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SOME CONTINUITY AND MEASURABILITY RESULTS ON SPACES OF MEASURES

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0. Introduction.

Consider the following problem: Let X, Y, Z be three analytic spaces and let $f\colon X \times Y \to Z$ be a universally measurable mapping. Associate with f the mapping

$$F\colon M_+(X) \times Y \to M_+(Z)$$

($M_+(X)$ denoting all non-negative finite Borel measures on X) which maps $(\mu, y) \in M_+(X) \times Y$ to the image measure of μ under f_y, where $f_y(x) := f(x,y)$. Can we claim that F again is universally measurable, and do stronger regularity properties of f such as Borel measurability, a-measurability or continuity also extend to F? Questions of this kind arise naturally in connection with stochastic sequential machines, and it is the purpose of this paper to give positive answers to them.

It turns out that the main problem indeed is the following: is the canonical map $M_+(X) \times M_+(Y) \to M_+(X \times Y)$, sending (μ, ν) to its product measure $\mu \otimes \nu$, a continuous mapping? This is known to be true if X and Y are separable metric spaces (cf. [3, III. Lemma 1.1]) and also for regular τ-smooth measures on completely regular spaces (Ditlev Monrad, unpublished), but both these results do not answer the question for all analytic spaces.

Our main result (Theorem 1 below) shows that this map is well-defined (in a canonical way) and continuous in a very general situation, namely for τ-smooth measures on arbitrary (not necessarily Hausdorff) topological spaces. A similar result is true for denumerable products of probability measures, see Theorem 2. In the second part we prove a result which might be of some independent interest: for any measurable space $(X, \mathscr{B})$ and a universally measurable subset $A \subseteq X$ the mapping $\mu \mapsto \mu(A)$ is universally measurable on the space of finite non-negative measures on $\mathscr{B}$, see Theorem 4 below.

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1. The main result.

Let X be a topological space, not necessarily Hausdorff. $\mathscr{B}(X)$ denotes the Borel σ-field of X, i.e. the σ-field generated by the topology. We denote by $M_+(X)$ the convex cone of all totally finite Borel measures on X, that is the set of all σ-additive functions $\mu: \mathscr{B}(X) \to [0, \infty[$. A measure $\mu \in M_+(X)$ is called *τ-smooth* iff $\mu(G) = \sup_\alpha \mu(G_\alpha)$ for every net of open subsets $\{G_\alpha\}$ filtering up to G. Let $M_+(X, \tau)$ be the subset of all τ-smooth measures of $M_+(X)$. We call a measure $\mu \in M_+(X)$ *regular* iff

$$\mu(B) = \sup\{\mu(F) : F \subseteq B, \ F \text{ closed}\}$$

holds for all $B \in \mathscr{B}(X)$, and if X is a Hausdorff space, then μ is called *tight* iff

$$\mu(B) = \sup\{\mu(K) : K \subseteq B, \ K \text{ compact}\}$$

is true for all $B \in \mathscr{B}(X)$. $M_+(X, r, \tau)$ denotes all regular τ-smooth measures and $M_+(X, t)$ all tight measures on X. The relation $M_+(X, t) \subseteq M_+(X, r, \tau)$ holds for any Hausdorff space (cf. P 15 in [5]).

The weak topology on $M_+(X, \tau)$ is the weakest topology such that the function $\mu \mapsto \int f \, d\mu$ is lower semi-continuous (l.s.c.) for every bounded l.s.c. function $f: X \to \mathsf{R}$. If X is Hausdorff then $M_+(X, t)$ is again Hausdorff, see Theorem 11.2 in [5].

Let now two topological spaces X and Y be given. On the product space $X \times Y$ we have two σ-fields, $\mathscr{B}(X) \otimes \mathscr{B}(Y)$ and $\mathscr{B}(X \times Y)$. It is easy to see that always $\mathscr{B}(X) \otimes \mathscr{B}(Y) \subseteq \mathscr{B}(X \times Y)$ but in general equality does not hold.

