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SOME OBSERVATIONS ON BESOV AND LIZORKIN–TRIEBEL SPACES

BJÖRN JAWERTH

0. Introduction.

In this note we consider the Besov spaces $\dot{B}_p^{sq}$ and the Lizorkin–Triebel spaces $\dot{F}_p^{sq}$. In various respects we complete earlier results. In particular, we find the dual spaces when $0<p<1$ (section 4). For $\dot{B}_p^{sq}$ the dual was previously only known when $0<q\leqq 1$ (Flett [6]). We also determine the trace of $\dot{F}_p^{sq}$ (section 5), obtaining in this way a result analogous to the one in [7] for $\dot{B}_p^{sq}$. Finally, we give an extension of Hardy's inequality to $\dot{F}_p^{0q}$ (section 3). In our treatment, based on Szasz' theorem and an imbedding theorem, this becomes almost a triviality even in the special case $q=2$ corresponding to the Hardy space H_p.

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1. Definition of the spaces.

To define the spaces to be studied we choose a sequence $\{\varphi_\nu\}_{\nu\in Z}$ of testfunctions such that

$$(1.1)\qquad \begin{cases} \varphi_\nu \in \mathscr{S}_0 \\ \operatorname{supp}\hat{\varphi}_\nu = \{1.5^{-1}\cdot 2^\nu \leqq |\xi| \leqq 1.5\cdot 2^\nu\} \\ |\hat{\varphi}_\nu(\xi)| \geqq C_\varepsilon > 0 \qquad \text{if } 2^\nu(1.5-\varepsilon)^{-1} \leqq |\xi| \leqq 2^\nu(1.5-\varepsilon) \\ |D_\alpha\hat{\varphi}_\nu(\xi)| \leqq C_\alpha|\xi|^{-|\alpha|} \quad \text{for every multiindex } \alpha\,. \end{cases}$$

Here and in what follows $\mathscr{S}_0$ is the space of rapidly decreasing functions whose Fourier transforms vanish together with all their derivatives at the origin. $\mathscr{S}'_0$ is its dual space. It is easy to see that $\mathscr{S}'_0$ in fact can be identified to the space of tempered distributions $\mathscr{S}'$ modulo polynomials.

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DEFINITION 1.1. Let s be real, $0<p,q\leqq\infty$. The Besov space $\dot{B}_p^{sq}$ is the space of all $f\in\mathscr{S}'_0$ such that

$$\|f\|_{\dot{B}_p^{sq}} \equiv \left(\sum_\nu (2^{\nu s}\|\varphi_\nu * f\|_{L_p})^q\right)^{1/q} < \infty .$$

DEFINITION 1.2. Let s be real, $0<p<\infty$, $0<q\leqq\infty$. The Lizorkin–Triebel space $\dot{F}_p^{sq}$ is the space of all $f\in\mathscr{S}'_0$ such that

$$\|f\|_{\dot{F}_p^{sq}} \equiv \left\|\left(\sum_\nu |2^{\nu s}\varphi_\nu * f|^q\right)^{1/q}\right\|_{L_p} < \infty .$$

REMARK 1.1. We emphasize that we in these homogeneous spaces work modulo polynomials.

From the definitions we at once get the imbeddings

(1.2) $$\dot{B}_p^{sq} \to \dot{F}_p^{sq} \to \dot{B}_p^{sp} \qquad \text{if } q\leqq p$$

$$\dot{B}_p^{sp} \to \dot{F}_p^{sq} \to \dot{B}_p^{sq} \qquad \text{if } q\geqq p$$

and especially

(1.3) $$\dot{B}_p^{sp} = \dot{F}_p^{sp} \quad \text{if } 0<p\leqq\infty .$$

Furthermore, using Littlewood–Paley theory it is possible to prove (see [8])

(1.4) $$\dot{F}_p^{02} \approx H_p$$

where H_p is the Hardy space on $\mathbf{R}^n$.

