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ON COMPLETELY MONOTONE FUNCTIONS ON $C_+(X)$

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1. Introduction.

Let X be a completely regular Hausdorff space, let $C(X)$ denote the vector space of all bounded realvalued continuous functions on X and $M(X)$ the vector space of all real Radon measures on X. The positive cones in $C(X)$ and $M(X)$ are denoted by $C_+(X)$ and $M_+(X)$.

Under pointwise addition the cone $C_+(X)$ becomes a 2-divisible abelian semigroup in the sense of [1]. As in [1] we define the character semigroup $\hat{S}$ of $S:=C_+(X)$ by $\varrho \in \hat{S}$ if and only if $\varrho\colon S \to [0,1]$ and

$$\varrho(0) = 1 \tag{1.1}$$

$$\varrho(f+g) = \varrho(f)\varrho(g) \quad \forall f,g \in S . \tag{1.2}$$

In the topology of pointwise convergence and with pointwise multiplication $\hat{S}$ becomes a compact topological abelian semigroup.

Let L denote all functionals $\lambda\colon C_+(X) \to [0,\infty]$ satisfying

$$\lambda(0) = 0 \tag{1.3}$$

and

$$\lambda(f+g) = \lambda(f)+\lambda(g) \quad \forall f,g \in C_+(X) . \tag{1.4}$$

Each λ satisfying (1.3) and (1.4) is increasing and hence positive homogeneous, i.e.

$$\lambda(af) = a\lambda(f) \quad \forall a \geqq 0,\ \forall f \in C_+(X) , \tag{1.5}$$

with the usual conventions $0\cdot\infty=0$ and $a\cdot\infty=\infty$, $\forall a>0$. Equipped with the topology of pointwise convergence L becomes compact and

$$\lambda(\cdot) \mapsto e^{-\lambda(\cdot)}$$

is a homeomorphism of L onto $\hat{S}$.

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We shall consider the following subsets of L:

$$L_0 := \{\lambda \in L \mid \lambda(f) < \infty,\ \forall f \in C_+(X)\}$$

and, if $Y \subseteq X$

$$L_Y := \left\{\lambda \in L_0 \,\middle|\, \exists \mu \in M_+(Y) \text{ such that } \lambda(f) = \int_Y f\, d\mu\ \forall f \in C_+(X)\right\}.$$

Let w^* be the *weak topology* on $M(X)$, that is, $w^* = \sigma(M(X), C(X))$, then the map

$$\mu \mapsto \int_X \cdot\, d\mu \tag{1.6}$$

is a homeomorphism of $M_+(X)$ onto L_X.

Let X denote the Stone–Čech compactification of X and let $\tilde{f}$ denote the unique continuous extension of f to βX, for all $f \in C(X)$. Then the map

$$\tilde{\mu} \mapsto \int_{\beta X} \tilde{\cdot}\, d\tilde{\mu} \tag{1.7}$$

is a homeomorphism of $(M_+(\beta X), w^*)$ onto L_0.

A function φ: $C_+(X) \to \mathbf{R}$ is *completely monotone* if and only if φ is bounded and

$$\sum_{i=j}^{n} \sum_{j=1}^{n} c_i c_j \varphi(f_i + f_j) \geqq 0$$

for all $n \in \mathbf{N}$, $c_1, \ldots, c_n \in \mathbf{R}$ and $f_1, \ldots, f_n \in C_+(X)$, (cf. [2] and Theorem 4.2 in [1]). From [2] we know that every completely monotone function φ: $C_+(X) \to \mathbf{R}$ has a unique *representing measure* $\xi \in M_+(L)$ in the sense that

$$\varphi(f) = \int_L e^{-\lambda(f)}\, d\xi(\lambda),\ \forall f \in C_+(X)\,. \tag{1.8}$$

Our aim in the following will be to establish a connection between continuity properties of φ and the concentration of the measure ξ to some "nice" subsets of L. A very special result of this type has already been proved in Theorem 6.1 of [1]. There X is the finite set $\{1, \ldots, p\}$ with discrete topology, $C_+(X)$ can be identified with $\mathbf{R}_+^p$ and L with $[0, \infty]^p$ and it is shown that a completely monotone functions on $\mathbf{R}_+^p$ is continuous if and only if the representing measure is concentrated on $\mathbf{R}_+^p$.

If we consider the dual pair $(C(X), M(X))$, two natural topologies on $C(X)$ arise, the *weak topology*, denoted by w, and the *Mackey topology*, which we shall denote by m. We shall need two further topologies. First we define the L_1-*topology* τ on $C(X)$ by the family of seminorms

$$r_{K,\sigma}(f) := \int_K \left| \int_X f\, d\mu \right| d\sigma(\mu) \tag{1.9}$$

where K runs through all w^*-compact, uniformly tight subsets of $M(X)$ and σ runs through $M_+(M(X), w^*)$. There is a simpler description of this topology, but first we need a lemma:

LEMMA 1.1. *Let $\sigma \in M_+(M(X), w^*)$ and suppose that the function $\mu \mapsto |\mu|(X)$ is σ-integrable, then*

$$\lambda(A) := \int_{M(X)} |\mu|(A)\, d\sigma(\mu), \qquad A \in \mathscr{B}(X)$$

is a positive finite τ-smooth measure on $(X, \mathscr{B}(X))$, and for every bounded Borel functions g on X we have

$$\int_X g\, d\lambda = \int_{M(X)} \left(\int_X g\, d|\mu| \right) d\sigma(\mu)\,.$$

If σ satisfies

$$\sigma(M(X)) = \sup \{\sigma(K) \mid K \text{ uniformly tight and closed}\}$$

then λ is a Radon measure on X.

