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TOTALLY DISCONNECTED SET NON-REMOVABLE FOR LIPSCHITZ CONTINUOUS BOUNDED ANALYTIC FUNCTIONS

NGUYEN XUAN UY

In this paper we exhibit an example of a totally disconnected compact set E of the complex plane, for which there exists a non-constant bounded analytic function on the complement of E, satisfying a Lipschitz condition

$$|f(z)-f(w)| \leqq M|z-w| \quad \text{for all } z \text{ and } w \text{ in } \mathbf{C}\setminus E .$$

1.

Let E be a compact subset of the complex plane $\mathbf{C}$ and let $0<\alpha\leqq 1$. We denote by $\Gamma_\alpha(E)$ the class of bounded analytic functions defined on $\mathbf{C}\setminus E$ which satisfy a Lipschitz condition of order α:

$$|f(z)-f(w)| \leqq M|z-w|^\alpha \quad \text{for all } z \text{ and } w \text{ in } \mathbf{C}\setminus E .$$

We will say that E is removable for Γ_α if $\Gamma_\alpha(E)$ consists only of constants. In [2, Corollary, p. 37], Dolženko has proved, for $0<\alpha<1$, that E is removable for Γ_α if and only if the $(1+\alpha)$-Hausdorff dimensional measure of E is zero. In this paper we are concerned about the problem of characterizing removable-sets for Γ_1.

It is clear that if the interior $\mathring{E}\neq\emptyset$, then E is non-removable for Γ_1, since, if we take a disc $\Delta(a,r)\subset E$ and define

$$f(z) = \begin{cases} \dfrac{1}{z-a} & \text{if } |z-a|\geqq r , \\[2ex] \dfrac{\bar{z}-\bar{a}}{r^2} & \text{if } |z-a|<r , \end{cases}$$

then $f\in\Gamma_1(E)$. In [4, Problem III 5.1], Garnett has asked if this non-empty interior condition can be improved. This problem has previously been solved by Dolženko in his same paper mentioned above. He proved that if $E=$

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$[0,1]\times L$, then E is removable for Γ_1 if and only if L has a zero length (see [4, Corollary, p. 39]).

In the following we show that there exists a totally disconnected compact set which is non-removable for Γ_1. We will be involved with the Riesz capacity, therefore, we give here some of its background. The terminology and notations are from Landkof [5].

Let μ be a positive measure and $0<\alpha<2$. The Riesz potential of order α of μ is denoted by U_α^μ, where

$$U_\alpha^\mu(z) = \int \frac{d\mu(\zeta)}{|\zeta - z|^{2-\alpha}} .$$

The Riesz capacity of order α of a compact set E is defined by the relation

$$C_\alpha(E) = \sup\{\mu(E):\ S_\mu \subset E,\ U_\alpha^\mu \leqq 1\} .$$

If E is arbitrary, then the inner capacity and outer capacity of E are defined respectively as

$$\underline{C}_\alpha(E) = \sup\{C_\alpha(K):\ K \text{ compact, } K\subset E\}$$

and

$$\bar{C}_\alpha(E) = \inf\{C_\alpha(V):\ V \text{ open, } V\subset E\} .$$

Clearly $\underline{C}_\alpha(E)\leqq \bar{C}_\alpha(E)$ If $\underline{C}_\alpha(E)=\bar{C}_\alpha(E)$, we say that E is capacitable and denote this value by $C_\alpha(E)$. If some property holds for all points of $\mathbf{C}$ with a possible exception of a set of zero outer capacity, then we say that this property holds quasi-everywhere. It is well known that every Borel set is capacitable and that if $C_\alpha(E)<\infty$, then there exists a unique measure μ, called the equilibrium measure of E, satisfying the following properties

i) $S_\mu \subset \bar{E}$
i) $\|\mu\| = C_\alpha(E)$
iii) $U_\alpha^\mu \leqq 1$ and $U_\alpha^\mu = 1$ quasi-everywhere on E.

2.

We are now ready to prove the following theorem.

THEOREM 1. *There exists a totally disconnected compact set non-removable for* Γ_1.

The proof is based on the following theorem due to Carleson (see [1, Theorem p. 314]).

THEOREM 2. *If E is a compact set contained in the interior of the closed unit disc Δ such that the capacity condition*

$$C_\alpha(\Delta\backslash E) < C_\alpha(\Delta)$$

holds for some $\alpha \in (0,2)$, then there exists a non-constant bounded analytic function with bounded derivative on $\mathbf{C}\backslash E$.

Now recall that, $\alpha>0$, the α-dimensional Hausdorff measure of a set E is defined as

$$\Lambda_\alpha(E) = \lim_{\delta\downarrow 0} \Lambda_\alpha^\delta(E),$$

where

$$\Lambda_\alpha^\delta(E) = \inf\left\{\sum r_j^\alpha : E \subset \Delta(a_j, r_j),\ r_j \leqq \delta\right\}.$$

Note that when $\alpha=1$ and $\alpha=2$, $\Lambda_\alpha(E)$ are essentially the length and the area of E.