THEOREM 1. *Let X and Y be two (not necessarily Hausdorff) topological spaces. For every $\mu \in M_+(X, \tau)$ and $\nu \in M_+(Y, \tau)$ there exists a uniquely determined measure $\mu \hat{\otimes} \nu \in M_+(X \times Y, \tau)$ which extends the product measure $\mu \otimes \nu$ on $\mathscr{B}(X) \otimes \mathscr{B}(Y)$. The mapping*

$$T: M_+(X, \tau) \times M_+(Y, \tau) \to M_+(X \times Y, \tau)$$

defined by $T(\mu, \nu) := \mu \hat{\otimes} \nu$ has the following properties:

(i) *T is continuous*
(ii) *$T(M_+(X, r, \tau) \times M_+(Y, r, \tau)) \subseteq M_+(X \times Y, r, \tau)$*
(iii) *$T(M_+(X, t) \times M_+(Y, t)) \subseteq M_+(X \times Y, t)$ if X and Y are Hausdorff spaces.*

PROOF. Every finite positive measure on $\mathscr{B}(X) \otimes \mathscr{B}(Y)$ has at most one extension to a τ-smooth measure on $X \times Y$, therefore we only have to show the existence of this extension.

Fix any point $x \in X$ and $\nu \in M_+(Y, \tau)$. The image measure ϱ of ν under the continuous mapping

$$Y \to X \times Y$$
$$y \mapsto (x, y)$$

is τ-smooth and clearly $\varrho(A \times B) = \varepsilon_x(A)\nu(B)$ for all $A \in \mathscr{B}(X)$ and $B \in \mathscr{B}(Y)$ [ε_x denoting the one-point measure in x], hence $\varrho = \varepsilon_x \hat{\otimes} \nu$. In a first step we show that the mapping

$$X \times M_+(Y, \tau) \to M_+(X \times Y, \tau)$$
$$(x, \nu) \mapsto \varepsilon_x \hat{\otimes} \nu$$

is continuous.

Assume that $x_\alpha \to x$ and $\nu_\alpha \to \nu$. Let $G_1, \ldots, G_n \subseteqq X$ and $H_1, \ldots, H_n \subseteqq Y$ be open and put $U := \bigcup_{i=1}^n (G_i \times H_i)$. We shall show

$$\liminf_\alpha \varepsilon_{x_\alpha} \otimes \nu_\alpha(U) \geqq \varepsilon_x \otimes \nu(U) .$$

This holds trivially if $x \notin \bigcup_{i=1}^n G_i$. Suppose now that

$$I := \{i \leqq n : x \in G_i\} \neq \emptyset .$$

Then there exists α_0 such that $x_\alpha \in \bigcap_{i \in I} G_i$ for all $\alpha \geqq \alpha_0$ and for those α we get

$$\varepsilon_{x_\alpha} \otimes \nu_\alpha(U) \geqq \varepsilon_{x_\alpha} \otimes \nu_\alpha\Big(\bigcup_{i \in I} (G_i \times H_i)\Big) = \nu_\alpha\Big(\bigcup_{i \in I} H_i\Big),$$

hence

$$\liminf_\alpha \varepsilon_{x_\alpha} \otimes \nu_\alpha(U) \geqq \liminf_\alpha \nu_\alpha\Big(\bigcup_{i \in I} H_i\Big)$$
$$\geqq \nu\Big(\bigcup_{i \in I} H_i\Big) = \varepsilon_x \otimes \nu(U) .$$

Every open set $U \subseteqq X \times Y$ has the form $U = \bigcup_{\lambda \in \Lambda} (G_\lambda \times H_\lambda)$ for suitable open sets $G_\lambda \subseteqq X$ and $H_\lambda \subseteqq Y$. The measure $\varepsilon_x \hat{\otimes} \nu$ being τ-smooth, we can find, given $\varepsilon > 0$, finitely many $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that

$$\varepsilon_x \otimes \nu\Big(\bigcup_{i=1}^n (G_{\lambda_i} \times H_{\lambda_i})\Big) > \varepsilon_x \hat{\otimes} \nu(U) - \varepsilon .$$

From this we get

$$\liminf_\alpha \varepsilon_{x_\alpha} \hat{\otimes} \nu_\alpha(U) \geqq \liminf_\alpha \varepsilon_{x_\alpha} \otimes \nu_\alpha\Big(\bigcup_{i=1}^n (G_{\lambda_i} \times H_{\lambda_i})\Big)$$
$$\geqq \varepsilon_x \otimes \nu\Big(\bigcup_{i=1}^n (G_{\lambda_i} \times H_{\lambda_i})\Big) > \varepsilon_x \hat{\otimes} \nu(U) - \varepsilon .$$

By the Portmanteau theorem (cf. [5, Theorem 8.1]) we can conclude that $\varepsilon_{x_\alpha}\hat{\otimes}\nu_\alpha \to \varepsilon_x\hat{\otimes}\nu$.