REMARK. $\mathscr{B}(X)$ of course denotes the Borel σ-algebra of X. The notion of τ-smoothness may be found in [7, p. XII], and from P 16 p. XIII in [7] it follows that if X can be homeomorphically embedded as a universally measurable subset of a compact space Y, then every τ-smooth finite measure on X is a Radon measure (e.g. if X is analytic, or if X is σ-compact, or if X is locally compact or if X is complete in the sense of Čech). From Proposition 1 in [5] we know that the function $\mu \mapsto |\mu|(A)$ is Borel on $M(X)$ for every Borel set $A \subseteq X$, it is lower semicontinuous if A is open.

PROOF OF LEMMA 1.1. In the first part we only need to show τ-smoothness of λ. Let a collection of open sets $G_\alpha \subseteq X$ filter up to G. Then the lower semicontinuous functions $\mu \mapsto |\mu|(G_\alpha)$ filter up to $\mu \mapsto |\mu|(G)$ and applying P 15 of [7] we get $\lambda(G) = \sup \lambda(G_\alpha)$.

The second part is proved in a straightforward manner, taking into account that λ is inner regular w.r.t. the closed subsets of X, cf. P 16 in [7].

COROLLARY 1.2. *The L_1-topology τ on $C(X)$ is generated by the seminorms*

$$q_\mu(f) := \int_X |f|\,d\mu$$

where μ runs through $M_+(X)$.

PROOF. Let K be a w^*-compact and uniformly tight subset of $M(X)$ and σ a positive finite Radon measure on $M(X)$, then

$$\mu(A) := \int_K |v|(A)\,d\sigma(v) \qquad A \in \mathscr{B}(X)$$

is a finite positive Radon measure on X by Lemma 1.1, and

$$r_{K,\sigma}(f) \leqq \int_K \left(\int_X |f|\,dv\right) d\sigma(v) = \int_X |f|\,d\mu\,.$$

If $\mu \in M_+(X)$, then there exists a measurable function $\alpha\colon X \to [0,1]$ such that $\{\alpha \geqq \varepsilon\}$ is compact for all $\varepsilon > 0$, and

$$\int_X \frac{1}{\alpha}\,d\mu < \infty\,.$$

Let $\psi(x) := \alpha(x)\delta_x$, where δ_x is the one point measure in x, then ψ is a Borel map from X into $M_+(X)$, $\psi(X)$ is uniformly tight and $K := \overline{\psi(X)}$ is therefore w^*-compact and uniformly tight. Let $d\lambda := (1/\alpha)d\mu$ and $\sigma := \lambda \circ \psi^{-1}$, then σ is a finite positive Radon measure on $M(X)$, and

$$\begin{aligned} q_\mu(f) &= \int_X |f|\,d\mu = \int_X \alpha(x)|f(x)|\,d\lambda(x) \\ &= \int_X \left|\int_X f(y)\,d(\psi(x))(y)\right| d\lambda(x) = \int_K \left|\int_X f\,dv\right| d(\lambda\circ\psi^{-1})(v) \\ &= \int_K \left|\int f\,dv\right| d\sigma(v)\,. \end{aligned}$$

This shows that $\{q_\mu\}$ and $\{r_{K,\sigma}\}$ generate the same topology.

We shall need a fourth topology on $C(X)$. This is the socalled *strict topology* on $C(X)$, which we denote by β. The strict topology is generated by the seminorms

$$p_\alpha(f) := \|\alpha f\|_X = \sup_{x\in X} |\alpha(x)f(x)|$$

where α runs through all bounded measurable functions on X which vanish at infinity, i.e. $\{|\alpha| \geqq \varepsilon\}$ is relatively compact for all $\varepsilon > 0$. This topology was first introduced by C. R. Buck for locally compact spaces and later generalized by many authors to general completely regular Hausdorff spaces (see e.g. [4]).

From Theorem 2 in [4] we know that $w \subseteqq \beta \subseteqq m$, and from Corollary 1.2 it follows easily that $w \subseteqq \tau \subseteqq \beta$, therefore we have

$$w \subseteqq \tau \subseteqq \beta \subseteqq m \tag{1.10}$$

and we shall leave to the reader to prove that $\tau \neq w$ and $\tau \neq \beta$ if X is infinite.

