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STRONGLY FINITE VON NEUMANN ALGEBRAS

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1. Introduction.

Let M be a von Neumann algebra with predual M_*, and let G be any group of *-automorphisms of M. For each $\varphi \in M_*$, we denote the orbit $\{\varphi \circ \alpha : \alpha \in G\}$ of φ under G by $O_G(\varphi)$. We say that M is *G-finite* if for each non-zero positive b in M, there is a G-invariant normal state φ of M such that $\varphi(b) \neq 0$. When G is the group $I(M)$ of all inner automorphisms of M, we write $O(\varphi)$ for $O_G(\varphi)$. In this case, G-finiteness is equivalent to the usual notion of finiteness in a von Neumann algebra. [14, Theorem 2.5.4, p. 97].

Following work of F. Yeadon, who considered in [17] the case $G = I(M)$, E. Størmer showed in [16] that M is G-finite if and only if for each $\varphi \in M_*$, $O_G(\varphi)$ is weakly relatively compact in M_*. We shall call an algebra M *strongly G-finite* if for each $\varphi \in M_*$, the orbit $O_G(\varphi)$ is relatively compact in M_* in its norm topology. In [5] and [10] the authors have considered examples of strongly G-finite algebras for various choices of G and M. The purpose of this paper is to show that if $G = I(M)$, this condition is very restrictive. In fact, M is strongly finite if and only if M is a direct sum of von Neumann subalgebras each of which is either abelian or finite dimensional.

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2. Preliminaries and some notation.

Let $\mathscr{B}(M)$ denote the space of all bounded linear operators from M into M, and let σ denote the ultraweak topology on M. Let M_1 be the unit ball of M. By the *p-topology* on $\mathscr{B}(M)$ we shall mean the topology of pointwise convergence when M has the σ-topology. (See [6].) The unit ball of $\mathscr{B}(M)$ is p-compact [8]. By the *u-topology* on $\mathscr{B}(M)$, we shall mean the topology which has for a basis at $\alpha \in \mathscr{B}(M)$ the family of all subsets of the form

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$$\{\beta : (\beta-\alpha)(M_1)\subseteqq U\},$$

where U is a σ-open neighborhood of zero in M. (See [1] and [6].) Note that if $\varphi \in M_*$, then the map $\alpha \to \varphi\circ\alpha$ is continuous from the u-topology into the norm topology.

Let Aut (M) denote the group of all *-automorphisms of M. If G is any subgroup of Aut (M), then by $\bar{G}$ we shall always mean the p-closure of G in $\mathscr{B}(M)$. If T: $M \to N$ is a bounded linear map between two Banach spaces, we denote by T^* the adjoint mapping from N^* to M^*.

3. Some characterizations and combinatorial properties.

Throughout this section we assume that G is a group of *-automorphisms of the von Neumann algebra M, and after the proof of Theorem 3.3, we shall assume that $G=I(M)$. Let M_*^a be the subset of M_* consisting of all φ such that $O_G(\varphi)$ is relatively compact in M_* with respect to the norm topology.

LEMMA 3.1. *The set M_*^a is a norm-closed linear subspace of M_*.*

PROOF. That M_*^a is a linear subspace of M_* is clear. Let $\varphi \in M_*$ be a point of closure of M_*^a. It suffices to show that $O_G(\varphi)$ is totally bounded, and this follows from a standard $\varepsilon/3$-argument together with the total boundedness of the orbits in M_*^a.

The algebra M is strongly G-finite if and only if $M_*^a=M_*$, and it follows from [16] that a strongly G-finite algebra is G-finite. The following list of characterizations of strong G-finiteness follows from our recent work in [5] and [10].

THEOREM 3.2. *The following are equivalent*:

(1) *M is strongly G-finite*;
(2) *$\bar{G}$ is a set of one-to-one maps*;
(3) *$\bar{G}$ is a set of *-automorphisms of M*;
(4) *$\bar{G}$ is a compact topological group when equipped with the p-topology*;
(5) *G is relatively compact in* Aut (M) *equipped with the p-topology*;
(6) *G is relatively compact in* Aut (M) *equipped with the u-topology*;
(7) *the p-topology and the u-topology coincide on $\bar{G}$*;
(8) *G is equicontinuous on M_1 with respect to the σ-topology.*

PROOF. The equivalences of (2), (3), (4), (5), (6), (7) are proved by Green [5], and that of (1) and (8) is proved by Lau [10].

If (4) and (7) hold, then for each $\varphi \in M_*$, the set $\{\varphi\circ\alpha : \alpha \in \bar{G}\}$ is norm compact in M_*. Since

$$O_G(\varphi) \subseteqq \{\varphi\circ\alpha : \alpha \in \bar{G}\},$$

(1) holds.