The continuity of $Y \to X \times Y$, $y \mapsto (x,y)$ shows that the section

$$A_x = \{y \in Y: \ (x,y) \in A\}$$

belongs to $\mathscr{B}(Y)$ if $A \in \mathscr{B}(X \times Y)$. If $U \subseteq X \times Y$ is open then $(x,\nu) \mapsto \nu(U_x) = \varepsilon_x\hat{\otimes}\nu(U)$ is l.s.c. on $X \times M_+(Y,\tau)$; therefore $x \mapsto \nu(A_x)$ is Borel on X for any $\nu \in M_+(Y,\tau)$ and $A \in \mathscr{B}(X \times Y)$, it is furthermore l.s.c. if A is open.

Now we define $\mu\hat{\otimes}\nu$ for $\mu \in M_+(X,\tau)$ and $\nu \in M_+(Y,\tau)$ by

$$\mu\hat{\otimes}\nu(A) := \int_X \nu(A_x)\,d\mu(x), \qquad A \in \mathscr{B}(X \times Y)\,.$$

To show that $\mu\hat{\otimes}\nu$ is τ-smooth let U_α be open sets in $X \times Y$ filtering up to U. For any $x \in X$ then the open sections $(U_\alpha)_x$ filter up to U_x implying

$$\nu(U_x) = \sup_\alpha \nu((U_\alpha)_x) \quad \text{for all } x \in X\,.$$

But as stated above the functions $x \mapsto \nu((U_\alpha)_x)$ are l.s.c.; therefore we may apply P 15 of [5] to get

$$\mu\hat{\otimes}\nu(U) = \sup_\alpha \mu\hat{\otimes}\nu(U_\alpha)\,.$$

The unicity of a τ-smooth extension of $\mu\otimes\nu$ gives us

$$\mu\hat{\otimes}\nu(A) = \int_X \nu(A_x)\,d\mu(x) = \int_Y \mu(A^y)\,d\nu(y)$$

for any $A \in \mathscr{B}(X \times Y)$, where $A^y = \{x \in X : (x,y) \in A\}$. From this it is seen immediately that for any bounded or non-negative Borel function f on $X \times Y$ we have

$$\int_{X \times Y} f\,d(\mu\hat{\otimes}\nu) = \int_Y \int_X f(x,y)\,d\mu(x)\,d\nu(y) = \int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x)\,.$$

Now let f be a non-negative l.s.c. function on $X \times Y$. From the continuity of $(x,\nu) \mapsto \varepsilon_x\hat{\otimes}\nu$ we get that

$$X \times M_+(Y,\tau) \to \overline{\mathbf{R}_+}$$

$$(x,\nu) \mapsto \int_Y f(x,y)\,d\nu(y)$$

is also l.s.c., and applying this result a second time

$$M_+(X,\tau) \times M_+(Y,\tau) \to \overline{\mathbf{R}_+}$$

$$(\mu, \nu) \mapsto \int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x) = \int_{X \times Y} f\, d(\mu \hat{\otimes} \nu)$$

is l.s.c., too.

Let us assume $\mu_\alpha \to \mu$ and $\nu_\alpha \to \nu$ and let $U \subseteqq X \times Y$ be open. Then

$$\liminf_\alpha \mu_\alpha \hat{\otimes} \nu_\alpha(U) = \liminf_\alpha \int 1_U\, d(\mu_\alpha \hat{\otimes} \nu_\alpha)$$

$$\geqq \int 1_U\, d(\mu \hat{\otimes} \nu) = \mu \hat{\otimes} \nu(U),$$

therefore $\mu_\alpha \hat{\otimes} \nu_\alpha \to \mu \hat{\otimes} \nu$ by the Portmanteau theorem. We have proved (i).