Let us list some other known properties of the spaces (cf. [9], [15]): The imbeddings from $\mathscr{S}_0$ and into $\mathscr{S}'_0$ are continuous:

(1.5) $$\mathscr{S}_0 \to \dot{B}_p^{sq},\ \dot{F}_p^{sq} \to \mathscr{S}'_0 .$$

(1.6) If $0<p,q<\infty$ then $\mathscr{S}_0$ is dense in $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$.

(1.7) $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$ are complete.

(1.8) The Riesz potential $I^s=(-\Delta)^{s/2}$ is an isomorphism from $\dot{B}_p^{s_0q}$ onto $\dot{B}_p^{s_0-s,q}$ and from $\dot{F}_p^{s_0q}$ onto $\dot{F}_p^{s_0-s,q}$.

2. An imbedding theorem.

The following imbedding theorem will later play a decisive rôle:

THEOREM 2.1. *Let* $s_0>s_1$, $0<p_0<p_1<\infty$, $0<q,r\leqq\infty$. *If*

$$s_0 - n/p_0 = s_1 - n/p_1$$

then

(i) $\dot{B}^{s_0 q}_{p0} \to \dot{B}^{s_1 q}_{p1}$

(ii) $\dot{F}^{s_0 q}_{p0} \to \dot{F}^{s_1 r}_{p1}$

(iii) $\dot{F}^{s_0 q}_{p0} \to \dot{B}^{s_1 p_0}_{p1}$

PROOF. We may assume $s_0 = 0$ (cf. (1.8)).

(i) From the easily verified inequality

$$\|\varphi_\nu * f\|_{L_\infty} \leqq C2^{\nu n/p} \|\varphi_\nu * f\|_{L_p} \tag{2.1}$$

we get by Hölder's inequality

$$\|\varphi_\nu * f\|_{L_{p_1}} \leqq C2^{\nu n(1/p_0 - 1/p_1)} \|\varphi_\nu * f\|_{L_{p_0}} .$$

This readily gives (i).

(ii) It suffices to take $q = \infty$ and $\|f\|_{\dot{F}^{0\infty}_{p0}} = 1$. By (2.1)

$$\|\varphi_\nu * f\|_{L_\infty} \leqq C2^{\nu n/p_0} \|f\|_{\dot{F}^{0\infty}_{p0}} = C2^{\nu n/p_0} .$$

Therefore for any fixed integer N

$$\left(\sum_{-\infty}^{N} |2^{\nu s_1} \varphi_\nu * f|^r \right)^{1/r} \leqq C2^{nN/p_1} \leqq t \tag{2.2}$$

if $t \approx C2^{nN/p_1}$. On the other hand, since $s_1 < 0$

$$\left(\sum_{N}^{\infty} |2^{\nu s_1} \varphi_\nu * f|^r \right)^{1/r} \leqq C2^{s_1 N} \sup_\nu |\varphi_\nu * f| \tag{2.3}$$

$$\leqq C t^{1 - p_1/p_0} \sup_\nu |\varphi_\nu * f| .$$

Combining (2.2) and (2.3) we get

$$\|f\|^{p_1}_{\dot{F}^{s_1 r}_{p_1}} = p_1 \int_0^\infty t^{p_1 - 1} \left| \left\{ \left(\sum_{-\infty}^{\infty} |2^{\nu s_1} \varphi_\nu * f|^r \right)^{1/r} > t \right\} \right| dt$$

$$\leqq p_1 \int_0^\infty t^{p_1 - 1} \left| \left\{ \sup_\nu |\varphi_\nu * f| > C t^{p_1/p_0} \right\} \right| dt$$

$$\leqq C \int_0^\infty t^{p_0 - 1} \left| \left\{ \sup_\nu |\varphi_\nu * f| > t \right\} \right| dt = C \|f\|^{p_1}_{\dot{F}^{0\infty}_{p0}} .$$

This proves (ii).