From Theorem 3 in [6] one easily deduces the following form of Riesz' representation theorem:

THEOREM 1.3. (D. Pollard and F. Topsøe [6]). *Let λ: $C_+(X) \to [0, \infty[$ be additive and suppose that λ satisfies*

(1.3.1) $\forall \varepsilon > 0 \; \exists \delta > 0 \; \exists$ *C compact $\subseteqq X$ such that $\lambda(f) \leqq \varepsilon$ whenever $0 \leqq f \leqq 1$ and $f \leqq \delta$ on C.*

Then there exists a unique measure $\mu \in M_+(X)$ representing λ, that is,

$$\lambda(f) = \int_X f \, d\mu, \qquad \forall f \in C_+(X) .$$

REMARK. Note that (1.3.1) holds if there exists $\{f_n\} \subseteqq C_+(X)$ with the following two properties:

$$\{f_n \leqq 1\} \text{ is compact for all } n \geqq 1 , \tag{1.3.2}$$

$$\lim_{n \to \infty} \lambda(f_n) = 0 . \tag{1.3.3}$$

2. Concentration of L_0 or L_X.

Let X be a completely regular Hausdorff space and φ a completely monotone function on $C_+(X)$, with representing measure ξ. We shall give necessary and sufficient conditions for ξ to be concentrated on L_0 or L_X. First we need a measurability lemma:

LEMMA 2.1. *If Y is a Borel subset of the Stone–Čech compactification βX, then L_Y is a Borel subset of L. The subset L_0 is open in L.*

PROOF. From $L_0 = \{\lambda \in L \mid \lambda(1) < \infty\}$ follows that L_0 is open. Identifying L_0 with $M_+(\beta X)$, see (1.7), we get

$$L_Y = \{\tilde{\mu} \in M_+(\beta X) \mid \tilde{\mu}(\beta X \setminus Y) = 0\}$$

and from the fact that $\tilde{\mu} \mapsto \tilde{\mu}(\beta X \setminus Y)$ is Borel on $M_+(\beta X)$ by Proposition 1 in [5] we deduce that L_Y is Borel in L_0, hence in L.

THEOREM 2.2. *Let* $\varphi\colon C_+(X) \to \mathbf{R}$ *be completely monotone with representing measure* ξ *(see* (1.8.)*). Then the following* 3 *statements are equivalent*:

$$\xi(L \setminus L_0) = 0\,, \tag{2.2.1}$$

$$\lim_{t\to 0} \varphi(tf) = \varphi(0), \quad \forall f \in C_+(X)\,, \tag{2.2.2}$$

$$\lim_{t\to 0} \varphi(t) = \varphi(0)\,, \tag{2.2.3}$$

where t *also denotes the constant function equal to* t.

PROOF. (2.2.1) $\Rightarrow$ (2.2.2): Let $f \in C_+(X)$ and define $F_t(\lambda) := e^{-t\lambda(f)}$. Then

$$0 \leqq F_t(\lambda) \leqq 1 \quad \text{and} \quad \lim_{t\to 0} F_t(\lambda) = 1$$

for all $\lambda \in L_0$. Hence the assumption implies that

$$\varphi(tf) = \int_{L_0} F_t(\lambda)\, d\xi(\lambda) \to \xi(L_0) = \varphi(0)\,,$$

as t tends to zero.

(2.2.2) $\Rightarrow$ (2.2.3): Obvious.

(2.2.3) $\Rightarrow$ (2.2.1): Let $F_t(\lambda)$ be defined as above but with f replaced by the constant 1. If $\lambda \in L \setminus L_0$ then $\lambda(1) = \infty$, therefore we get

$$\lim_{t\to 0} F_t(\lambda) = 1_{L_0}(\lambda) \quad \text{for all } \lambda \in L\,.$$

Hence by assumption

$$\xi(L) = \varphi(0) = \lim_{t\to 0} \int_L F_t(\lambda)\, d\xi(\lambda) = \xi(L_0)$$

and so $\xi(L \setminus L_0) = 0$.

THEOREM 2.3. *Let* $\varphi\colon C_+(X) \to \mathbf{R}$ *be a completely monotone function with representing measure* ξ, *and let* ϱ *be a topology on* $C(X)$ *satisfying* $\tau \subseteqq \varrho \subseteqq \beta$. *Then the following* 6 *statements are equivalent*:

(2.3.1) $\exists\, Y$ *σ-compact* $\subseteqq X$ *such that* $\xi(L \setminus L_Y) = 0$,
(2.3.2) $\xi(L) = \sup\{\xi(K) \mid K \subseteqq M_+(X)$ compact and uniformly tight$\}$,
(2.3.3) φ *is uniformly* ϱ*-continuous*,
(2.3.4) φ *is* ϱ*-continuous at* 0,

(2.3.5) *$\varphi \mid B$ is ϱ-continuous at 0, where $B=\{f \in C(X) \mid 0 \leqq f \leqq 1\}$,*

(2.3.6) *$\forall \varepsilon>0\ \exists \delta>0\ \exists\ C$ compact $\subseteqq X$ such that $\varphi(0)-\varphi(f) \leqq \varepsilon$ whenever $f \in B$ and $f \leqq \delta$ on C.*

Note that we identify $M_+(X)$ with L_X in (2.3.2) (*see* (1.6)).

