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QUASIANALYTIC VECTORS AND DERIVATIONS OF OPERATOR ALGEBRAS

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Abstract.

We show that an unbounded derivation δ of a W^*- or C^*-algebra implements a group of automorphisms if there exists a state ω on the algebra (normal in the W^*-case) such that ω defines a faithful cyclic representation and $\omega \circ \delta = 0$, and δ has a dense set of quasianalytic elements. In the W^*-case it is enougth to require a dense set of quasi-analytic elements on the Hilbert space.

1. Introduction.

Our concern in this paper is two-fold. We answer in various settings, the questions as to when a derivation of an operator algebra gives rise to an automorphism group of the algebra and when the set on which we "know" the derivation forms a core for the generator of the automorphism group. As the title of the paper indicates we deal with the case in which quasi-analytic vectors exist in abundance for the derivation.

Nussbaum [10] introduced the idea of quasi-analytic vectors for a closed symmetric operator on a Hilbert space, showing that a dense set of such vectors guarantees self-adjointness. Thus he generalized the analytic vector theorem. (An excellent discussion of quasi-analytic vectors can be found in Chernoff's address [5].) Other results are available (the Stieltjes vector case) if one assumes the given operator is semi-bounded; see [6] and [11].

Since self-adjointness guarantees a corresponding unitary group, the first question we raised above asks for analogues of Nussbaum's result.

The second problem we mentioned may be rephrased, in the case of symmetric operators, as follows. Suppose the operator has a self-adjoint extension; when is the closure of the original operator equal to this extension? One criterion, which seems to have been rediscovered many times in the literature, is to show that the unitary group generated by the extension leaves

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the domain of the original operator invariant. The earliest reference we know of for this result is Singer's thesis [14]. A Banach space version appears in a paper of de Leeuw [9].

Aided by quasi-analyticity we are able to show a variant of this result (lemma 1) by working with the graph of the extension and thus obtain our core results.

In the setting of quasi-analytic vectors the results of this paper give an affirmative answer to a problem raised in [3]. The problem has already been resolved in other cases, [3], [4], [8]. We also obtain a C^* algebra theorem under a mild additional assumption. Indeed some such restriction is necessary, as an example given in [1] shows that even a dense set of analytic vectors is generally insufficient to ensure that a closed derivation is a generator.

2. Definitions and fundamentals.

We need the following definition:

DEFINITION 1. Let X be a Banach space and T a (possibly unbounded) linear operator on X with domain $D(T)$, dense in X. Let $C^\infty(T)$ denote the set $\{x \in X : x \in D(T^n);\ n=1,2,\ldots\}$. Let M_n be a sequence of positive numbers with $M_n^{1/n}$ non-decreasing and normalized so that $M_0=1$[1]. Define

$$C\{M_n\} = \{x \in C^\infty(T):\ \exists\,\lambda>0 \text{ with } \|T^n x\| \leqq \lambda^n M_n\}$$

The elements of $C\{M_n\}$ are said to be quasi-analytic vectors if $\Sigma_{n=1}^{\infty} M_n^{-1/n}=\infty$ and $C\{M_n\}$ is called a quasi-analytic class in this case. In general, a vector x is said to be quasi-analytic if it lies in some $C\{M_n\}$. Note that x is analytic if $\liminf_{n\to\infty} n\cdot M_n^{-1/n}>0$.

When referring to a function f in $C^b(\mathsf{R})$ (the bounded, continuous, complex valued function on R) quasi-analyticity is with respect to the operator of differentiation on $C^b(\mathsf{R})$, equipped with the supremum norm.

In Theorems 1 and 3 we obtain some results on cores for unbounded derivations. These results are based on the following lemma

LEMMA 1. *Let X be a Banach-space, F a norm-closed linear subspace of the dual of X such that either F is the dual of X or X is the dual of F, and let $\sigma(X,F)$ be the topology on X induced by F. Let $t \to \tau_t$ be a one-parameter group of isometries of X such that*

1) *$t \to \tau_t(x)$ is $\sigma(X,F)$ continuous for $x \in X$*
2) *$x \to \tau_t(x)$ is $\sigma(X,F)$ continuous for all $t \in \mathsf{R}$.*

[1] The non-decreasing nature of $M_n^{1/n}$ is not an essential restriction. In fact there always exists $\overline{M_n}$ with $C\{\overline{M_n}\}=C\{M_n\}$ and $(\overline{M_n})^{1/n}$ non-decreasing [12].