(iii) By (2.1) it follows that

$$\dot{F}^{s_0 \infty}_{p'} \to \dot{B}^{s', \infty}_{p_1}$$

if $s'-n/p_1=s_0-n/p'$. Hence by interpolation

$$(\dot{F}_{p'}^{s_0\infty},\dot{F}_{p''}^{s_0\infty})_{\theta p_0}\to(\dot{B}_{p_1}^{s',\infty},\dot{B}_{p_1}^{s'',\infty})_{\theta p_0}$$

or by the lemma below

$$\dot{F}_{p_0}^{s_0\infty}\to\dot{B}_{p_1}^{s_1p_0}$$

if $1/p_0=(1-\theta)/p'+\theta/p''$, $s_1=(1-\theta)s'+\theta s''$ $(0<\theta<1)$.

LEMMA 2.1. *Let* $0<p,q\leqq\infty$. *Concerning real interpolation we have*

$$\text{(i)}\qquad(\dot{B}_p^{s',q'},\dot{B}_p^{s'',q''})_{\theta q}=\dot{B}_p^{sq}$$

if $s=(1-\theta)s'+\theta s''$ $(0<\theta<1;\ s'\neq s'')$

$$\text{(ii)}\qquad(\dot{F}_{p'}^{s\infty},\dot{F}_{p''}^{s\infty})_{\theta p}=\dot{F}_p^{s\infty}$$

if $1/p=(1-\theta)/p'+\theta/p''$ $(0<\theta<1)$.

PROOF. (i) is well-known; see [9, chapter 11]. We do not detail the proof of (ii). Roughly speaking, one first shows that $\dot{F}_p^{s\infty}$ is a retract of $H_p(l_\infty)$. Then it is just to invoke a vector valued version of the Fefferman–Rivière–Sagher theorem [4] on interpolation of Hardy spaces.

REMARK 2.1. If $p\geqq 1$ then (i) of theorem 2.1 is of course essentially the classical Besov imbedding theorem. If $0<p<1$ it is also well-known; see Peetre [9, chapter 11] from where our proof is taken over.

If $q=2$ and $p_0,p_1>1$ then (ii) is "Sobolev's theorem on fractional integration" (cf. [13, chapter 5]). (ii) also contains results of for example Stein–Weiss [14], Peetre [8] and Triebel [15].

(iii) with $q=2$ goes back to Hardy–Littlewood. Our proof is based on ideas in [8].

3. An extension of Hardy's inequality.

In this brief section we consider an application of theorem 2.1. Recall the n-dimensional version of Szasz' theorem (cf. [9, chapter 6]):

LEMMA 3.1. *Let* $\hat{f}$ *denote the Fourier transform of* f. *Then*

$$\|\hat{f}\|_{L_p}\leqq C\|f\|_{\dot{B}_2^{n(1/p-1/2),p}}\qquad\text{if } 0<p\leqq 2\,.$$

Now it is a simple matter to verify the following

THEOREM 3.1. *Let* $0<p<2$ *and* $0<q\leqq\infty$. *Then*

$$\left(\int |\hat{f}(\xi)|^p/|\xi|^{n(2-p)}\,d\xi\right)^{1/p} \leqq C\|f\|_{\dot{F}_p^{0q}}. \tag{3.1}$$

PROOF. Let $g=I^{-n(2-p)/p}f$. By theorem 2.1:(iii) and (1.8) we have

$$\|g\|_{\dot{B}_2^{n(1/p-1/2),p}} \leqq C\|f\|_{\dot{F}_p^{0q}}.$$

Hence, using the lemma, we get

$$\|\hat{g}\|_{L_p} \leqq C\|f\|_{\dot{F}_p^{0q}}$$

which is the desired inequality.

REMARK 3.1. If $p=2$ we must take $q\leqq 2$ in view of (1.4) and Plancherel's theorem.

REMARK 3.2. The case $q=2$, $0<p\leqq 2$ (the Hardy space case) is well-known; (3.1) is then sometimes called Hardy's inequality. For $\mathbf{T}^1$ it was proved by Paley for $1<p\leqq 2$, for $p=1$ by Hardy and for $0<p<1$ by Hardy–Littlewood. For $\mathbf{R}^n$ the case $p=1$ has been obtained by Fefferman [3] and Björk [1] and $0<p<1$ is due to Peetre [8].