Proof. (2.3.1) $\Rightarrow$ (2.3.2): First we note that $L_Y \subseteqq L_X \subseteqq L_0$ and therefore $\xi(L \backslash L_0)=0$. Let $\{C_n\}$ be compact sets in X with $C_1=\varnothing$ and $C_n \uparrow Y$; then we may define

$$F_n(\nu) := \nu(\beta X \backslash C_n) \quad \text{for } \nu \in L_0 = M_+(\beta X).$$

Then $F_n\colon L_0 \to [0, \infty[$ is Borel and $\lim_{n \to \infty} F_n(\nu)=0$ for all $\nu \in L_Y$. From Lemma 2.1 we know that $L_Y \in \mathscr{B}(L)$, and since $\xi(L \backslash L_Y)=0$ by assumption, we have that $F_n \to 0$ a.e. $[\xi]$. Hence by Egoroff's theorem we can find for any given $\varepsilon>0$ a sequence $a_1 \geqq a_2 \geqq \ldots \geqq 0$ of positive numbers such that

$$\lim_{n \to \infty} a_n = 0,$$

$$\xi(\{\nu \mid F_n(\nu) \leqq a_n, \forall n \geqq 1\}) \geqq \xi(L)-\varepsilon.$$

Now since F_n is lower semi-continuous on L_0 (see Proposition 1 in [5]) and $F_1(\nu)=\nu(\beta X)$, we have that

$$K := \{\nu \in L_0 \mid F_n(\nu) \leqq a_n, \forall n \geqq 1\}$$

is a compact uniformly tight subset of $M_+(X)$ with $\xi(K) \geqq \xi(L)-\varepsilon$. Hence (2.3.2) holds.

(2.3.2) $\Rightarrow$ (2.3.3): Given $\varepsilon>0$ choose $K \subseteqq M_+(X)$ compact and uniformly tight so that $\xi(K) \geqq \xi(L)-\varepsilon/4$. We claim that $|\varphi(f)-\varphi(g)|<\varepsilon$ whenever $r_{K,\xi}(|f-g|)<\varepsilon/2$ (see (1.9) for the definition of $r_{K,\xi}$). In fact, if $r_{K,\xi}(|f-g|)<\varepsilon/2$ for two functions $f, g \in C_+(X)$, then

$$\begin{aligned}
|\varphi(f)-\varphi(g)| &\leqq \varphi(f)-\varphi(f \vee g)+\varphi(g)-\varphi(f \vee g) \\
&= \int_L (e^{-\lambda(f)}-e^{-\lambda(f \vee g)})\, d\xi(\lambda)+\int_L (e^{-\lambda(g)}-e^{-\lambda(f \vee g)})\, d\xi(\lambda) \\
&\leqq \int_L [1-e^{-\lambda(f \vee g-f)}]\, d\xi(\lambda)+\int_L [1-e^{-\lambda(f \vee g-g)}]\, d\xi(\lambda) \\
&\leqq \frac{\varepsilon}{2}+\int_K \lambda(f \vee g-f)\, d\xi(\lambda)+\int_K \lambda(f \vee g-g)\, d\xi(\lambda) \\
&= \frac{\varepsilon}{2}+\int_K \lambda(2(f \vee g)-f-g)\, d\xi(\lambda)
\end{aligned}$$

$$= \frac{\varepsilon}{2} + \int_K \lambda(|f-g|)\,d\xi(\lambda)$$

$$= \frac{\varepsilon}{2} + r_{K,\xi}(|f-g|) < \varepsilon .$$

This shows that φ is uniformly τ-continuous. (2.3.3) follows because τ is weaker than ϱ by assumption.

(2.3.3) $\Rightarrow$ (2.3.4) $\Rightarrow$ (2.3.5): Obvious.

(2.3.5) $\Rightarrow$ (2.3.6): Since ϱ is weaker than β, we have that $\varphi \mid B$ is β-continuous at 0. Let $\varkappa$ be the topology on $C(X)$ of uniform convergence on compact subsets of X, then by Proposition 1 in [4], $\varkappa$ coincides with β on B. Hence $\varphi \mid B$ is $\varkappa$-continuous at 0 and this evidently implies (2.3.6).

(2.3.6) $\Rightarrow$ (2.3.1): First we note that (2.3.6) implies that $\lim_{t\to 0}\varphi(t)=\varphi(0)$, therefore $\xi(L\setminus L_0)=0$ by Theorem 2.1. Now let

$$M_n := \{\nu \in L_0 \mid \nu(1) \leqq n\} .$$

Let $f \in C_+(X)$ and define

$$F_t(\lambda) := \frac{1}{t}(1-e^{-t\lambda(f)}) \quad \text{for } t>0,\ \lambda \in L_0 .$$

