. The original definition of von Neumann's corresponds to $M=1$.

The intersection of all spectral sets (with $M=1$) for T equals the spectrum of T, but this does by no means imlies that $\sigma(T)$ is a spectral set. If T is normal, however, then $\sigma(T)$ is readily seen to be a spectral set. In [1, p. 933] it is stated that, conversely, if $\sigma(T)$ is a spectral set, then T is a normal operator. This is erroneous (unless $\dim H < \infty$, see [3, p. 440]). We shall give two simple counterexamples that, in conjunction, will suggest a correct statement.

First, if T is similar to a normal operator N, the similarity being implemented by a boundedly invertible operator S, then $T=SNS^{-1}$ need *not* by normal, but $\sigma(T)$ $(=\sigma(N))$ is a spectral set for T (with $M \leqq \|S\| \cdot \|S^{-1}\|$), because $u(T)=Su(N)S^{-1}$ for $u \in R(\Lambda)$. On the other hand, the unilateral shift, with spectrum the closed unit disc, does admit its spectrum as a spectral set (with $M=1$) by virtue of the von Neumann inequality for contractions. But this shift is similar to no normal operator. Explanation: the spectrum of this shift is too big. Looking into the opposite direction, a compact set $\Lambda \subset \mathbf{C}$ will be called *thin* if the above algebra $R(\Lambda)$ is dense in the familiar Banach algebra $C(\Lambda)$. The characterization of thin sets still is a major open problem in the

Received February 5, 1976.

theory of rational approximation, [4]. Suffice it to state that sets of planar measure zero are thin.

The present note aims to establish the following result.

THEOREM. *Let T be a bounded-linear operator in a complex Hilbert space H, possessing a spectral set Λ which is thin. Then T is similar to a normal operator.*

REMARK. For the application we have in mind (to appear elsewhere) Λ often has the property that the polynomials p constitute a dense set $P(\Lambda)$ of $C(\Lambda)$. If that happens it is important to notice that the theorem remains valid if we replace the assumption that the thin set Λ be a spectral set by $\sigma(T) \subset \Lambda$ and

$$\sup\{\|p(T)\| : p \in P(\Lambda) \text{ and } \|p\|_\Lambda \leqq 1\} < \infty .$$

2.

We start with the case that $\sigma(T) = \Lambda$. The following lemma is no more than an elementary fact about spectral operators, although in a disguised form. Since it appears in [2, p. 2222, Ex. 2] as an exercise, we shall present a short proof here.

LEMMA. *Let T be an operator for which the set $\sigma(T)$ is both spectral and thin. Then T is similar to a normal operator.*

PROOF. The operator T affords the representation $u \to u(T)$ of the algebra $R(\sigma(T))$ into the Hilbert space H. The set $\sigma(T)$ being spectral by assumption, there is a constant M such that

$$\|u(T)\| \leqq M\|u\|_{\sigma(T)} \qquad u \in R(\sigma(T)) ,$$

showing that this representation is continuous (for the sup-norm on $R(\sigma(T))$ and the operator norm). Since the set $\sigma(T)$ is thin, this representation is immediately extended by continuity to a continuous representation of the entire Banach algebra $C(\sigma(T))$ into H. The fundamental theorem [2, XVII, 2.5] asserts that this representation comes from a unique strongly σ-additive spectral Borel measure E on $\sigma(T)$ via

$$f(T) = \int_{\sigma(T)} f(\lambda)\, dE(\lambda) \qquad f \in C(\sigma(T)) .$$

In other words, T is a spectral operator of scalar type in the sense of N. Dunford. The values $E(\delta)$, δ a Borel set in Λ, are bounded projections on H, but possibly skew ones. At any rate, H can be renormed in such a manner as to turn all $E(\delta)$ into orthogonal projections simultaneously. In fact, by virtue of [2, XV, 6.4.] there is a bounded selfadjoint operator S, with a bounded inverse on H, such that $N = S^{-1}TS$ is normal. It follows that T is similar to the normal operator N.

3.

Whereas $\sigma(T) = \Lambda$ in the Lemma, the theorem deals with the more general case that $\sigma(T) \subset \Lambda$. The set Λ being spectral by hypothesis, we do know that $\|u(T)\|$ remains bounded for $\|u\|_\Lambda \leqq 1$, but we have no control whatsoever of $\|u(T)\|$ on the unit ball of $R(\sigma(T))$. We shall reduce, however, the general situation to that of Lemma by "doubling" the Hilbert space H to its direct turn $H \oplus H$. But this trick will only work if $\dim H = \infty$ and, in order to cover the important case that $\dim H < \infty$, we shall use an auxiliar separable complex Hilbert space K and rather work in $\mathscr{H} = H \oplus K$.