4. The duals of $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$.

The duals of $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$ may be considered as subspaces of $\mathscr{S}_0'$ because of (1.5) and (1.6). It turns out that to characterize them exactly if $0<p<1$ is to a large extent just another application of theorem 2.1.

We begin by describing the dual of $\dot{B}_p^{sq}$:

THEOREM 4.1. *Let s be real and* $0<p<1$. *Then*

$$\text{(i)}\quad (\dot{B}_p^{sq})' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty} \quad \text{if } 0<q\leqq 1$$

$$\text{(ii)}\quad (\dot{B}_p^{sq})' \approx \dot{B}_\infty^{-s+n(1/p-1),q'} \quad \text{if } 1<q<\infty$$

where $1/q+1/q'=1$.

PROOF. For simplicity we take $s=0$. By theorem 2.1:(i) we have

$$\dot{B}_p^{0q} \to \dot{B}_1^{-n(1/p-1),1} \quad \text{if } 0<q\leqq 1$$

$$\dot{B}_p^{0q} \to \dot{B}_1^{-n(1/p-1),q} \quad \text{if } 1<q<\infty.$$

If we use the well-known fact (cf. [9, chapter 3])

$$(\dot{B}_1^{sq})' \approx \dot{B}_\infty^{-s,q} \quad \text{if } 1\leqq q<\infty$$

we therefore find

$$\text{(i}'\text{)} \quad \dot{B}_{\infty}^{n(1/p-1),\infty} \to (\dot{B}_p^{0q})' \quad \text{if } 0<q\leqq 1$$

$$\text{(ii}'\text{)} \quad \dot{B}_{\infty}^{n(1/p-1),q} \to (\dot{B}_p^{0q})' \quad \text{if } 1<q<\infty .$$

In order to prove the converse inclusions we fix a $f \in (\dot{B}_p^{0q})'$ with $\|f\|_{(\dot{B}_p^{0q})'}=1$. (i) is a consequence of (i′) and

$$\|f\|_{\dot{B}_{\infty}^{n(1/p-1),\infty}} = \sup_{\nu}\sup_{h} |\langle f, 2^{\nu n(1/p-1)}\varphi_\nu(\cdot-h)\rangle| \leqq C\cdot 1 .$$

Thus there only remains to verify the second half of (ii). Let $\{\varphi_\nu\}$ be a sequence of testfunctions satisfying in addition to (1.1) also $\sum_\nu \varphi_\nu=\delta$ (which is no restriction). Then we obviously have

$$f = \sum_\nu \varphi_\nu * f \equiv \sum_\nu a_\nu .$$

Assume with no loss of generality that $\operatorname{supp}\hat{a}_\nu$ and $\operatorname{supp}\varphi_\mu$ are disjoint if $\nu \neq \mu$. (Just multiply $\hat{f}$ by a suitable function.) For a fixed integer $N>0$ we define the sequence $\{b_\nu\}_{\nu=-N}^{N}$ by

$$b_\nu(x) = \varepsilon_\nu 2^{\nu n(1/p-1)}\|a_\nu\|_{\dot{B}_{\infty}^{n(1/p-1),\infty}}^{q'-1}\varphi_\nu(x-h_\nu), \qquad |\varepsilon_\nu|=1$$

where $\{h_\nu\}$ and the arguments of $\{\varepsilon_\nu\}$ are at our disposal. (That indeed $\|a_\nu\|_{\dot{B}_{\infty}^{n(1/p-1),\infty}}<\infty$ can be seen in the same way as for (i)). Clearly, in view of our assumption on the supports,