Then we have

$$\lim_{t\to 0} F_t(\lambda) = \lambda(f) \quad \forall \lambda \in L_0$$

$$0 \leqq F_t(\lambda) \leqq \lambda(f) \leqq \|f\|_X \lambda(1)$$

and this implies that

$$\mu_n(f) := \int_{M_n} \lambda(f)\,d\xi(\lambda) = \lim_{t\to 0} \int_{M_n} F_t(\lambda)\,d\xi(\lambda)$$

for all $f \in C_+(X)$. Let for $A \in \mathscr{B}(\beta X)$

$$\tilde{\mu}_n(A) := \int_{M_n} \lambda(A)\,d\xi(\lambda)$$

then by Lemma 1.1 $\tilde{\mu}_n$ is a positive Radon measure on βX with

$$\mu_n(f) = \int_{\beta X} \bar{f}\,d\tilde{\mu}_n \quad \forall f \in C(X) .$$

Now we use the elementary inequality

$$x \leqq (1+a)(1-e^{-x}) \quad \text{for } 0 \leqq x \leqq a$$

to conclude that

$$F_t(\lambda) \leqq \lambda(f) \leqq (n+1)(1-e^{-\lambda(f)})$$

for $f \in B$ and $\lambda \in M_n$. Hence we get

$$\mu_n(f) \leqq (n+1)\int_{M_n} (1-e^{-\lambda(f)})\,d\xi(\lambda) \leqq (n+1)(\varphi(0)-\varphi(f))$$

for all $f \in B$. The assumption (2.3.6) now implies that μ_n satisfies (1.3.1) and by Theorem 1.3 we have that X is $\tilde{\mu}_n$-measurable and $\tilde{\mu}_n(\beta X \backslash X)=0$.

Hence we can find a σ-compact subset $Y \subseteq X$ such that $\tilde{\mu}_n(\beta X \backslash Y)=0$ for all $n \geqq 1$. But then we have

$$\xi(\{\lambda \in M_n \mid \lambda(\beta X \backslash Y)>0\}) = 0 \qquad \forall n \geqq 1$$

and since $M_n \uparrow L_0$ and $\xi(L \backslash L_0)=0$, we finally get

$$\xi(L \backslash L_Y) = \xi(L_0 \backslash L_Y) = \xi(\{\lambda \in L_0 \mid \lambda(\beta X \backslash Y)>0\}) = 0$$

which proves our theorem.

3. The Lévy continuity theorem on $M_+(X)$.

Let X be a completely regular Hausdorff space. Then $M_+(M_+(X))$ denotes the set of positive finite Radon measures on $(M_+(X), w^*)$, and $M_t(M_+(X))$ denotes the set of all $\sigma \in M_+(M_+(X))$ satisfying

(3.1) $\quad \sigma(M_+(X)) = \sup\{\sigma(K) \mid K \text{ compact and uniformly tight}\}$.

Note that $M_t(M_+(X)) = M_+(M_+(X))$ if X is semi-Radonian (see Theorem 10 in [5]).

If $\sigma \in M_+(M_+(X))$, then we define its *Laplace transform* $\hat{\sigma}$ by

$$\hat{\sigma}(f) := \int_{M_+(X)} \exp\left(-\int_X f\,dv\right) d\sigma(v)$$

for $f \in C_+(X)$. Note that $\hat{\sigma}$ is completely monotone on $C_+(X)$. If X is σ-compact, then the set of all Laplace transforms of measures on $M_+(X)$ is characterised by those completely monotone functions φ on $C_+(X)$ which satisfy one of the continuity properties (2.3.3)–(2.3.6) stated in Theorem 2.3.

We shall consider $M_+(M_+(X))$ and $M_t(M_+(X))$ equipped with their weak topologies, coming from the space $C(M_+(X), w^*)$. Let ψ denote the map $M_+(X) \to L$ given by (1.6), and let

$$\Psi(\sigma) := \sigma \circ \psi^{-1}$$

be the corresponding map from $M_+(M_+(X))$ to $M_+(L)$. It is easily checked that

(3.2) Ψ is a homeomorphism of $M_+(M_+(X))$ onto

$$M_X(L) := \{\xi \in M_+(L) \mid \xi^*(L \setminus L_X) = 0\}$$

(see e.g. Corollary 9 in [3, p. 244]).

THEOREM 3.1. *Let $\{\sigma_\alpha\}$ be a net in $M_+(M_+(X))$ satisfying*

(3.1.1) $\sup_\alpha \sigma_\alpha(M_+(X)) < \infty$,
(3.1.2) $\hat{\sigma}_\alpha(f) \to \varphi(f)$ *for all* $f \in C_+(X)$, *where* $\varphi \mid B$ *is* β*-continuous at* 0, $B := \{f \in C(X) \mid 0 \leqq f \leqq 1\}$.

Then there exists a measure $\sigma \in M_t(M_+(X))$ whose Laplace transform is φ and $\sigma_\alpha \to \sigma$ weakly.

PROOF. Let $A := \sup_\alpha \sigma_\alpha(M_+(X))$ and let

$$M_A := \{\xi \in M_+(L) \mid \xi(L) \leqq A\}.$$

Then $\xi_\alpha := \Psi(\sigma_\alpha) \in M_A$ for all α, and M_A is a compact subset of $M_+(L)$. If ξ is a limit point of $\{\xi_\alpha\}$, then

$$\begin{aligned}\hat{\xi}(f) &= \int_L e^{-\lambda(f)}\,d\xi(\lambda) = \lim_\alpha \int_L e^{-\lambda(f)}\,d\xi_\alpha(\lambda) \\ &= \lim_\alpha \hat{\sigma}_\alpha(f) = \varphi(f).\end{aligned}$$

Hence $\hat{\xi}$ is a completely monotone function on $C_+(X)$, with representing measure ξ. Since a measure on L is uniquely determined by its Laplace transform (see Corollary 2.5 of [1]), we find that $\{\xi_\alpha\}$ admits at most one limit point in $M_+(L)$. Hence $\xi = \lim_\alpha \xi_\alpha$ exists and $\hat{\xi} = \varphi$.