If μ and ν are regular (tight) then

$$\mathscr{M} := \{A \in \mathscr{B}(X \times Y) : \mu \hat{\otimes} \nu(A) = \sup\{\mu \hat{\otimes} \nu(F) : F \subseteqq A \text{ closed (compact)}\}\}$$

is a monotone class containing $\{B \times C : B \in \mathscr{B}(X), C \in \mathscr{B}(Y)\}$ and also finite disjoint unions of this family. Hence by the theorem about monotone classes $\mathscr{B}(X) \otimes \mathscr{B}(Y) \subseteqq \mathscr{M}$. Let $U \subseteqq X \times Y$ be open and $W \subseteqq X \times Y$ be closed. Then using τ-smoothness of $\mu \hat{\otimes} \nu$ we get $U \cap W \in \mathscr{M}$ and the same for finite disjoint unions of sets of this form. Hence indeed $\mathscr{M} = \mathscr{B}(X \times Y)$ and this finishes the proof.

Theorem 1 extends by induction to the product of finitely many topological spaces. That it also extends to denumerable products, is shown in the next theorem.

THEOREM 2. *Let $X_1, X_2, \ldots$ be a sequence of (not necessarily Hausdorff) topological spaces, and let $\mu_i \in M_+^1(X_i, \tau)$ be probability measures on X_i, $i = 1, 2, \ldots$. Then there exists a uniquely determined τ-smooth measure $\hat{\otimes}_{i \in \mathbb{N}} \mu_i$ on $X := \prod_{i=1}^\infty X_i$ extending the product measure $\otimes_{i \in \mathbb{N}} \mu_i$ on $\otimes_{i \in \mathbb{N}} \mathscr{B}(X_i)$. The mapping $T: \prod_{i=1}^\infty M_+^1(X_i, \tau) \to M_+^1(X, \tau)$ defined by $T((\mu_i)) := \hat{\otimes}_{i \in \mathbb{N}} \mu_i$ has the following properties:*

(i) *T is continuous*
(ii) *$T(\prod_{i=1}^\infty M_+^1(X, r, \tau)) \subseteqq M_+^1(X, r, \tau)$*
(iii) *$T(\prod_{i=1}^\infty M_+^1(X_i, t)) \subseteqq M_+^1(X, t)$ if all the X_i's are Hausdorff spaces.*

PROOF. Let $\pi_n: X \to \prod_{i=1}^n X_i$ be the natural projection. Every open set $U \subseteqq X$ has the form $U = \bigcup_{n=1}^\infty \pi_n^{-1}(G_n)$ where $G_n \subseteqq \prod_{i=1}^n X_i$ is open for all n and where $\pi_n^{-1}(G_n)$ increases. Therefore, denoting

$$\mathscr{A}_n := \pi_n^{-1}\left(\mathscr{B}\left(\prod_{i=1}^n X_i\right)\right)$$

$\mathscr{A}_\infty := \bigcup_{n=1}^\infty \mathscr{A}_n$ is an algebra generating $\mathscr{B}(X)$. From Theorem 1 we get in the usual way a finitely additive set function μ: $\mathscr{A}_\infty \to [0,1]$ which is σ-additive on each $\mathscr{A}_n$, namely

$$\mu(\pi_n^{-1}(B_n)) := \hat{\bigotimes_{i \leqq n}} \mu_i(B_n), \qquad B_n \in \mathscr{B}\left(\prod_{i=1}^n X_i\right).$$

To show that μ has a σ-additive extension to $\mathscr{B}(X)$, it has to be proved that $\{A_j\}_1^\infty \subseteqq \mathscr{A}_\infty$, $A_j \downarrow \varnothing$ implies $\mu(A_j) \to 0$ and this in fact can be seen by an easy adaptation of the proof of the product-measure-theorem, see for ex. [1, p. 139/140]. The unique σ-additive extension to $\mathscr{B}(X)$ we denote also by μ.