$$\begin{aligned} f\left(\sum_{-N}^{N} b_\nu\right) &= \sum_{-N}^{N} f(b_\nu) \\ &\geqq \sum_{-N}^{N} \|a_\nu\|_{\dot{B}_{\infty}^{n(1/p-1),\infty}}^{q'-1}\langle a_\nu, \varepsilon_\nu 2^{\nu n(1/p-1)}\varphi_\nu(\cdot-h_\nu)\rangle \\ &\geqq C\sum_{-N}^{N} \|a_\nu\|_{\dot{B}_{\infty}^{n(1/p-1),\infty}}^{q'} = C\sum_{-N}^{N} (2^{\nu n(1/p-1)}\|a_\nu\|_{L_\infty})^{q'} \end{aligned}$$

if $\{h_\nu\}$ and the arguments of $\{\varepsilon_\nu\}$ are chosen properly. But on the other hand

$$f\left(\sum_{-N}^{N} b_\nu\right) \leqq 1\cdot\left\|\sum_{-N}^{N} b_\nu\right\|_{\dot{B}_p^{0q}} \leqq \left(\sum_{-N}^{N} (2^{\nu n(1/p-1)}\|a_\nu\|_{L_\infty})^{q'}\right)^{1/q} .$$

Hence,

$$\left(\sum_{-N}^{N} (2^{\nu n(1/p-1)}\|a_\nu\|_{L_\infty})^{q'}\right)^{1/q'} \leqq C$$

and the proof is complete if we let $N\to\infty$.

We turn our attention to $\dot{F}_p^{sq}$. To determine its dual almost becomes a triviality when knowing both theorem 2.1 and theorem 4.1:

THEOREM 4.2. *Let s be real and* $0<p<1$. *Then*

$$(\dot{F}_p^{sq})' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty} \quad \text{if } 0<q<\infty .$$

PROOF. From (1.2) we deduce that

$$(\dot{F}_p^{sq})' \to (\dot{B}_p^{sq})' \quad \text{if } q \leqq p$$

$$(\dot{F}_p^{sq})' \to (\dot{B}_p^{sp})' \quad \text{if } q \geqq p .$$

Conversely, theorem 2.1:(ii) or (iii) yields

$$(\dot{B}_1^{s-n(1/p-1),1})' \to (\dot{F}_p^{sq})'$$

Invoking theorem 4.1 we see that the Besov spaces have the same dual and thus also $\dot{F}_p^{sq}$:

$$(\dot{F}_p^{sq})' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty} .$$

REMARK 4.1. Part (i) of theorem 4.1 for $\mathbf{T}^1$ is due to Flett [6]. It was extended to $\mathbf{R}^n$ by Peetre [11].

By (1.4) theorem 4.2 includes the fact

$$(H_p)' \approx \dot{B}_\infty^{n(1/p-1),\infty} \quad \text{if } 0<p<1 .$$

For $\mathbf{T}^1$ this is a famous result of Duren–Romberg–Shields [2]. The extension to $\mathbf{R}^n$ is by Walsh [16] (cf. also Fefferman–Stein [5], Peetre [11], Rivière [13]).

5. The trace of $\dot{F}_p^{sq}$.

Let us denote a point $x \in \mathbf{R}^n$ by $x=(x',x_n)$, where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}^1$. Identify $\mathbf{R}^{n-1}$ with the hyperplane $x_n=0$ in $\mathbf{R}^n$ and consider the trace operator

$$\mathrm{Tr}\colon\ \mathscr{S}_0(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^{n-1})$$

defined by

$$\mathrm{Tr}\, f(x') = f(x',0) .$$

THEOREM 5.1. *Let* $0<p<\infty$, $0<q\leqq\infty$ *and* $s>1/p+\max(0,(n-1)(1/p-1))$. *Then the trace operator can be extended so that*

$$\mathrm{Tr}\colon\ \dot{F}_p^{sq}(\mathbf{R}^n) \to \dot{B}_p^{s-1/p,p}(\mathbf{R}^{n-1}) . \tag{5.1}$$

Conversely, there is an operator Sr

$$\text{Sr}: \dot{B}_p^{s-1/p,p}(\mathbf{R}^{n-1}) \to \dot{F}_p^{sq}(\mathbf{R}^n) \tag{5.2}$$

so that $\text{Tr} \circ \text{Sr} = \text{Id}$.