If (8) holds, then it follows from [7, D14.1] that $(\bar{G}, p)$ is a compact topological semigroup with jointly continuous multiplication. Since G is a group, it follows easily that $\bar{G}$ is also a group. Hence (4) holds.

The next theorem gives some geometric characterizations of strong G-finiteness.

THEOREM 3.3. *The following are equivalent:*

(1) *M is strongly G-finite;*
(2) *for each $x \in M$, the σ-closure $C(x)$ of $\{\alpha(x) : \alpha \in G\}$ is minimal with respect to being σ-closed and invariant under G;*
(3) *each $x \in M_1$ is an almost periodic point of the transformation group (G, M_1);*
(4) *each α in $\bar{G}$ is an isometry.*

PROOF. (1) $\Rightarrow$ (3). By condition (8) in Theorem 3.2, G acts equicontinuously on (M_1, σ). Condition (3) now follows from Proposition 4.4, Corollary 5.4, and Corollary 5.5 of [4].

(3) $\Rightarrow$ (2). By Proposition 2.5 of [4], each point x in M_1 generates a minimal set. The same argument works for any ball in M, so (2) holds.

(2) $\Rightarrow$ (4). Let $\alpha \in \bar{G}$ be such that $\alpha(a)=0$ for some $a \in M$. Then $0 \in C(a)$. Since $C(a)$ is minimal, $a=0$. Consequently $\bar{G}$ consists only of one-to-one maps. By (2) $\Leftrightarrow$ (3) in Theorem 3.2, each $\alpha \in \bar{G}$ is an isometry.

(4) $\Rightarrow$ (1). Condition (4) implies that each $\alpha \in \bar{G}$ is one-to-one. By (1) $\Leftrightarrow$ (2) in Theorem 3.2, M is strongly G-finite.

From now on we assume that $G=I(M)$. Clearly any abelian or finite dimensional von Neumann algebra is strongly finite, and we show in the remainder of this section that subalgebras, quotients, direct summands, and direct sums of strongly finite algebras are strongly finite. Let M^u be the unitary group of M, and for each $u \in M^u$, let T_u be the automorphism of M defined by $T_u(a)=u^*au$.

LEMMA 3.4. *If M is strongly finite, and if N is a von Neumann subalgebra of M with the same identity as M, then N is strongly finite.*

PROOF. Let $\varphi \in N_*$, and let $\hat{\varphi} \in M_*$ be a normal extension of φ to M (see [12, p. 317]). Since $N^u \subseteqq M^u$, $\{\hat{\varphi}\circ T_u : u \in N^u\}$ is relatively compact in the norm topology of M_*. Since restriction to N is a norm-continuous map from M_* onto N_*, it follows that $\{\varphi \circ T_u : u \in N^u\}$ is relatively compact in the norm topology of N_*.

LEMMA 3.5. *If M is strongly finite, and if p is any projection of M, then pMp is also strongly finite.*

PROOF. Let $\alpha(x)=pxp$, $x \in M$, and let $N=pMp$. If $v \in N^u$, then $v+(1-p) \in M^u$. It follows that if $\varphi \in N_*$, then

$$\{T_v^*(\varphi) : v \in N^u\} \subseteqq \{T_{\alpha(u)}^*(\varphi) : u \in M^u\} .$$

Since $T_{\alpha(u)}^*(\varphi)$ is the restriction to N of $T_u^*(\varphi \circ \alpha)$, the result follows from the relative compactness of $\{T_u^*(\varphi \circ \alpha) : u \in M^u\}$.

PROPOSITION 3.6. *If M is strongly finite, then any von Neumann subalgebra N of M is also strongly finite.*

PROOF. Let p be the identity of N. Then $N \subseteqq pMp$, and the proposition now follows from the last two lemmas.

PROPOSITION 3.7. *Let M and N be von Neumann algebras, and let α be an ultraweakly continuous *-homomorphism from M onto N. If M is strongly finite, then N is also strongly finite.*

PROOF. Let $I=\{x \in M : \alpha(x)=0\}$, and let z be a central projection in M such that $I=Mz$. Then N is *-isomorphic to $M(1-z)$, which is strongly finite by Proposition 3.6. We may therefore assume that α is a *-isomorphism. In that case,

$$N^u = \{\alpha(u) : u \in M^u\} ,$$

and for each $u \in M^u$, we have

$$T_{\alpha(u)}^* = (\alpha^{-1})^* \circ T_u^* \circ \alpha^* .$$

The result then follows from the norm continuity of $(\alpha^{-1})^*$.