Now we have to show that μ is τ-smooth. For that reason let the net U_λ of open subsets in X filter up to U. As already remarked there exist open sets $G_n^\lambda \subseteqq \prod_{i=1}^n X_i$ such that $U_\lambda = \bigcup_{n=1}^\infty \pi_n^{-1}(G_n^\lambda)$. Put

$$H_n^\lambda := \bigcup \left\{ W \subseteqq \prod_{i=1}^n X_i \, : \, W \text{ open and } \pi_n^{-1}(W) \subseteqq U_\lambda \right\},$$

then $G_n^\lambda \subseteqq H_n^\lambda$, H_n^λ is open, $U_\lambda = \bigcup_{n=1}^\infty \pi_n^{-1}(H_n^\lambda)$ and furthermore $\{H_n^\lambda\}_\lambda$ filters up to his union for every $n \in \mathbf{N}$. From

$$U = \bigcup_\lambda U_\lambda = \bigcup_{n=1}^\infty \pi_n^{-1}\left(\bigcup_\lambda H_n^\lambda\right)$$

we get $\mu(U) = \sup_\lambda \mu(U_\lambda)$, using first the σ-additivity of μ and then the τ-smoothness of the finite products $\mu_1 \hat{\otimes} \cdots \hat{\otimes} \mu_n$. Hence μ is the uniquely determined τ-smooth extension of $\otimes_{i \in \mathbf{N}} \mu_i$ to $\mathscr{B}(X)$.

The continuity of T will follow if we can show that for every open set $U \subseteqq X$ the mapping

$$\prod_{i=1}^\infty M_+^1(X_i, \tau) \to \mathbf{R}$$

$$(\mu_i) \mapsto \hat{\bigotimes_{i \in \mathbf{N}}} \mu_i(U)$$

is l.s.c. Writing $U = \bigcup_{u=1}^\infty \pi_n^{-1}(G_n)$, $G_n \subseteqq \prod_{i=1}^n X_i$ open, and with increasing $\pi_n^{-1}(G_n)$, we get this from Theorem 1.

If all the μ_i's are regular, then $\mu = \hat{\otimes}_{i \in \mathbf{N}} \mu_i$ is regular on $\mathscr{A}_\infty$, hence on $\mathscr{B}(X)$. If all the μ_i's are tight measures on Hausdorff spaces X_i, then μ is regular on $\mathscr{A}_\infty$ and

$$\mu(X) = \sup\{\mu(K) : K \subseteqq X \text{ compact}\}.$$

This implies that μ is tight on $\mathscr{A}_\infty$ and therefore $\mu \in M_+(X, t)$.

2. Measurability for measures on analytic spaces.

Let X be a topological space and define $\varphi_B\colon M_+(X) \to \mathsf{R}$ by $\varphi_B(\mu) := \mu(B)$, where B is a Borel set in X. These functions can hardly be expected to be continuous, but they are l.s.c. if B is open, and then a standard argument shows that they are always Borel. We assume now that X is an analytic space, i.e. a Hausdorff space which is the continuous image of some polish space. Then if A is an analytic subset of X or if A is a-measurable (i.e. belongs to the σ-field spanned by the analytic subsets of X) the function φ_A is well defined, and we want to show that it still has some measurability property.

First we must prove two lemmas. Let Y be a Hausdorff space and consider on

$$\mathscr{K}(Y) := \{K \subseteqq Y : K \text{ is compact}\}$$

the topology generated by

$$\{\{K \in \mathscr{K}(Y) : K \subseteqq G\} : G \subseteqq Y \text{ open}\} \quad \text{and}$$

$$\{\{K \in \mathscr{K}(Y) : K \cap G \neq \emptyset\} : G \subseteqq Y \text{ open}\} .$$

It is easy to see that this topology is again Hausdorff. We are going to use the very important fact that $\mathscr{K}(Y)$ is a polish space in this topology, if Y is polish (cf. [2, Chapter 3]).

LEMMA 1. *Let $f\colon Y \to X$ be a continuous mapping between two Hausdorff spaces. Then*

$$X \times \mathscr{K}(Y) \to \mathsf{R}$$

$$(x, K) \mapsto 1_{f(K)}(x)$$

is upper semi-continuous (*u.s.c.*).