Proof. In proving (5.1) we shall for convenience assume $0<p\leqq 1$ (with minor modifications the same proof also works for $p>1$). It also suffices to take $q=\infty$.

Let $f \in \dot{F}_p^{sq}(\mathbf{R}^n)$. If $q<\infty$ we can extend Tr by continuity, since $\mathscr{S}_0$ is then dense in $\dot{F}_p^{sq}(\mathbf{R}^n)$. For $q=\infty$ this is no longer applicable. However, for all q we can define Tr by

$$\text{Tr}\, f(x') = \sum_{\nu \in \mathbf{Z}} \varphi_\nu * f(x',0) \equiv \sum_{\nu\in\mathbf{Z}} a_\nu$$

where $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ is a sequence of testfunctions on $\mathbf{R}^n$ satisfying (1.1) and $\sum_\nu \varphi_\nu = \delta$. Obviously, it is an extension of our original Tr (in fact, as is easily seen, the unique one). That the sum has a limit and thus that Tr is well-defined follows from the completeness of $\dot{B}_p^{sq}$ and (5.3) below.

We need two lemmata; the first is for $p\geqq 1$ just Minkowski:

Lemma 5.1. *If* $f, g \in \mathscr{S}_0'$ *and* $\text{supp}\,\hat{f}$, $\text{supp}\,\hat{g} \subset \{|\xi|\leqq r\}$ *then*

$$\|f*g\|_{L_p} \leqq Cr^{n(1/p-1)}\|f\|_{L_p}\|g\|_{L_p} \quad \text{if } 0<p\leqq 1$$

$$\|f*g\|_{L_p} \leqq \|f\|_{L_1}\|g\|_{L_p} \quad \text{if } 1\leqq p\leqq\infty\,.$$

For a proof we refer the reader to [9, chapter 11]. The second lemma is also a result by Peetre [10] (we shall only need it for $q=\infty$ when it is easiest to prove):

Set

$$\varphi_\nu^a f(x) = \sup_{|x-y|\leqq 2^{-\nu}a} |\varphi_\nu * f(y)|$$

for a fixed $a\geqq 0$.

Lemma 5.2. *Let s be real and* $0<p<\infty$, $0<q\leqq\infty$. *Then*

$$\|f\|_{\dot{F}_p^{sq}} \approx \left\| \left(\sum_\nu |2^{\nu s}\varphi_\nu^a f|^q \right)^{1/q} \right\|_{L_p}.$$

Now the proof of (5.1) is easily accomplished. If $\{\varphi_\nu'\}_{\nu\in\mathbf{Z}}$ is a sequence of testfunctions on $\mathbf{R}^{n-1}$ satisfying (1.1), then

$$\operatorname{Tr} f * \varphi'_\nu = \sum_{\mu \geqq \nu - 1} a_\mu * \varphi'_\nu .$$

Consequently,

$$\|\operatorname{Tr} f * \varphi'_\nu\|^p_{L_p(\mathbf{R}^{n-1})} \leqq \sum_{\mu \geqq \nu - 1} \|a_\mu * \varphi'_\nu\|^p_{L_p(\mathbf{R}^{n-1})}$$

since

$$(x+y)^p \leqq x^p + y^p$$

when $x, y \geqq 0$ and $0 < p \leqq 1$. By lemma 5.1 we have

$$\|a_\mu * \varphi'_\nu\|_{L_p(\mathbf{R}^{n-1})} \leqq C 2^{\mu(t-1/p)} \|a_\mu\|_{L_p(\mathbf{R}^{n-1})} \|\varphi'_\nu\|_{L_p(\mathbf{R}^{n-1})}$$

with $t = 1/p + (n-1)(1/p - 1)$. But

$$\|\varphi'_\nu\|_{L_p(\mathbf{R}^{n-1})} \approx 2^{-\nu(t-1/p)} .$$

Hence,

$$2^{\nu(s-1/p)p} \|\varphi'_\nu * \operatorname{Tr} f\|^p_{L_p(\mathbf{R}^{n-1})}$$
$$\leqq C \sum_{\mu \geqq \nu - 1} 2^{(\nu-\mu)(s-t)p} 2^{-\mu} \|2^{\mu s} a_\mu\|^p_{L_p(\mathbf{R}^{n-1})}$$