PROPOSITION 3.8. *Let $\{M_i : i \in \mathscr{I}\}$ be a family of von Neumann algebras, and let $M=\sum_{i\in\mathscr{I}} \oplus M_i$. Then M is strongly finite if and only if each M_i is strongly finite.*

PROOF. Recall that M_* is a set of functions on $\mathscr{I}$, and that we may identify $(M_i)_*$ with the subset of all those functions which vanish everywhere except at the index i. It is easy to check that if $f \in (M_i)_*$, then this identification makes the orbits of f under $(M_i)^u$ and under M^u coincide. If now $f \in M_*$ and $k \in \mathscr{I}$, let $f_k \in (M_k)_*$ be defined by $f_k = \delta_{ik} f$. Then f is a norm limit of linear combinations of the f_k. By Lemma 3.1, M is strongly finite whenever all the M_k are strongly finite. The converse follows from Proposition 3.6.

4. Strong finiteness and type.

THEOREM 4.1. *If M is a strongly finite von Neumann algebra, then M is finite and of type* I.

PROOF. By [16] or [17], M is finite. Since every von Neumann algebra of type II_1 contains the hyperfinite factor N, it suffices by Proposition 3.6 to show that N is not strongly finite. Let F be the algebra of all 2×2 complex matrices, and let e_2 be the identity of F. Let $N_1 = F$ and $N_{k+1} = N_k \otimes F$. We regard N_k as a subalgebra of N_{k+1} and N as the weak closure of $\bigcup_{k=1}^{\infty} N_k$. Let p be the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and let $p_k \in N_k$ be the projection $e_2 \otimes e_2 \otimes \ldots \otimes e_2 \otimes p$. We claim that if q is a σ-cluster point of $\{p_k\}$, then $q = \frac{1}{2}e$, where e is the identity of N. Let τ be the normal tracial state of N, and let $a \in \bigcup_{k=1}^{\infty} N_k$. For all sufficiently large k,

$$\tau(p_k a) = \tau(p_k)\tau(a) = \tfrac{1}{2}\tau(a),$$

so that $\tau(qa) = \frac{1}{2}\tau(a)$, that is, $\tau((q - \frac{1}{2}e)a) = 0$. By the σ-density of $\bigcup_{k=1}^{\infty} N_k$, this last equality holds for any $a \in N$. But τ is faithful, so $q - \frac{1}{2}e = 0$, and the claim is proved.

For each k, choose now a unitary $u_k \in N_k$ such that $p_k = T_{u_k}(p_1)$. By compactness, $\{T_{u_k}\}$ has at least one p-cluster point, say T, in $\overline{I(N)}$. But then $T(p_1)$ is a σ-cluster point of $\{p_k\}$, so $T(p_1) = \frac{1}{2}e$. If now N were strongly finite, then by condition (3) of Theorem 3.2, T would be a *-automorphism of N. But this would imply that $\frac{1}{2}e$ is a projection, which is absurd.

Let Γ be a discrete group. For each $a \in \Gamma$, let $\lambda(a)$ be the isometry of $l_2(\Gamma)$ defined by

$$[\lambda(a)f](x) = f(a^{-1}x), \qquad x \in \Gamma.$$

Let $\mathrm{VN}(\Gamma)$ be the von Neumann algebra generated by $\{\lambda(a) : a \in \Gamma\}$.

COROLLARY 4.2. *If $\mathrm{VN}(\Gamma)$ is strongly finite, then the center of Γ has finite index in Γ.*

PROOF. Let G denote the group of all inner automorphisms of VN (Γ) induced by $\{\lambda(a) : a \in \Gamma\}$, and let $M = \mathrm{VN}(\Gamma)$. Since $\overline{I(M)}$ is a group of *-automorphisms, $\bar{G}$ is also a group of *-automorphisms of M. By [5, Theorem 2.4] each conjugacy class in Γ is finite. By Theorem 4.1 and a result in [9] (see also [15]), Γ contains an abelian subgroup of finite index. An application of [11, Corollary 4.4] shows that the center of Γ must have finite index.

EXAMPLE 4.3. There exist finite von Neumann algebras of type I which are not strongly finite. Let H be the semi-direct product of the integers $\mathbf{Z}$ by the two element group $\{\pm 1\}$. We write $\mathbf{Z}$ additively, so that the product in H is given by

$$(m,a)(n,b) = (m+an, ab), \qquad m,n \in \mathbf{Z};\ a,b \in \{\pm 1\}\ .$$

It is easy to check that the infinite set $\{(2k, -1) \in H : k \in \mathbf{Z}\}$ is a conjugacy class in H, so by Corollary 4.2, VN (H) cannot be strongly finite. However, by [9] or [15], VN (H) is of type I, and since H is discrete, VN (H) is also finite.

