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SIDON SETS AND FOURIER–STIELTJES TRANSFORMS OF SOME PRIME L-IDEALS

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Let G be an infinite compact abelian group and $\hat{G}$ be its dual group. $M(G)$ denotes the measure algebra on G. By Taylor [11], there is a compact topological semigroup S, and we can consider that $M(G) \subset M(S)$ and $\hat{S}$, the set of all continuous semicharacters on S, is identified with the maximal ideal space of $M(G)$. The reader is assumed to be familiar with the Taylor's structure semigroup. The Gelfand transform $\hat{\mu}$ of $\mu \in M(G)$ is given by $\hat{\mu}(f) = \int f\,d\mu$ ($f \in \hat{S}$). We can consider $\hat{G} \subset \hat{S}$ and $\hat{\mu}|_{\hat{G}}$ is the Fourier–Stieltjes transform of $\mu \in M(G)$. The closure of $\hat{G}$ in $\hat{S}$ is denoted by $\bar{\hat{G}}$. Brown [1] shows that there are many idempotents in $\bar{\hat{G}} \backslash \hat{G}$, where $f \in \hat{S}$ is called an idempotent if $f^2 = f$. For $\mu \in M(G)$, we put

$$L^1(\mu) = \{\lambda \in M(G)\,;\ \lambda \text{ is absolutely continuous with respect to } \mu\}\ .$$

For an idempotent $f \in \hat{S}$, we put

$$J(f) = \{x \in S\,;\ f(x) = 0\}$$

and

$$I(f) = \{\mu \in M(G)\,;\ \mu \text{ is concentrated on } J(f)\}\ .$$

Then $I(f)$ is a prime L-ideal, where a closed ideal I of $M(G)$ is called a prime L-ideal if $L^1(\lambda) \subset I$ for $\lambda \in I$ and

$$I^\perp = \{\mu \in M(G)\,;\ \mu \text{ is singular with } I\}$$

is a subalgebra. $E \subset \hat{G}$ is called a Sidon set if $M(G)\hat{}\,|_E = l^\infty(E)$, where $l^\infty(E)$ is the set of all bounded functions on E. Let

$$M_c(G) = \{\mu \in M(G)\,;\ \mu \text{ is continuous}\}\ .$$

Hartman [8] and Wells [12] show that $M_c(G)\hat{}\,|_E = l^\infty(E)$ for every Sidon set E. And Brown [2] shows that Riesz products, using lacunary sequences, show that if E is an infinite subset of $\hat{G}$ then $\bar{E} \backslash \hat{G}$ contains f such that $|f|^2 \neq |f|$. In

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this paper, we give a generalization of Hartman–Wells' theorem that $I(\chi)\hat{}|_E = l^\infty(E)$ for every Sidon set $E \subset \hat{G}$ and for every idempotent $\chi \in \bar{\hat{G}} \backslash \hat{G}$. As a corollary, we show that if E is an infinite Sidon set and $f \in \bar{E} \backslash \hat{G}$, then $|f|^2 \neq |f|$.

1.

For a finite subsets A, B of $\hat{G}$, we put

$$AB = \{xy\,;\ x \in A,\ y \in B\}$$

and $|A|$ denotes the cardinal number of A. Throughout the rest of this paper, let $\chi \in \bar{\hat{G}} \backslash \hat{G}$ and $\chi^2 = \chi$. For $\mu \in M(G)$, we write $\mu = \mu_1 + \mu_2$, where $\mu_1 \in I(\chi)$ and $\mu_2 \perp I(\chi)$.

THEOREM 1 (cf. [9, pp 48–50]). *Let E be a subset of $\hat{G}$ such that*

$$\sup\{\min(|A|, |B|)\,;\ AB \subset E\} < \infty\,.$$

For $\mu \in M(G)$, we have $\hat{\mu}_2(\hat{G}) \subset \hat{\mu}(\hat{G} \backslash E)^-$, where $\hat{\mu}(\hat{G} \backslash E)^-$ is the closure of $\hat{\mu}(\hat{G} \backslash E)$ in the complex number plane.

PROOF. Since $\chi \in \bar{\hat{G}} \backslash \hat{G}$, there is a net $\{\gamma_\alpha\}_{\alpha \in A} \subset \hat{G}$ such that $\gamma_\alpha \to \chi$ in $\hat{S}$. Since $\chi = 1$ a.e. μ_2, we have

$$\int |\gamma_\alpha - 1|\, d|\mu_2| \to 0 \tag{1}$$

by Taylor ([11, 5.1.5a]). Let $\gamma_0 \in \hat{G}$ and $\varepsilon > 0$. We first show that, using Graham's method in [6], there is a subsequence of distinct elements $\{\gamma_{\alpha_1}, \gamma_{\alpha_2}, \ldots\}$ in $\{\gamma_\alpha\}_{\alpha \in A}$ such that:

$$|\hat{\mu}_1(\gamma_0 \gamma_{\alpha_n} \gamma_{\alpha_m}^{-1})| < \varepsilon \qquad \text{for } n > m; \tag{2}$$

$$\int |\gamma_{\alpha_n} - 1|\, d|\mu_2| < \frac{1}{n}; \tag{3}$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots\,. \tag{4}$$

By (1), there is $\alpha_1 \in A$ such that $\int |\gamma_{\alpha_1} - 1|\, d|\mu_2| < 1$.