Define

$$\delta(x) = \lim_{t\to 0}\frac{1}{t}(\tau_t(x)-x), \qquad x \in D(\delta) \subseteqq X$$

where $D(\delta)$ is the set of $x \in X$ such that the limit exists. $D(\delta)$ is then a $\sigma(X,F)$-dense subspace of X, and δ is $\sigma(X,F)$-closed. Assume that $D \subseteqq D(\delta)$ is a subset of X such that:

i) *D is $\sigma(X,F)$-dense in X*
ii) $\delta(D) \subseteqq D$
iii) *All elements in D are quasianalytic for δ.*

Then the linear span L of D is a core for δ in the $\sigma(X,F)$-topology.

PROOF. Condition 2 and the well known resolvent formula [11]:

$$(1+\alpha\delta)^{-1}(x) = \int_0^\infty e^{-t}\tau_{-\alpha t}(x)\,dt, \qquad x \in X,\ \alpha \in \mathbf{R}$$

implies that $(1+\alpha\delta)^{-1}$ is $\sigma(X,F)$-closed, since the adjoint of $(1+\alpha\delta)^{-1}$ exists as an operator on F by the condition on the pair (X,F). We here make use of the fact that a weakly continuous group of isometries of a Banach space is strongly continuous [15], and a straightforward Riemann integral argument.

Hence δ is $\sigma(X,F)$-closed. Let δ_0 be the $\sigma(X,F)$ closure of the restriction of δ to L. We have to show $\delta_0 = \delta$.

Consider the subset of the graph of δ_0 given by

$$G_{00} = \{(x,\delta(x)) \mid x \in D\}$$

and the map of G_{00} into $X \times X$ (with the product $\sigma(X,F)$-topology) given by $\tau_t \times \tau_t$:

$$(x,\delta(x)) \to (\tau_t(x), \tau_t(\delta(x))) .$$

Let G_0 be the graph of δ_0, i.e.

$$G_0 = \{(x,\delta(x)) \mid x \in D(\delta_0)\} .$$

CLAIM 1.

$$(\tau_t \times \tau_t)(G_0) \subseteqq G_0, \qquad t \in \mathbf{R} .$$

To prove this, pick a $\psi \in G_0^\perp$ which is continuous in the $\sigma(X,F)\times\sigma(X,F)$ topology, i.e. $\psi = (\psi_1,\psi_2)$ where $\psi_1,\psi_2 \in F$ [7, V.3.11] and

$$\psi_1(x)+\psi_2(\delta(x)) = 0, \qquad x \in D .$$

For a fixed $x \in D$, define

$$g(t) = \psi_1(\tau_t(x)) + \psi_2(\tau_t\delta(x)) .$$

Then $g \in C^\infty(\mathsf{R})$ and

$$g^{(n)}(t) = \psi_1(\tau_t(\delta^n(x))) + \psi_2(\tau_t(\delta^{n+1}(x))) .$$

From this we have

$$\|g^{(n)}\|_\infty \leqq C(\|\delta^n(x)\| + \|\delta^{n+1}(x)\|) .$$

Two remarks are now necessary. First Chernoff, [5], notes that Carleman's inequality

$$\sum_{\nu=2}^{\infty} a_\nu^{1-1/\nu} \leqq \sum_{\nu=2}^{\infty} a_\nu + 2\left|\sum_{\nu=0}^{\infty} a_\nu\right|^{\frac{1}{2}}, \qquad a_\nu \geqq 0$$

implies that $\Sigma (M_{n+1})^{-1/n}$ and $\Sigma M_n^{-1/n}$ both diverge or converge simultaneously. This yields that $\delta(x)$ is a quasi-analytic element. Secondly M_n itself is nondecreasing (since $M_0 = 1$), and so $\delta(x)$ and x are in a common quasianalytic class. As a result $g(t)$ is quasi-analytic.

But

$$g^{(n)}(0) = \psi_1(\delta^n(x)) + \psi_2(\delta^{n+1}(x)) = 0$$

since $\delta\colon D \to D$ and $\psi \in G_0^\perp$. Hence by the Denjoy–Carleman theorem $g(t) = 0$ for all t [5]. Hence, by the bipolar theorem [7, V.3.12],

$$(\tau_t \times \tau_t)(G_{00}) \subseteqq G_0, \qquad t \in \mathsf{R} .$$

By taking linear combinations and closures, claim 1 follows.

