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A RELATION BETWEEN THE NUMBERS OF SINGULAR POINTS AND SINGULAR LINES OF A PLANE CLOSED CURVE

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1. Let γ denote a smooth closed curve in the affine plane, t_e and t_i the numbers of exterior and interior double tangents, respectively, d the number of double points and i the number of points of inflection of the curve. It has been proved that these numbers are connected by the equation

$$t_e - t_i = d + \tfrac{1}{2}i , \tag{1}$$

(see [1, p. 365], [2], [3, p. 166] and [4]). In the formula it is assumed that the double tangents are *double supporting lines*. A double supporting line s is called *exterior* or *interior* according as the neighboring arcs at the two points of contact are lying on the same side of s or on opposite sides.

We shall prove that if we allow the curve γ, besides the mentioned singular points and lines, to have c_1 *cusps of the first kind* and c_2 *cusps of the second kind* then the more general formula

$$t_e - t_i = d + \tfrac{1}{2}i + c_1 + \tfrac{1}{2}c_2 \tag{2}$$

is valid.

2. It is an obvious assumption for the curve γ that the numbers of singular points and singular lines are finite. A double supporting line (s_e or s_i) may be a double tangent or a tangent through a cusp or a line through two cusps (fig. 1). We assume that any line s is supporting line at only two points, and analogously that any double point is point of intersection between exactly two arcs with different tangents at the point. Moreover it is assumed that the tangent to the curve at a cusp point does not touch the curve at any other point. At each point $P \in \gamma$ the curve has a tangent p which varies continuously with P.

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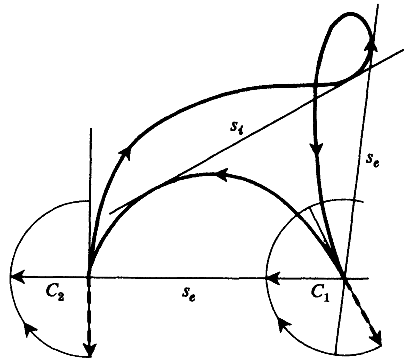

Fig. 1.

3. Now let γ be oriented such that there is at P a *positive* half-tangent p^+ and a *negative* half-tangent p^-. The two half-lines are opposite unless P lies at a cusp point C where p^+ and p^- coincide. Any other line through C may be oriented and divided by C into a positive and a negative half-line. This can be done as follows. With C as the center a half-circle is drawn having the tangent to γ at C as bounding diameter. For a line p through C the half-line which intersects the half-circle is p^+, the opposite half-line p^-. For a cusp of the first kind the half-circle may be chosen arbitrarily, and for a cusp of the second kind we place the half-circle in the half-plane which does not contain the curve in the neighbourhood of the cusp (fig. 2 and 3).

The positive half-tangents of γ and the positive half-lines with end point at a cusp together make a field (p^+) of positive half-lines. This field has a discontinuity when p^+ passes the tangent at a cusp, which may be removed by adding to (p^+) the half-line opposite to the half-tangent at the cusp. With these additions the field is continuous. The same properties are of course valid for the negative half-lines.

4. Let p be the tangent to γ at an ordinary point P. The numbers of points in common for γ and the half-tangents p^+ and p^- are denoted by N^+ and N^-, respectively. We consider the difference

$$N = N(p) = N^+ - N^- . \tag{3}$$

The value of the function $N(p)$ will change when p passes a double supporting line and when P passes a double point, a point of inflection and a cusp. When the set (P, p) has made a full circuit along the curve the sum of the changes taken with sign must be equal to zero.

5. If p passes an *exterior double supporting line* s_e either two points on p^+ will be gained or two points on p^- lost, i.e. N will increase by 2. Since s_e is passed twice any s_e gives a contribution to N of 4. If p passes an interior double supporting line s_i, then either two points on p^+ are lost or two points on p^- will be gained. Hence any s_i gives a contribution to N of -4, see the table below. (Consider in fig. 1 the s_e through C_1 and C_2. With the indicated orientation of the curve and of the half-circles the rotation of p about C_1 gives two new points on p^+ in passing C_2, and two points on p^- are lost when p turns about C_2 in passing C_1. When p passes the s_i in the figure two points are gained on p^- at each passing).