Then by (3.1.2) and Theorem 2.3 we conclude that $\xi = \Psi(\sigma)$ for some $\sigma \in M_t(M_+(X))$, and $\hat{\sigma} = \varphi$. Therefore by (3.2) we find that $\sigma_\alpha \to \sigma$ weakly.

THEOREM 3.2. *Let $\mathscr{K}$ be a subset of $M^1_+(M_+(X))$, the probability Radon measures on $M_+(X)$. Let again $B := \{f \in C_+(X) \mid 0 \leqq f \leqq 1\}$. Then we have*

(i) *If $\{\hat{\sigma} \mid B \mid \sigma \in \mathscr{K}\}$ is β-equicontinuous at* 0, *then $\mathscr{K}$ is a relatively compact subset of $M_t(M_+(X))$.*
(ii) *If $\mathscr{K}$ is uniformly tight and X is a Prohorov space (see e.g.* [5] *for this notion), then $\{\hat{\sigma} \mid B \mid \sigma \in \mathscr{K}\}$ is β-equicontinuous at* 0.

PROOF. (i). Follows immediately from Theorem 3.1.

(ii). Let $\varepsilon>0$ be given. There exists by assumption a compact set $K\subseteqq M_+(X)$ such that

$$\sup\{\sigma(M_+(X)\backslash K) \mid \sigma\in\mathscr{K}\} < \frac{\varepsilon}{3}.$$

X being a Prohorov space we can find a compact subset C of X such that

$$\sup\{\nu(X\backslash C) \mid \nu\in K\} < \frac{\varepsilon}{3}.$$

From the compactness of K we deduce that $A:=\sup\{\nu(X) \mid \nu\in K\}$ is finite. Now suppose that $f\in B$, $f\,|\,C<\varepsilon/3A$ and $\sigma\in\mathscr{K}$. Then we get

$$1-\hat{\sigma}(f) = \int_{M_+(X)}(1-e^{-\lambda(f)})\,d\sigma(\lambda) \leqq \frac{\varepsilon}{3}+\int_K(1-e^{-\lambda(f)})\,d\sigma(\lambda)$$

and for $\lambda\in K$

$$1-e^{-\lambda(f)} \leqq \lambda(f) \leqq \frac{\varepsilon}{3}+\int_C f\,d\lambda \leqq \frac{2\varepsilon}{3}$$

hence $1-\hat{\sigma}(f)\leqq\varepsilon$, showing the required β-equicontinuity at 0.

The next theorem might be more useful for applications. Note that if $\sigma\in M_+(M_+(X))$, then its Laplace transform is defined in a natural way on all non-negative Borel functions on X, in particular on Borel subsets of X.

THEOREM 3.3. *Let $\mathscr{K}\subseteqq M_+^1(M_+(X))$ satisfy the following two conditions*

(3.3.1) $\lim_{A\to\infty}\sup_{\sigma\in\mathscr{K}}\sigma(\{\nu\in M_+(X) \mid \nu(C)>A\}) = 0,\ \forall\, C\subseteqq X$ *compact,*

(3.3.2) $\lim_{C}\sup_{\sigma\in\mathscr{K}}(1-\hat{\sigma}(X\backslash C)) = 0,$

where the limit is taken along the net of compact subsets of X. Then $\mathscr{K}$ is a relatively compact subset of $M_+(M_+(X))$.

PROOF. Let $0<\varepsilon<1$, $0<\delta<1$; then $1-e^{-\delta}\geqq\frac{1}{2}\delta$ and

$$1-\hat{\sigma}(X\backslash C) = \int_{M_+(X)}(1-e^{-\lambda(X\backslash C)})\,d\sigma(\lambda)$$

$$\geqq \int_{\{\lambda \mid \lambda(X\backslash C)\geqq\frac{1}{2}\delta\}}(1-e^{-\lambda(X\backslash C)})\,d\sigma(\lambda) \geqq \frac{\delta}{4}\sigma\left(\left\{\lambda \,\middle|\, \lambda(X\backslash C)\geqq\frac{\delta}{2}\right\}\right).$$

By (3.3.2) there exists a compact set $C \subseteq X$ such that

$$\sup_{\sigma \in \mathscr{K}} \frac{\delta}{4} \sigma\left(\left\{\lambda \;\middle|\; \lambda(X \backslash C) \geqq \frac{\delta}{2}\right\}\right) \leqq \frac{\varepsilon\delta}{24}$$

hence

$$\inf_{\sigma \in \mathscr{K}} \sigma\left(\left\{\lambda \;\middle|\; \lambda(X \backslash C) < \frac{\delta}{2}\right\}\right) \geqq 1 - \frac{\varepsilon}{6}$$

and applying (3.3.1) we find $A \in \mathbf{R}$ such that

$$\inf_{\sigma \in \mathscr{K}} \sigma(\{\lambda \mid \lambda(C) \leqq A\}) \geqq 1 - \frac{\varepsilon}{6}.$$