Suppose that there exists an n-distinct subset $\{\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}\} \subset \{\gamma_\alpha\}_{\alpha \in A}$ such that

$$|\hat{\mu}_1(\gamma_0 \gamma_{\alpha_j} \gamma_{\alpha_i}^{-1})| < \varepsilon \qquad \text{for } i < j \leqq n\,,$$

$$\int |\gamma_{\alpha_j} - 1|\, d|\mu_2| < \frac{1}{j} \qquad (j \leqq n)$$

and $\alpha_1<\alpha_2<\ldots<\alpha_n$. Since $\gamma_0^{-1}\gamma_{\alpha_i}\mu_1\in I(\chi)$ and $|(\gamma_0^{-1}\gamma_{\alpha_i}\mu_1)\hat{}(\gamma_\alpha)|\to 0$ $(\alpha\to\infty)$ $(i=1,2,\ldots,n)$, there is $\alpha_{n+1}\in A$ such that $\alpha_n<\alpha_{n+1}$,

$$\int|\gamma_{\alpha_{n+1}}-1|\,d|\mu_2|<\frac{1}{n+1}\quad\text{and}\quad\sum_{i=1}^{n}|\hat{\mu}_1(\gamma_0\gamma_{\alpha_{n+1}}\gamma_{\alpha_i}^{-1})|<\varepsilon\,.$$

This completes the inductive step and establishes (2), (3) and (4). By (3), there is a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}\subset\{\alpha_n\}_{n=1}^{\infty}$ such that $\gamma_{\alpha_{n_k}}\to 1$ $(k\to\infty)$ a.e. $|\mu_2|$. We may assume that $\gamma_{\alpha_n}\to 1$ $(n\to\infty)$ a.e. $|\mu_2|$. By Egorov's theorem [7, p. 88], there is a Borel subset F such that $|\mu_2|(F^c)<\varepsilon$ and $\gamma_{\alpha_n}\to 1$ $(n\to\infty)$ uniformly on F. Then there is $N>0$ such that $|\gamma_{\alpha_n}-1|<\varepsilon$ on F for every $n>N$. Then for $n>k>N$, we have $|\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1}-1|<2\varepsilon$ on F and

$$\begin{aligned}|\hat{\mu}_2(\gamma_0)-\hat{\mu}(\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})&\leqq|\hat{\mu}_2(\gamma_0)-\hat{\mu}_2(\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})|\\&\quad+|\hat{\mu}_1(\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})|\\&\leqq|(\mu_2|_F)\hat{}(\gamma_0)-(\mu_2|_F)\hat{}(\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})|\\&\quad+|(\mu_2|_{F^c})\hat{}(\gamma_0)-(\mu_2|_{F^c})\hat{}(\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})|+\varepsilon\qquad\text{(by (2))}\\&\leqq\left|\int\gamma_0(1-\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1})\,d\mu_2|_F\right|+2\varepsilon+\varepsilon\\&\leqq\int|1-\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1}|\,d|\mu_2||_F+3\varepsilon\\&\leqq 2\varepsilon\|\mu_2\|+3\varepsilon\,,\end{aligned}$$

where $\mu_2|_F$ means the restriction measure of μ_2 to F. Here suppose that $\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1}\in E$ for every $n>k>N$. For $m>N$, we put

$$A_m=\{\gamma_{\alpha_{2m}},\gamma_{\alpha_{2m+1}},\ldots,\gamma_{\alpha_{3m-1}}\}\gamma_0$$

and

$$B_m=\{\gamma_{\alpha_m}^{-1},\gamma_{\alpha_{m+1}}^{-1},\ldots,\gamma_{\alpha_{2m-1}}^{-1}\}\,.$$

Then we have $|A_m|=|B_m|=m$ and $A_mB_m\subset E$. This contradicts the assumption of E. So that there is $n>k>N$ such that $\gamma_0\gamma_{\alpha_n}\gamma_{\alpha_k}^{-1}\in\hat{G}\backslash E$. This completes the proof.

It is well known that if $E\subset\hat{G}$ is a Sidon set then

$$\sup\{\min(|A|,|B|);\ AB\subset E\}<\infty$$

([9, p. 8]).

COROLLARY 2 (cf. [4] and [5]). *Let $E \subset \hat{G}$ be a Sidon set, then*

$$\|\hat{\mu}_2\|_\infty \leqq \sup\{|\hat{\mu}(\gamma)| \; ; \; \gamma \in \hat{G}\backslash E\} ,$$

where $\|\hat{\mu}_2\|_\infty = \sup\{|\hat{\mu}_2(\gamma)| \; ; \; \gamma \in \hat{G}\}$.