Now, if G is the graph of δ, then clearly $G_0 \subseteqq G$. To prove the lemma, we must show

CLAIM 2. $D(\delta_0) = D(\delta)$; that is $G_0 = G$. By claim 1, $\tau_t(D(\delta_0)) = D(\delta_0)$ for all t. By using the technique in [2, Theorem 3] one shows that this implies:

$$D(\delta_0) \subseteqq \overline{(1+\alpha\delta)(D(\delta_0))}, \qquad \alpha \in \mathsf{R}$$

where the closure is in the $\sigma(X,F)$ topology. Thus the range of $1+\alpha\delta_0$ is dense for all $\alpha \in \mathsf{R}$. Hence, as $(1+\alpha\delta)^{-1}$ exists as a $\sigma(X,F)$-continuous operator, it follows that $D(\delta_0)$ is a core for δ. Since δ_0 is closed, claim 2 follows.

3. Quasi-analytic elements in a von Neumann-algebra.

Given a self-adjoint operator H, we write $\tau_t(A)$ for $e^{iHt}Ae^{-iHt}$ when A is any fixed operator. If B is a subset of a vector space we write B^{L} for its linear span.

THEOREM 1. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\delta(\cdot)=[iH, \cdot]$ a derivation of $\mathcal{M}$ implemented by a self-adjoint operator H. Assume further that there exists a subset $\mathcal{B} \subseteq D(\delta)$ such that $\mathcal{B}$ is weakly dense in $\mathcal{M}$ and all elements in $\mathcal{B}$ are quasi-analytic (in operator norm) for δ. Then*

$$e^{itH} \mathcal{M} e^{-itH} = \mathcal{M} .$$

Furthermore if $\delta(\mathcal{B}) \subseteq \mathcal{B}$, then $\mathcal{B}^{\mathrm{L}}$ is a core (in the σ-weak topology) for $\hat{\delta}=[iH, \cdot]$ defined on the set $\{A \in \mathcal{M} : [iH, A] \in \mathcal{M}\}$,

REMARK. Before entering the proof we note that $\hat{\delta}$ is the generator of $t \to \tau_t$ by [3; theorem 5].

PROOF. Define $U_t = e^{itH}$. Let $A \in \mathcal{B}$, $A' \in \mathcal{M}'$, $\varphi, \psi \in \mathcal{H}$ and consider the function

$$g(t) = (\psi, [U_t A U_{-t}, A']\varphi) .$$

Then $g \in C^\infty(\mathbf{R})$ and

$$\frac{d^n g}{dt^n}(t) = (\psi, [U_t \delta^n(A) U_{-t}, A']\varphi) .$$

Since

$$\left\| \frac{d^n g}{dt^n} \right\|_\infty \leqq \|\psi\| \, \|\varphi\| \, \|A'\| \, \|\delta^n(A)\|$$

we have that g is quasi-analytic ($A \in \mathcal{B}$).

Moreover

$$\frac{d^n g}{dt^n}(0) = (\psi, [\delta^n(A), A']\varphi) = 0$$

Hence, by the Denjoy–Carleman theorem, $g(t) \equiv 0$. Since $A' \in \mathcal{M}'$ was arbitrary, we conclude that

$$U_t A U_t^* \in \mathcal{M}, \qquad t \in \mathbf{R} .$$

By the density of $\mathcal{B}$ we see that

$$U_t \mathcal{M} U_t^* = \mathcal{M}$$

To complete the theorem, let $\hat{\delta}$ be as in the statement of the theorem and note that by [3, theorem 5], $\hat{\delta}$ is the generator of the group $\tau_t(A) = U_t A U_t^*$. We can now apply lemma 1 with $X = \mathcal{M}$, $F = \mathcal{M}_*$, to get the last statement of the theorem.

4. Quasi-analytic elements in the representation Hilbert space of a von Neumann algebra.

THEOREM 2. *Assume that $\mathcal{M}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic unit vector Ω, and $\delta(\cdot)=[iH,\cdot]$ is a derivation of $\mathcal{M}$ implemented by a self-adjoint operator H, such that $H\Omega=0$. Further suppose there exists a subset $\mathcal{B}\subseteqq D(\delta)$ such that*

(i) *$\mathcal{B}$ is strongly dense in $\mathcal{M}$*
(ii) *$\mathcal{B}$ is closed under multiplication and the *-operation*
(iii) *$\delta(\mathcal{B})\subseteqq\mathcal{B}$*
(iv) *$\mathcal{B}\Omega$ consists of quasi-analytic elements for H and any four such elements lie in a common quasi-analytic class.*