Inserting this into the definition of Besov spaces (definition 1.1) and using Minkowski's inequality for sums we find

$$\|f\|^p_{\dot{B}_p^{s-1/p,p}(\mathbf{R}^{n-1})} \leqq C \sum_{\nu \leqq 1} 2^{\nu(s-t)p} \sum_\mu 2^{-\mu} \|2^{\mu s} a_\mu\|^p_{L_p(\mathbf{R}^{n-1})}$$
$$\leqq C \sum_\mu 2^{-\mu} \|2^{\mu s} a_\mu\|^p_{L_p(\mathbf{R}^{n-1})}$$

since $s > t$. However,

$$\sum_\mu 2^{-\mu} \|2^{\mu s} a_\mu\|^p_{L_p(\mathbf{R}^{n-1})} \leqq \sum_\mu \int_{2^{-\mu}}^{2^{-\mu+1}} \|2^{\mu s} \varphi^2_\mu f(\cdot, x_n)\|^p_{L_p(\mathbf{R}^{n-1})} dx_n$$
$$\leqq C \left\| \sup_\mu |2^{\mu s} \varphi^2_\mu f| \right\|^p_{L_p(\mathbf{R}^n)} .$$

Thus by lemma 5.2

(5.3) $$\|f\|_{\dot{B}_p^{s-1/p,p}(\mathbf{R}^{n-1})} \leqq C \|f\|_{\dot{F}_p^{s,\infty}(\mathbf{R}^n)} .$$

This concludes the proof of (5.1).

We turn to (5.2). Now we can take q very small, at least $q \leqq p$. Let $\{\varphi'_\nu\}_{\nu \in \mathbf{Z}}$ and $\{\psi_\nu\}_{\nu \in \mathbf{Z}}$ be testfunctions on $\mathbf{R}^{n-1}$ and $\mathbf{R}^1$ respectively, satisfying (1.1) as well as

$$\sum_\nu \varphi'_\nu = \delta; \quad \psi_\nu(x_n) = 2^\nu \psi_0(2^\nu x_n), \quad \psi_0(0) = 1 .$$

Again in order to avoid some trivial technical nuisances we assume that

$$\text{(5.4)} \qquad \varphi'_\nu * \varphi'_\mu * f = 0 \quad \text{if } \nu \neq \mu$$

$$\psi_\nu * \psi_\mu = 0 \quad \text{if } \nu \neq \mu$$

However, without any loss of generality we may assume that the testfunctions $\{\varphi_\nu\}_{\nu \in \mathbf{Z}}$ on $\mathbf{R}^n$ are of the form

$$\text{(5.5)} \qquad \varphi_\nu = \varphi'_\nu \otimes \sum_{\mu \leqq \nu} \psi_\mu + \sum_{\mu \leqq \nu} \varphi'_\mu \otimes \psi_\nu .$$

Put

$$\operatorname{Sr} f(x', x_n) = \sum_\mu 2^{-\mu} \varphi'_\mu * f(x') \otimes \psi_\mu(x_n) .$$

Clearly,

$$\operatorname{Sr} f(x', 0) = f(x')$$

that is, $\operatorname{Tr} \circ \operatorname{Sr} = \operatorname{Id}$. Because of (5.4) and (5.5) we see that

$$\operatorname{Sr} f * \varphi_\nu = 2^{-\nu+1} \varphi'_\nu * \varphi'_\nu * f \otimes \psi_\nu * \psi_\nu .$$