Let Λ be a thin spectral set for T. Take a bounded normal operator N in K with $\sigma(N) = \Lambda$. (For instance, let (λ_n) be a dense sequence in Λ, (e_n) an orthonormal base for K and put $Ne_n = \lambda_n e_n$. This diagonal operator is clearly normal, $\|N\| \leqq \max\{|\lambda| : \lambda \in \Lambda\}$ and $\sigma(N)$ equals the closure Λ of $\{\lambda_n : n \in \mathsf{N}\}$.)

Consider the diagonal operator

$$\mathscr{T} = \begin{pmatrix} T & O \\ O & N \end{pmatrix}$$

in $\mathscr{H}$. The operator $\mathscr{T} - \lambda$ has a bounded inverse on $\mathscr{H}$ if and only if both diagonal entries $T - \lambda I$ and $N - \lambda I$ have that property (on H and K, respectively). Hence, $\sigma(\mathscr{T}) = \Lambda$. We next check that Λ is a spectral set for $\mathscr{T}$. For

$$u(\mathscr{T}) = \begin{pmatrix} u(T) & O \\ O & u(N) \end{pmatrix}$$

we have $\|u(\mathscr{T})\|^2 = \|u(T)\|2 + \|u(N)\|2$. Let M be the constant pertaining to the spectral set Λ of T, that is $\|u(T)\| \leqq M\|u\|_\Lambda$ on $R(X)$. For the normal operator N the spectrum Λ is a spectral set with $M = 1$. Hence

$$\|u(\mathscr{T})\| \leqq (M^2+1)^{\frac{1}{2}} \|u\|_\Lambda \qquad u \in R(\Lambda),$$

showing that Λ is a spectral set for $\mathscr{T}$, indeed.

We infer from the Lemma that $\mathscr{T}$ is a scalar operator on $\mathscr{H}$. Let $\mathscr{E}$ be the corresponding resolution of the identity, *i.e.*

$$(*) \qquad f(\mathscr{T}) = \int_\Lambda f(\lambda)\, d\mathscr{E}(\lambda) \qquad f \in C(\Lambda).$$

For each Borel set δ in Λ the operator $\mathscr{E}(\delta)$ can be represented by a matrix

$$\mathscr{E}(\delta) = \begin{pmatrix} E_{11}(\delta) & E_{12}(\delta) \\ E_{21}(\delta) & E_{22}(\delta) \end{pmatrix},$$

where $E_{11}(\delta)$ is a bounded operator mapping H into H. In matrix form (*) reads

$$\begin{pmatrix} f(T) & O \\ O & f(N) \end{pmatrix} = \int_{\Lambda} f(\lambda)\, d \begin{pmatrix} E_{11}(\lambda) & E_{12}(\lambda) \\ E_{21}(\lambda) & E_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} \int_{\Lambda} f(\lambda)\, dE_{11}(\lambda) & * \\ * & * \end{pmatrix}$$

Comparing the entries, we obtain $E_{12}(\delta)=E_{21}(\delta)=0$ for every Borel set $\delta \subset \Lambda$. But this implies that both E_{11} and E_{22} are projection-valued measures and for the north-west corners we get

$$f(T) = \int_{\Lambda} f(\lambda)\, dE_{11}(\lambda) \qquad f \in C(\Lambda).$$

Hence, T is of scalar type and the theorem is proved by another appeal to [2, XV, 6.4.].

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part II (Interscience Tracts in Pure and Applied Mathematics 7), Interscience Publ. Inc., New York, 1963.
2. N. Dunford and J. T. Schwartz, *Linear operators*, Part III (Interscience Tracts in Pure and Applied Mathematics 7), Interscience Publ. Inc., New York, 1971.
3. F. Riesz et B. Sz.-Nagy, *Lecons d'analyse fonctionelle*, 2me Ed., Budapest, 1953.
4. L. Zalcman, *Analytic capacity and rational approximation*, Lecture Notes in Mathematics 50, Springer-Verlag, Berlin, Heidelberg, New York, 1968.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ANTWERP U.I.A.
B 2610 WILRIJK
BELGIUM