PROOF. Let $x_\alpha \to x_0$ and $K_\alpha \to K_0$. We want to show that

$$\limsup 1_{f(K_\alpha)}(x_\alpha) \leqq 1_{f(K_0)}(x_0) .$$

This is trivially true if $x_0 \in f(K_0)$. Assume $x_0 \notin f(K_0)$, then there exist open disjoint set $G, H \subseteqq X$ such that $x_0 \in G$ and $f(K_0) \subseteqq H$. The open set $\{K \in \mathscr{K}(Y) : K \subseteqq f^{-1}(H)\}$ contains K_0, hence $K_\alpha \subseteqq f^{-1}(H)$ for all $\alpha \geqq \alpha_1, \alpha_1$ suitably chosen; furthermore $x_\alpha \in G$ for all $\alpha \geqq \alpha_2, \alpha_2$ suitable. Choose $\alpha_3 \geqq \alpha_1$, $\alpha_3 \geqq \alpha_2$. For all $\alpha \geqq \alpha_3$ we have $x_\alpha \notin f(K_\alpha)$ and therefore

$$\limsup 1_{f(K_\alpha)}(x_\alpha) = 0 .$$

LEMMA 2. *Let X be an analytic space, Y a polish space and $f\colon Y \to X$ a continuous mapping. Then*

$$M_+(X) \times \mathscr{K}(Y) \to \mathsf{R}$$
$$(\mu, K) \mapsto \mu(f(K))$$

is a u.s.c.-function.

PROOF. By Lemma 1 the function $\varrho \mapsto \int 1_{f(L)}(x)\, d\varrho(x, L)$ is u.s.c. on $M_+(X \times \mathscr{K}(Y))$. On the other hand the canonical map $M_+(X) \times M_+(\mathscr{K}(Y)) \to M_+(X \times \mathscr{K}(Y))$ is continuous by Theorem 1. This implies that

$$(\mu, K) \mapsto \int 1_{f(L)}(x)\, d(\mu \otimes \varepsilon_K)(x, L)$$
$$= \int 1_{f(K)}(x)\, d\mu(x) = \mu(f(K))$$

is u.s.c. on $M_+(X) \times \mathscr{K}(Y)$.

We recall that a real valued function f defined on an analytic space X is by definition an *S-function* iff $\{f > t\}$ is analytic for all $t \in \mathsf{R}$.

THEOREM 3. *Let X be an analytic space and suppose that A is an analytic subset of X (respectively that A is a-measurable). Then φ_A is an S-function (respectively an a-measurable function) on $M_+(X)$.*

PROOF. $M_+(X)$ is again an analytic space, cf. [4, Appendix, Theorem 7]. Suppose $A \subseteqq X$ is analytic, then there exists a polish space P and a continuous surjection f: $P \to A$. One easily can see that

$$\mu(A) = \sup\{\mu(f(K)) : K \in \mathscr{K}(P)\} \quad \text{for all } \mu \in M_+(X).$$

Hence, if $t \in \mathsf{R}$,

$$\{\mu \in M_+(X) : \mu(A) > t\} = \operatorname{proj}\{(\mu, K) \in M_+(X) \times \mathscr{K}(P) : \mu(f(K)) > t\}$$

is an analytic subset of $M_+(X)$, being the projection of a Borel set in the analytic space $M_+(X) \times \mathscr{K}(P)$. Therefore φ_A is an S-function on $M_+(X)$. The second statement follows immediately.

REMARK. For a polish space X the above result has been proved by M. Traki, see [6]. H. Wiltmann has generalized Traki's method and gave independently another proof of Theorem 3.

The next result we want to show, namely that φ_A is universally measurable on $M_+(X)$ if A is a universally measurable subset of X, is not a topological statement. We only assume $(X, \mathscr{B})$ to be an abstract measurable space and denote by $\mathscr{B}_u$ the σ-field of universally measurable subsets of X with respect to $\mathscr{B}$, i.e. the intersection of all completions $\mathscr{B}_\mu$ where $\mu \in M^1_+(X,\mathscr{B})$, the set of all probability measures on $(X,\mathscr{B})$. The space $M_+(X,\mathscr{B})$ of all totally finite measures on $(X,\mathscr{B})$ is equipped with the σ-field $\mathscr{M}$ generated by the functions φ_B, $B \in \mathscr{B}$.