COROLLARY 3. *Let $E \subset \hat{G}$ be a Sidon set, then $I(\chi)\hat{}|_E = l^\infty(E)$.*

PROOF. By Drury's theorem [3], there is $\mu \in M(G)$ such that $\hat{\mu} = 1$ on E and $|\hat{\mu}| \leqq \frac{1}{2}$ on $\hat{G}\backslash E$. By Corollary 2, $\|\hat{\mu}_2\|_\infty \leqq \frac{1}{2}$ and $|\hat{\mu}_1| \geqq \frac{1}{2}$ on E. Then $(\mu_1 * M(G))\hat{}|_E = \hat{\mu}_1 M(G)\hat{}|_E = l^\infty(E)$. Since $\mu_1 \in I(\chi)$ and $I(\chi)$ is a closed ideal, we have $\mu_1 * M(G) \subset I(\chi)$.

COROLLARY 4. *Let $\chi_1, \chi_2, \ldots, \chi_n \in \bar{\hat{G}}\backslash \hat{G}$ and $\chi_1^2 = \chi_1, \chi_2^2 = \chi_2, \ldots, \chi_n^2 = \chi_n$. We put $I = I(\chi_1) \cap \ldots \cap I(\chi_n)$, then $\hat{I}|_E = l^\infty(E)$ for every Sidon set E.*

PROOF. By Corollary 3, there are $\mu_1 \in I(\chi_1)$, $\mu_2 \in I(\chi_2), \ldots, \mu_n \in I(\chi_n)$ such that $\hat{\mu}_1 = \hat{\mu}_2 = \ldots = \hat{\mu}_n = 1$ on E. Then $\mu = \mu_1 * \mu_2 * \ldots * \mu_n \in I$ and $\hat{\mu} = 1$ on E. Since I is a closed ideal, we have $\hat{I}|_E = l^\infty(E)$.

COROLLARY 5. *If E is an infinite Sidon set and $f \in \bar{E}\backslash \hat{G}$, then there are no idempotents $\pi \in \bar{\hat{G}}\backslash \hat{G}$ such that $\pi(x) \geqq |f(x)|$ for every $x \in S$.*

PROOF. Let $f \in \bar{E}\backslash \hat{G}$. Suppose that there is an idempotent $\pi \in \bar{\hat{G}}\backslash \hat{G}$ such that $\pi(x) \geqq |f(x)|$ for every $x \in S$. By Corollary 3, there is $\mu \in I(\pi)$ such that $\hat{\mu} = 1$ on E. Since $\hat{\mu}(\pi) = \hat{\mu}(f) = 0$, we have $f \notin \bar{E}$. This is a contradiction.

COROLLARY 6. *Let $E \subset \hat{G}$ be an infinite Sidon set. Then $|f|^2 \neq |f|$ for every $f \in \bar{E}\backslash \hat{G}$.*

PROOF. If $f \in \bar{\hat{G}}\backslash \hat{G}$ and $|f|^2 = |f|$, then $|f| \in \bar{\hat{G}}$. By Corollary 5, the result follows.

REFERENCES

1. G. Brown, *Idempotents in the closure of the characters*, Bull. London Math. Soc. 4 (1972), 43–46.
2. G. Brown, *Riesz products and generalized characters*, Proc. London Math. Soc. (3) 30 (1975), 209–233.
3. S. W. Drury, *Sur les ensembles de Sidon*, C. R. Acad. Sci. Paris, Sér. A 271 (1970), 162–163.
4. C. F. Dunkl and D. E. Ramirez, *C*-algebras generated by Fourier–Stieltjes transforms*, Trans. Amer. Math. Soc. 164 (1972), 435–441.

5. I. Glicksberg and I. Wik, *The range of Fourier–Stieltjes transforms of parts of measures*, in *Conference on harmonic analysis*, ed. D. Gulick and R. L. Lipsman (Lecture Notes in Mathematics 266), 73–77, Springer-Verlag, Berlin · Heidelberg · New York, 1972.
6. C. C. Graham, *Non Sidon sets in the support of a Fourier–Stieltjes transform*, to appear in Studia Math.
7. P. R. Halmos, *Measure theory*, D. Van Nostrand Co., New York, 1950.
8. S. Hartman, *Interpolction par les mesures diffuses*, Colloq. Math. 26 (1972), 339–343.
9. J. M. López and K. A. Ross, *Sidon sets*, Marcel Dekker, Inc., New York, 1975.
10. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure and Applied Mathematics 12), Interscience, New York, 1962.
11. J. L. Taylor, *Measure algebras*, CBMS Regional Conf. Ser. Math., 1972.
12. B. B. Wells, *Restrictions of Fourier transforms of continuous measures*, Proc. Amer. Math. Soc. 38 (1973), 92–94.

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