	S_e	S_i	D	I	C_1	C_2
$N(p)$	$+4$	-4	-4	-2	-4	-2

6. If P passes a *double point* or a *point of inflection* a common point of γ and p will go from p^+ to p^- and N will decrease by 2. Since a double point is passed twice and a point of inflection only once when P traverses γ, we get for each of the points the contribution to N indicated in the table for D and I.

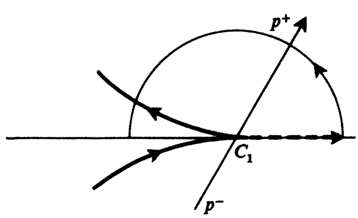

Fig. 2.

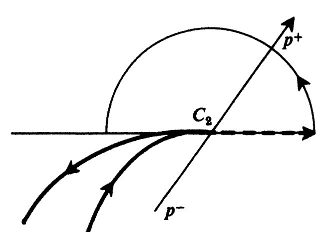

Fig. 3.

7. At a cusp point C a neighbourhood may be composed by two small convex arcs with endpoint at C, without common points and lying on opposite sides of the tangent at C or on the same side according as C is a cusp of the first or of the second kind (fig. 2 and 3).

Let C_1 denote a *cusp of the first kind.* We consider the points on the mentioned convex arcs which are gained or lost on p^+ and p^- when P passes C_1 and p the tangent at C_1. When P tends to C_1 one point on p^+ is lost. When the rotation about C_1 begins one point on p^- is gained, and when the rotation ends a new point on p^+ is lost. Finally, when P leaves C_1 a point on p^- will appear. Hence a cusp of the first kind contributes to N with the number -4. The same result would be obtained rounding off the cusp by means of a small convex arc. By the rounding off the cusp will disappear and two points of inflection appear.

8. Next, let C_2 denote a *cusp of the second kind*, and let P pass it from the interior to the exterior arc (fig. 1 and 3). When P tends to C_2 the tangent p at P intersects the exterior arc at two points, one on p^+ and one on p^-. Both of them disappear when P reaches C_2. By the rotation about C_2 two points on p^- are gained at once while no point on p^+ appears in the neighbourhood of C_2. When P leaves C_2 along the exterior arc no new point on p^+ appears (in the neighbourhood of C_2). Hence, when P passes a cusp of the second kind the number N will decrease by 2. The same result will (of course) be obtained if P passes C_2 in the opposite direction. The result is in accordance with the fact that rounding off the cusp in this case implies the appearance of only *one* point of inflection.

9. In order to show the desired relation (2) we apply the obtained results of the table in section 5. Let t_e, t_i, d, i, c_1 and c_2 denote the numbers of exterior double supporting lines, interior double supporting lines, double points, points of inflection and cusps of the first and the second kind, respectively. Using a remark in the end of section 4 we find the equation

$$4t_e - 4t_i - 4d - 2i - 4c_1 - 2c_2 = 0$$

or

$$t_e - t_i = d + \tfrac{1}{2}i + c_1 + \tfrac{1}{2}c_2 ,$$

which is the equation (2).

10. Thus we have proved formula (2) for an arbitrary closed curve in the affine plane which satisfy the assumptions in section 2 (In fig. 1 we have $t_e = 5$, $t_i = 2$, $d = i = c_1 = c_2 = 1$). But there does not exist a curve corresponding to any arbitrary combination of singularities whose numbers satisfy equation (2). If for example $d = i = c_1 = c_2 = 0$ then (2) is reduced to $t_e = t_i$. It is known that a closed curve without double points, points of inflection and cusps is a convex curve. Hence in this case only the solution $t_e = t_i = 0$ corresponds to a curve.

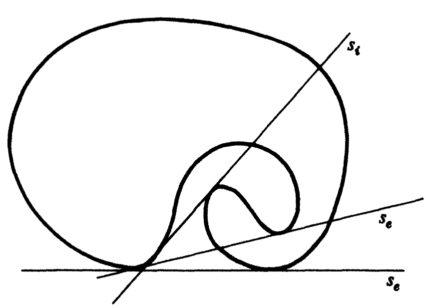

Fig. 4.

If $t_i = 1$ we must have $t_e \geqq 2$. Fig. 4 shows a curve where $t_i = 1$, $t_e = 2$, $i = 2$ and $d = c_1 = c_2 = 0$. Steiner's hypocycloid is an example of a curve with $t_e = 3$, $c_1 = 3$ and $t_i = d = i = c_2 = 0$.

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