THEOREM 4. *Let $(X,\mathscr{B})$ be a measurable space and suppose that $A \subseteqq X$ is universally measurable. Then φ_A is universally measurable on $(M_+(X,\mathscr{B}),\mathscr{M})$.*

PROOF. Let $t \in \mathrm{R}$ and $\varrho \in M_+(M_+(X),\mathscr{M})$. Without any restriction we may assume ϱ to be concentrated on

$$M^* := \{\mu \in M_+(X,\mathscr{B}) : \mu(X) \leqq 1\} .$$

We define a measure μ_ϱ on $(X,\mathscr{B})$ by

$$\mu_\varrho(B) := \int_{M^*} \mu(B)\, d\varrho(\mu) = \int \varphi_B \, d\varrho, \qquad B \in \mathscr{B}$$

and use the assumption on A to the effect that there exist $B_1, B_2 \in \mathscr{B}$ such that $B_1 \subseteqq A \subseteqq B_2$ and $\mu_\varrho(B_2 \setminus B_1) = 0$. We have

$$\{\mu : \mu(B_1) > t\} \subseteqq \{\mu : \mu(A) > t\} \subseteqq \{\mu : \mu(B_2) > t\}$$

and $\{\mu : \mu(B_i) > t\} \in \mathscr{M}$, $i = 1,2$. Furthermore

$$\{\mu : \mu(B_2) > t, \mu(B_1) \leqq t\} \subseteqq \{\mu : \mu(B_2 \setminus B_1) > 0\}$$

and $\varrho(\{\mu : \mu(B_2 \setminus B_1) > 0\}) = 0$. We conclude that $\{\mu : \mu(A) > t\}$ belongs to the ϱ-completion of $\mathscr{M}$, and therefore, ϱ being arbitrary, is universally measurable.

Now we are able to give a satisfactory answer to the problem mentioned in the introduction.

THEOREM 5. (i) *Let X, Y, Z be three analytic spaces and let $f\colon X \times Y \to Z$ be universally measurable. Consider the induced mapping*

$$F\colon M_+(X) \times Y \to M_+(Z)$$
$$(\mu, y) \mapsto \mu^{f_y}$$

where $f_y(x) := f(x,y)$ and μ^{f_y} is the image measure of μ under f_y. The following holds:

(1) *F is universally measurable.*
(2) *If f is a-measurable, Borel measurable or continuous, then F has the corresponding property.*

(ii) *Suppose that in* (i) *$Z = \mathrm{R}$ and that f is bounded.*

Consider the function

$$\hat{F}: M_+(X) \times Y \to \mathsf{R}$$

$$(\mu, y) \mapsto \int_X f(x, y)\, d\mu(x)\,.$$

The following holds:

(1′) $\hat{F}$ *is universally measurable.*

(2′) *If f is a-measurable, an S-function, Borel measurable, u.s.c. or continuous, then $\hat{F}$ has the corresponding property.*

PROOF. (i) An easy direct argument shows that f_y is universally measurable for all $y \in Y$, hence F is well defined. Let $\psi: M_+(X \times Y) \to M_+(Z)$ be the map which sends $\varrho \in M_+(X \times Y)$ to its image measure under f. The fact that $\mathscr{B}(M_+(Z))$ is generated by the maps $\mu \mapsto \mu(C)$, $C \in \mathscr{B}(Z)$ (cf. [4, Appendix, Theorem 8]) together with Theorem 4 shows that ψ is universally measurable. Furthermore μ is a-measurable, Borel or continuous if f has the resp. property. Observing that F is the composition of ψ with the canonical map $M_+(X) \times Y \to M_+(X \times Y)$ which is continuous by Theorem 1, part (i) is proved.

(ii) Choose $a, b \in \mathsf{R}$ such that $f(X \times Y) \subseteqq [a, b]$. The function $M_+([a,b]) \to \mathsf{R}, \nu \mapsto \int_{[a,b]} t\, d\nu(t)$ is continuous, hence (1′) and most of (2′) follows from (i). The only point remaining to be proved is that $\hat{F}$ is an S-function if f is. From Theorem 3 we get this if f is the indicator of an analytic subset of $X \times Y$, and this extends to general S-functions by standard arguments.

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