Then

$$e^{itH}\mathcal{M}e^{-itH} = \mathcal{M} .$$

PROOF. Define $U_t=e^{itH}$. Then it clearly suffices to show

$$U_t\mathcal{M}'U_t^* = \mathcal{M}', \qquad t\in \mathbf{R} .$$

To this end, pick $A'\in\mathcal{M}'$. By the cyclicity of Ω and the density of $\mathcal{B}$ in $\mathcal{M}$ it is enough to show

$$(B\Omega,[A,U_t^*A'U_t]C\Omega) = 0$$

for $A,B,C\in\mathcal{B}$ and $t\in\mathbf{R}$. Thus define

$$\begin{aligned} g(t) &\equiv (B\Omega,[A,U_t^*A'U_t]C\Omega)\\ &= (U_tA^*B\Omega,A'U_tC\Omega)-(U_tB\Omega,A'U_tAC\Omega)\\ &\equiv g_1(t)-g_2(t). \end{aligned}$$

By the assumptions, $A^*B\Omega$, $C\Omega$, $B\Omega$ and $AC\Omega$ all lie in a common quasi-analytic class $C\{M_n\}$, i.e. if ψ is any of the above four vectors,

$$\|H^n\psi\| \leqq M_n \quad \text{with } \sum (M_n)^{-1/n} = \infty .$$

We claim that

$$\left\|\frac{d^n}{dt^n}g_i\right\|_\infty \leqq 2^n\|A'\|M_n, \quad i=1,2 ,$$

from which we conclude that g is in the quasi-analytic class $C\{M_n\}$.
Consider

$$g_1(t) = (U_tA^*B\Omega,A'U_tC\Omega)$$

Clearly $g_1 \in C^\infty(\mathbf{R})$. In fact by Leibnitz' rule

$$\frac{d^n}{dt^n} g_1(t) = \sum_{k=0}^{n} \binom{n}{k} (U_t(iH)^k A^* B\Omega, A' U_t(iH)^{n-k} C\Omega)$$

Thus

$$\left| \frac{d^n}{dt^n} g_1(t) \right| \leqq \sum_{k=0}^{n} \binom{n}{k} \|H^k A^* B\Omega\| \, \|A'\| \, \|H^{n-k} C\Omega\|$$

$$\leqq \|A'\| \sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k} \leqq \|A'\| \sum_{k=0}^{n} \binom{n}{k} (M_k^{1/k})^k (M_{n-k}^{1/n-k})^{n-k}$$

$$\leqq \|A'\| \sum \binom{n}{k} (M_n^{1/n})^{k+n-k} = \|A'\| 2^n M_n$$

The next to last step employed the non-decreasing nature of $(M_n)^{1/n}$.

Having the quasi-analyticity of $g(t)$ we claim $g(t) \equiv 0$. For this we need only show that $g^{(n)}(0) = 0$ for all n. But

$$\frac{d^n}{dt^n} g(0) = \sum_{k=0}^{n} \binom{n}{k} ((iH)^k A^* B\Omega, A'(iH)^{n-k} C\Omega) -$$

$$- \sum_{k=0}^{\infty} \binom{n}{k} ((iH)^{n-k} B\Omega, A'(iH)^k AC\Omega)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (\delta^l(A^*) \delta^{k-l}(B)\Omega, A' \delta^{n-k}(C)\Omega) -$$

$$- \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (\delta^{n-k}(B)\Omega, A' \delta^l(A) \delta^{k-l}(C)\Omega)$$

$$= \left(A'^*\Omega, \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{(n-k)!\,(k-l)!\,l!} \delta^{k-l}(B^*) \delta^l(A) \delta^{n-k}(C)\Omega \right) -$$

$$- \left(A'^*\Omega, \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{(n-k)!\,(k-l)!\,l!} \delta^{n-k}(B^*) \delta^l(A) \delta^{k-l}(C)\Omega \right)$$

$$= (A'^*\Omega, \delta^n(B^* AC)\Omega)$$

$$- (A'^*\Omega, \delta^n(B^* AC)\Omega)$$

$$= 0\,.$$

This completes the proof of the theorem.

REMARK. Theorem 2 partially generalizes [3, theorem 7]. The authors understand that H. Araki has obtained a generalization of that theorem which also removes the separating condition on Ω (private communication).

5. Quasi-analytic elements in a C^*-algebra.

THEOREM 3. *Let $\mathscr{A}$ be a C^*-algebra, and ω a state of $\mathscr{A}$ defining a faithful cyclic representation. Let δ be a derivation of $\mathscr{A}$ such that $\omega \circ \delta = 0$. Assume that there exists a subset $\mathscr{B} \subseteq \mathscr{A}$ such that*

i) *$\mathscr{B}$ is norm-dense in $\mathscr{A}$.*
ii) *$\delta(\mathscr{B}) \subseteq \mathscr{B}$.*
iii) *The elements in $\mathscr{B}$ are quasi-analytic for δ.*

Then δ is closeable, its closure is equal to the closure of its restriction to the linear span of $\mathscr{B}$, and $\bar{\delta}$ is the generator of a strongly continuous one-parameter group of automorphisms of $\mathscr{A}$.