Putting

$$L_1 := \left\{\lambda \in M_+(X) \;\middle|\; \lambda(X \backslash C) < \frac{\delta}{2} \text{ and } \lambda(C) \leqq A\right\}$$

we have

$$\inf_{\sigma \in \mathscr{K}} \sigma(L_1) \geqq 1 - \frac{\varepsilon}{3}.$$

Now let $f \in C_+(X)$, $0 \leqq f \leqq 1$ and $\sup_{x \in C} f(x) < \varepsilon/3A$. Then for any $\sigma \in \mathscr{K}$ we get

$$1 - \hat{\sigma}(f) = \int_{M_+(X)} (1 - e^{-\lambda(f)})\, d\sigma(\lambda) \leqq \frac{\varepsilon}{3} + \int_{L_1} \lambda(f)\, d\sigma(\lambda)$$

and for $\lambda \in L_1$

$$\lambda(f) = \int_X f\, d\lambda \leqq \frac{\delta}{2} + \int_C f\, d\lambda \leqq \frac{\delta}{2} + \frac{\varepsilon}{3}.$$

Hence, choosing $\delta = \frac{2}{3}\varepsilon$, $1 - \hat{\sigma}(f) \leqq \varepsilon$, proving β-equicontinuity of $\{\hat{\sigma} | B \mid \sigma \in \mathscr{K}\}$ at 0. From Theorem 3.2 we get the desired conclusion.

4. Completely alternating functions on $C_+(X)$.

A class of functions on $C_+(X)$, closely connected to completely monotone functions, is that of completely alternating (or alternating of infinite order) functions. A function ψ: $C_+(X) \to [0, \infty]$ is *completely alternating* if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \psi(f_i + f_j) \leqq 0$$

for all $n \in \mathbf{N}$, $f_1, \ldots, f_n \in C_+(X)$ and $c_1, \ldots, c_n \in \mathbf{R}$ such that $\sum_{i=1}^{n} c_i = 0$, (see [1, Proposition 3.2 and Theorem 4.2]). One of the main results in [1] was the "Lévy–Khinchin"-representation for completely alternating functions (Theorem 3.7 in [1]). This uniquely determined representation has the form

$$\psi(f) = c + h(f) + \int_{L \setminus \{0\}} (1 - e^{-\lambda(f)}) \, d\xi_0(\lambda)$$

where $c \in [0, \infty[$, $h\colon C_+(X) \to [0, \infty[$ is additive and ξ_0 is a non-negative Radon-measure on $L \setminus \{0\}$. Observing that

$$L \setminus \{0\} = \{\lambda \in L \mid \lambda(1) > 0\}$$

we can write this representation in the folowing form

$$\psi(f) = c + \int_{\beta X} \tilde{f} \, d\varkappa + \int_{L \setminus \{0\}} \frac{1 - e^{-\lambda(f)}}{1 - e^{-\lambda(1)}} \, d\xi(\lambda) \tag{4.1}$$

where $\varkappa \in M_+(\beta X)$, $\xi \in M_+(L)$ and

$$\Delta_1 \psi(f) := \psi(f+1) - \psi(f) = \int_L e^{-\lambda(f)} \, d\xi(\lambda),$$

cf. the proof of Theorem 3.7 in [1].

Note that each completely alternating function ψ on $C_+(X)$ satisfies the inequalities

$$\alpha \psi(f) \leqq \psi(\alpha f) \quad \forall f \in C_+(X), \ \forall \alpha \in [0, 1] \tag{4.2}$$

$$\psi(\beta f) \leqq \beta \psi(f) \quad \forall f \in C_+(X), \ \forall \beta \in [1, \infty[\,. \tag{4.3}$$

This follows from (4.1) and from the fact that

$$1 - e^{-\alpha\lambda} \geqq \alpha(1 - e^{-\lambda}) \quad \forall \lambda \geqq 0, \ \forall \alpha \in [0, 1]$$

$$1 - e^{-\beta\lambda} \leqq \beta(1 - e^{-\lambda}) \quad \forall \lambda \geqq 0, \ \forall \beta \in [1, \infty[$$

which is easily established using Cauchy's mean value theorem. Another important property is subadditivity

$$\psi(f + g) \leqq \psi(f) + \psi(g) \quad \forall f, g \in C_+(X), \tag{4.4}$$

cf. Proposition 3.5 in [1].

THEOREM 4.1. *Let the completely alternating function* $\psi\colon C_+(X) \to [0, \infty[$ *have the representation* (4.1). *Then* $\xi(L \setminus L_0) = 0$ *if and only if* $\lim_{t \to 0} \psi(t) = \psi(0)$.

PROOF. We may and do assume $\psi(0)=c=0$. Suppose that $\lim_{t\to 0}\psi(t)=0$. By (4.4)

$$0 \leqq \psi(t+1)-\psi(1) \leqq \psi(t) \to 0 \quad \text{as } t \to 0\,,$$

hence

$$\Delta_1\psi(t) = \psi(t+1)-\psi(t) \to \psi(1) = \Delta_1\psi(0)$$

and we get $\xi(L\backslash L_0)=0$ from Theorem 2.2. The other direction follows immediately from (4.1).