Hence,

$$\|\operatorname{Sr} f\|^p_{\dot{F}^{sq}_p(\mathbf{R}^n)} \leqq C \left\| \left(\sum_\nu |2^{\nu(s-1)} \varphi'_\nu * \varphi'_\nu * f \otimes \psi_\nu * \psi_\nu|^q \right)^{1/q} \right\|^p_{L_p(\mathbf{R}^n)}$$

$$\leqq C \int_{-\infty}^{\infty} \left| \sum_\nu (2^{\nu(1p-1)} a_\nu \psi_\nu * \psi_\nu(x_n))^q \right|^{p/q} dx_n$$

with

$$a_\nu = 2^{\nu(s-1/p)} \|\varphi'_\nu * \varphi'_\nu * f\|_{L_p(\mathbf{R}^{n-1})} .$$

Inserting the trivial estimate

$$|\psi_\nu * \psi_\nu(x_n)| \leqq C 2^\nu \min(1, (2^\nu |x_n|)^{-j})$$

for an arbitrarily large j, gives with $r = p/q$

$$\|\operatorname{Sr} f\|^p_{\dot{F}^{sq}_p(\mathbf{R}^n)} \leqq C \int_{-\infty}^{\infty} \left| \sum_\nu (2^{\nu/p} a_\nu \min(1, (2^\nu |x_n|)^{-j}))^q \right|^{p/q} dx_n$$

$$\leqq C \sum_\mu \left(\sum_\nu (2^{\nu-\mu} a_\nu^p \min(1, (2^{\nu-\mu})^{-jp}))^{1/r} \right)^r$$

If we use Minkowski again, we see that

$$\|\mathrm{Sr}\, f\|^p_{\dot{F}^{sq}_p(\mathbf{R}^n)} \leqq C \sum_\nu a^p_\nu \leqq C\|f\|^p_{\dot{B}^{s-1/p,p}_p(\mathbf{R}^{n-1})}\,.$$

This is the desired inequality and thus the proof of the theorem is complete.

REMARK 5.1. Our proof of theorem 5.1 is essentially the same as in [7].

REFERENCES

1. J.-E. Björk, *Some conditions which imply that a translation invariant operator is continuous in the Hardy space* $H^1(\mathbf{R}^n)$, Pre-print.
2. P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on* H^p *spaces with* $0<p<1$, J. Reine Angew. Math. 238 (1969), 32–60.
3. C. Fefferman, talk delivered at the University of Maryland.
4. C. Fefferman, N. Rivière and Y. Sagher, *Interpolation between* H^p *spaces, the real method*, Trans. Amer. Math. Soc. 191 (1974), 75–81.
5. C. Fefferman and E. M. Stein, H^p *spaces of several variables*, Acta Math. 129 (1972), 137–193.
6. T. M. Flett, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. Appl. 39 (1972), 125–158.
7. B. Jawerth, *The trace of Sobolev and Besov spaces if* $0<p<1$. To appear in Studia Math.
8. J. Peetre, H^p *spaces*, Technical report, Lund, 1974.
9. J. Peetre, *New thoughts on Besov spaces*, Duke University Press, Durham, to appear.
10. J. Peetre, *On spaces of Triebel–Lizorkin type*, Ark. Mat. 13 (1975), 123–130.
11. J. Peetre, *Remarques sur les espaces de Besov. Le cas* $0<p<1$, C. R. Acad. Sci. Paris Sér. A 277 (1973), 947–949.
12. N. M. Rivière, *Classes of smoothness*, Lecture notes, 1972/73.
13. E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton Mathematical Series 30), Princeton University Press, Princeton, 1970.
14. E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables* I: *the theory of* H^p *spaces*, Acta Math. 103 (1960), 25–62.
15. H. Triebel, *Spaces of distributions of Besov type in Euclidean n-space, Duality, interpolation*, Ark. Mat. 11 (1973), 13–64.
16. T. Walsh, *The dual of* $H^p(\mathbf{R}^{n+1}_+)$ *for* $p<1$, Canad. J. Math. 25 (1973), 567–577.

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