PROOF. We may view $\mathscr{A}$ as represented on the Hilbert space defined by the couple $\{\mathscr{A}, \omega\}$, and that

$$\omega(A) = (\Omega, A\Omega), \qquad A \in \mathscr{A}$$

for a cyclic unit vector Ω. The invariance $\omega \circ \delta = 0$ implies that δ is closeable and there exists a symmetric operator H on $D(\delta)\Omega$ such that by [1]:

$$H\Omega = 0$$

$$\delta(A)\psi = [iH, A]\psi, \qquad A \in D(\delta),\ \psi \in D(\delta)\Omega\,,$$

From the assumptions it follows that $\mathscr{B}\Omega$ is a dense, invariant set of quasi-analytic vectors for H, and so by Nussbaum's theorem, [10], H is essentially self-adjoint on the linear span of $\mathscr{B}\Omega$. Hence, by theorem 1 applied to $\mathscr{M} = \mathscr{A}''$:

$$e^{itH}\mathscr{M}e^{-itH} = \mathscr{M}, \qquad t \in \mathbf{R}$$

Define

$$\tau_t(A) = e^{itH}Ae^{-itH}, \quad A \in \mathscr{M}, \quad t \in \mathbf{R}$$

We will first show that

$$\tau_t(\mathscr{A}) = \mathscr{A}, \qquad t \in \mathbf{R}$$

and that $t \to \tau_t|_{\mathscr{A}}$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathscr{A}$ whose infinitesimal generator $\hat{\delta}$ is an extension of δ. This follows from the following two remarks.

OBSERVATION 1. *If $A \in \mathscr{B}$, then $t \to \tau_t(A)$ is norm-continuous.*

PROOF. If $\eta \in \mathcal{M}_*$ (=the predual of $\mathcal{M}$, [13]), then:

$$\eta(\tau_t(A)) - \eta(A) = \int_0^t ds\, \eta(\tau_s(\delta(a)))$$

whence

$$|\eta(\tau_t(A) - A)| \leqq |t|\, \|\eta\|\, \|\delta(A)\| .$$

Thus we have

$$\|\tau_t(A) - A\| \leqq |t|\, \|\delta(A)\|$$

and observation 1 follows.

OBSERVATION 2. *If $A \in \mathcal{B}$, then*

$$\tau_t(A) \in \mathcal{A}, \qquad t \in \mathbf{R}$$

PROOF. Pick a linear functional $\eta \in \mathcal{M}^*$ which annihilates $\mathcal{A}$, that is, $\eta \in \mathcal{M}^* \cap \mathcal{A}^\perp$, and define

$$f(t) = \eta(\tau_t(A)) .$$

We show by induction that

$$\frac{d^n}{dt^n} f(t) = \eta(\tau_t(\delta^n(A))) .$$

Assume that this has been shown for n and consider

$$\begin{aligned} &\frac{d^n}{dt^n} f(t+h) - \frac{d^n}{dt^n} f(t) \\ &= \eta(\tau_{t+h}(\delta^n(A)) - \tau_t(\delta^n(A))) \\ &= \eta\left(\int_0^h ds\, \tau_{t+s}(\delta^{n+1}(A))\right) \end{aligned}$$

where the last integral exists as a Riemann integral by observation 1. Hence, by observation 1:

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{d^n}{dt^n} f(t+h) - \frac{d^n}{dt^n} f(t)\right) = \eta(\tau_t(\delta^{n+1}(A))) .$$

Thus

$$\left\|\frac{d^n}{dt^n} f\right\|_\infty \leqq \|\eta\|\, \|\delta^n(A)\| ,$$

so f is quasi-analytic. Also

$$\frac{d^n}{dt^n} f(0) = \eta(\delta^n(A)) = 0$$

since $\eta \in \mathscr{A}^\perp$. Again the Denjoy Carleman theorem gives

$$f(t) = \eta(\tau_t(A)) = 0, \quad t \in \mathbf{R}.$$

By the Hahn–Banach theorem this implies

$$\tau_t(A) \in \mathscr{A}, \quad t \in \mathbf{R}.$$

From observation 1 and 2 and the density of $\mathscr{B}$ in $\mathscr{A}$ it follows readily that

$$\tau_t(\mathscr{A}) = \mathscr{A}$$

and $t \to \tau_t|\mathscr{A}$ is strongly continuous.

Our theorem is completed by the preceding remark and another application of lemma 1. Here we take $X = \mathscr{A}$ and $F = \mathscr{A}^*$.

We remark that Chernoff obtains the core part of theorem 3 by different means, [6].

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