THEOREM 4.2. *Let* $\psi\colon C_+(X) \to [0,\infty[$ *be a completely alternating function with the representation* (4.1); *let* ϱ *be any topology on* $C(X)$ *satisfying* $\tau\subseteqq\varrho\subseteqq\beta$ *and put*

$$B := \{f\in C(X) \mid 0\leqq f\leqq 1\}\,.$$

Then the following 5 statements are equivalent:

(4.2.1) $\exists\, Y$ *σ-compact* $\subseteqq X$ *such that* $\varkappa(X\backslash Y)=0$ *and* $\xi(L\backslash L_Y)=0$

(4.2.2) $\psi\,|\,B$ *is τ-continuous at* 0

(4.2.3) $\psi\,|\,B$ *is uniformly ϱ-continuous*

(4.2.4) $\psi\,|\,B$ *is β-continuous at* 0

(4.2.5) $\forall\,\varepsilon>0\;\exists\,\delta>0\;\exists\,C$ *compact* $\subseteqq X$ *such that* $\psi(f)-\psi(0)\leqq\varepsilon$ *whenever* $f\in B$ *and* $f\leqq\delta$ *on* C.

PROOF. Again we assume $\psi(0)=c=0$.

(4.2.1) ⇒ (4.2.2): The function $f\mapsto\int_{\beta X}\tilde f\,d\varkappa$ is τ-continuous because $\varkappa\in M_+(X)$, cf. Corollary 1.2. By Theorem 2.2 there exist compact uniformly tight subsets $K_n\subseteqq L_Y\backslash\{0\}$ such that $\xi(L\backslash K_n)\leqq 1/n$. We define

$$\psi_n(f) := \int_{K_n}\frac{1-e^{-\lambda(f)}}{1-e^{-\lambda(1)}}\,d\xi(\lambda), \qquad f\in C_+(X),\ n\in \mathbf{N}\,.$$

By Theorem 2.3 $\{\psi_n\}$ is a sequence of τ-continuous completely alternating functions. Now

$$\sup_{f\in B}\frac{1-e^{-\lambda(f)}}{1-e^{-\lambda(1)}} \leqq 1 \qquad \text{for all } \lambda\in L$$

implying that ψ_n converges uniformly to $\psi(f)-\int f\,d\varkappa$ on B. Hence $\psi\,|\,B$ is τ-continuous.

(4.2.2) ⇒ (4.2.3): If f,g belong to $C_+(X)$ then applying the subadditivity (4.4) we have

$$\begin{aligned}|\psi(f)-\psi(g)| &\leqq \psi(f\vee g)-\psi(f)+\psi(f\vee g)-\psi(g)\\ &\leqq \psi((f-g)^+)+\psi((g-f)^+)\,.\end{aligned}$$

If now $\psi | B$ is τ-continuous at 0, then $\psi | B$ is uniformly τ-continuous, as one can see immediately from the definition of the τ-topology.

(4.2.3) $\Rightarrow$ (4.2.4): Obvious.

(4.2.4) $\Rightarrow$ (4.2.5): This can be seen as in Theorem 2.3.

(4.2.5) $\Rightarrow$ (4.2.1): Let $\varepsilon > 0$ and choose $\delta > 0$, $C \subseteqq X$ compact such that $\psi(f) < \varepsilon$ for all $f \in B$, $f \leqq \delta$ on C. For those f we get

$$\Delta_1 \psi(0) - \Delta_1 \psi(f) = \psi(1) - \psi(f+1) + \psi(f) \leqq \psi(f) < \varepsilon$$

hence there exists by Theorem 2.3 a σ-compact subset $Y_1 \subseteqq X$ such that $\xi(L \setminus L_{Y_1}) = 0$. The function $f \mapsto \int_{\beta X} \tilde{f} \, d\varkappa$ has of course also the continuity property of (4.2.5), hence by Theorem 1.3., $\varkappa$ belongs to $M_+(X)$ and therefore is concentrated on a σ-compact subset $Y_2 \subseteqq X$. The union $Y := Y_1 \cup Y_2$ fulfills condition (4.2.1).

REFERENCES

1. C. Berg, J. P. R. Christensen and P. Ressel, *Positive definite functions on abelian semigroups*, Math. Ann. (to appear).
2. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1954), 131–295.
3. J. Hoffmann-Jørgensen, *The theory of analytic spaces*, Århus University, Various Publications Series No. 10 (1970).
4. J. Hoffmann-Jørgensen, *A generalization of the strict topology*, Math. Scand. 30 (1972), 313–323.
5. J. Hoffmann-Jørgensen, *Weak compactness and tightness of subsets of $M(X)$*, Math. Scand. 31 (1972), 127–150.
6. D. Pollard and F. Topsøe, *A unified approach to Riesz type representation theorems*, Studia Math. 54 (1975), 173–190.
7. F. Topsøe, *Topology and measure*, Lecture Notes in Mathematics 133, Springer-Verlag, Berlin · Heidelberg · New York, 1970.

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