5. The main result.

By Theorem 4.1, the problem of finding all strongly finite von Neumann algebras is reduced to the corresponding problem for algebras of the form $Z \otimes M_n$, where Z is abelian and M_n is the algebra of all $n \times n$ complex matrices. Let μ_0 be normalized Haar measure on the circle group $\mathbf{T}$.

LEMMA 5.1. *The algebra $L^\infty(\mathbf{T}, \mu_0) \otimes M_2$ is not strongly finite.*

PROOF. Consider the algebra VN (H), where H is the group discussed in Example 4.3. As in [2, Chapitre 3, § 7, no 6], we identify VN (H) with an algebra (under convolution) of square-summable functions on H, and we may further identify VN $(\mathbf{Z})$ with the subalgebra of all functions in VN (H) which vanish off $\mathbf{Z}$. Let δ be the characteristic function of the singleton containing $(0, -1)$. Each function f in VN (H) may be written uniquely in the form $f = f_1 + \delta * f_2$, where f_1 and f_2 are in VN $(\mathbf{Z})$. To each such f, we associate the 2×2 matrix $L(f)$, where

$$L(f) = \begin{bmatrix} f_1 & \delta * f_2 * \delta \\ f_2 & \delta * f_1 * \delta \end{bmatrix}.$$

(This matrix arises naturally from the left multiplication of f on VN (H).) It is easy to check that the map $f \to L(f)$ is a normal *-isomorphism of VN (H) into the von Neumann algebra VN $(\mathbf{Z}) \otimes M_2$. Since VN $(\mathbf{Z}) \cong L^\infty(\mathbf{T}, \mu_0)$, there exists a normal *-isomorphism of VN (H) into $L^\infty(\mathbf{T}, \mu_0) \otimes M_2$. As VN (H) is not strongly finite, the lemma follows from Propositions 3.6 and 3.7.

LEMMA 5.2. *Let (X,μ) be a probability space with no atoms. Then there exists a von Neumann subalgebra of $L^\infty(X,\mu)$ which is *-isomorphic to $L^\infty(\mathsf{T},\mu_0)$.*

PROOF. Let n be a non-negative integer. By [3, Lemma 2.3, p. 57], there exists a partition $\{S_i^n : i=1,2,\ldots,n\}$ of X into pairwise disjoint measurable subsets such that each S_i^n has measure $1/n$. Let χ_i^n be the characteristic function of S_i^n, and let A be the C*-algebra generated by

$$\{\chi_i^n : \ i=1,\ldots,n; \ n=1,2,\ldots\} .$$

We may assume that $L^\infty(X,\mu)$ is operating by multiplication on $L^2(X,\mu)$. Let Y be the spectrum of A, and let ν be the spectral measure on Y determined by the constant function 1. Then $\nu(\chi_i^n)=\mu(S_i^n)=1/n$, and it follows easily that ν has no atoms. By [2, Proposition 1, p. 114], the weak closure of A is isomorphic to $L^\infty(Y,\nu)$. As Y is compact and metrizable, the result now follows from [13, Theorem 9, p. 327].

PROPOSITION 5.3. *Let $Z\cong L^\infty(X,\mu)$ be an abelian von Neumann algebra, and let $n>1$ be an integer. Then $Z\otimes M_n$ is strongly finite if and only if (X,μ) is purely atomic.*

PROOF. If (X,μ) is purely atomic, then $Z\otimes M_n$ is a direct sum of copies of M_n. Suppose then that (X,μ) has a non-trivial non-atomic part. By Lemma 5.2, some countably decomposable direct summand of Z contains a copy of $L^\infty(\mathsf{T},\mu_0)$. It follows that $Z\otimes M_n$ contains a copy of $L^\infty(\mathsf{T},\mu_0)\otimes M_2$, so by Lemma 5.1, $Z\otimes M_n$ is not strongly finite.

THEOREM 5.4. *A von Neumann algebra M is strongly finite if and only if M is a direct sum of von Neumann algebras each of which is either abelian or finite dimensional.*

PROOF. Combine Proposition 3.8, Theorem 4.1, and Proposition 5.3.

REMARK. Let $Z\cong L^\infty(X,\mu)$, and let Δ be the spectrum of Z. Then (X,μ) is purely atomic if and only if $Z_*\cap\Delta$ separates the points of Z. For suppose (X,μ) has a non-atomic part, and let Z_0 be a von Neumann subalgebra of Z which is *-isomorphic to $L^\infty(\mathsf{T},\mu_0)$. Since zero is the only linear functional on $L^\infty(\mathsf{T},\mu_0)$ which is both normal and multiplicative, each element of $Z_*\cap\Delta$ must annihilate Z_0.

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