LEMMA 1. *Suppose E is a Borel set with a finite length, i.e., $\Lambda_1(E)<\infty$. Then $C_\alpha(E)=0$ for every $\alpha \in (0,1)$.*

PROOF. Follows from the fact that if $C_\alpha(E)>0$, then $\Lambda_{2-\alpha}(E)>0$ (see [3, chapter VII, Theorem 2]) and that if $\beta>1$, then $\Lambda_\beta(E)=0$.

LEMMA 2. *If $E\subset\Delta$ and $\mathring{E}\neq\emptyset$, then*

$$C_\alpha(\Delta\backslash E) < C_\alpha(\Delta)$$

for all $\alpha \in (0,2)$.

PROOF. Let γ and ν respectively be the equilibrium measures of $\Delta\backslash E$ and Δ. Since $S_\gamma\subset\overline{\Delta\backslash E}$ and $S_\nu=\Delta$ (see 5, Appendix]) we have $\gamma\neq\nu$. Furthermore, since ν is the unique maximal measure with $U_\alpha^\nu\leqq 1$ and $\|\nu\|=C_\alpha(\Delta)$, we obtain $\|\gamma\|<C_\alpha(\Delta)$. Hence

$$C_\alpha(\Delta\backslash E) < C_\alpha(\Delta).$$

PROOF OF THEOREM 1. The idea is to construct a compact set E which satisfies the hypothesis of Theorem 2 and has the property that any two points z and w in $\mathbf{C}\backslash E$ can be joined in this set by a rectifiable curve such that its length does not exceed $M|z-w|$. This condition guarantees that any function with bounded derivative on $\mathbf{C}\backslash E$ satisfies a Lipschitz condition there.

Consider a closed square S situated in the interior of Δ. By Lemma 2 above

$$C_\alpha(\Delta\backslash S) < C_\alpha(\Delta), \quad 0<\alpha<2 .$$

Now fix $\alpha \in (0,1)$ and choose a sequence (a_n) of positive number such that

$$C_\alpha(\Delta\backslash S)+\sum_{n=1}^{\infty} a_n < C_\alpha(\Delta) .$$

We divide S into 4 squares by the two line segments parallel to the sides of S and passing through its center. Let L_1 be the union of these two line segments. By Lemma 1, $C_\alpha(L_1)=0$. Thus there exists a small open neighborhood V_1 of L_1 such that $C_\alpha(V_1)<\alpha_1$, because L_1 is capacitable. We may choose V_1 as a strip along L_1 so that $E_1=S\backslash V_1$ is the union of four disjoint squares of equal side. Suppose at stage $n\geqq 1$, L_n, V_n, E_n were defined and E_n consists of 4^n disjoint squares of equal side. We repeat the above process with each square of E_n and let L_{n+1} be the union of new line segments occurring in this step. Let V_{n+1} be a neighborhood of L_{n+1} with $C_\alpha(V_{n+1})<a_{n+1}$, and $E_{n+1}=E_n\backslash V_{n+1}$ is the union of 4^{n+1} disjoint squares with equal side. Thus $E_1\supset E_2\supset \ldots \supset E_n \supset \ldots$. We define

$$E = \bigcap_{n=1}^{\infty} E_n .$$

Then E is totally disconnected and

$$\Delta\backslash E = (\Delta\backslash S) \cup \left(\bigcup_{n=1}^{\infty} V_n\right).$$

So, by the sub-additivity of C_α, we obtain

$$\begin{aligned} C_\alpha(\Delta\backslash E) &\leqq C_\alpha(\Delta\backslash S)+\sum_{n=1}^{\infty} C_\alpha(V_n) \\ &< C_\alpha(\Delta\backslash S)+\sum_{n=1}^{\infty} a_n \\ &< C_\alpha(\Delta) , \end{aligned}$$

and Theorem 1 follows.

3.

It is known that if area $(E)=0$, then E is removable for Γ_1 (see [1, Theorem p. 312]). We don't know whether the converse is also true, this appears to be a more difficult problem. It seems possible that there exists a set of positive area but removable for Γ_1. There is a known result (see [4, III 4.7]) closely related to this problem which deserves to be mentioned here with proof.

THEOREM 3. *Suppose E is compact with area $(E)>0$. Let $g \in L^\infty$ such that $g=0$ on $\mathbf{C}\backslash E$* a.e. *Then the function*

$$f(z) = \iint_E \frac{g(\zeta)}{\zeta - z} d\xi d\eta, \qquad \zeta = \xi + i\eta$$

satisfies a Zygmund condition $|f(z+h)+f(z-h)-2f(z)| \leqq M|h|$ for all complex z and h.

PROOF. Observe that

$$f(z+h)+f(z-h)-2f(z) = 2h^2 \iint_E \frac{g(\zeta)\, d\xi d\eta}{(\zeta-z-h)(\zeta-z+h)(\zeta-z)} .$$

We estimate this integral over the sets

$$A = \left\{\zeta : \ |\zeta - z - h| \leqq \frac{|h|}{2}\right\},$$

$$B = \left\{\zeta : \ |\zeta - z + h| \leqq \frac{|h|}{2}\right\},$$

$$C = \left\{\zeta : \ |\zeta - z| \leqq \frac{|h|}{2}\right\},$$

$$D = \left\{\zeta : \ |\zeta - z| \geqq |\zeta - z - h|, |\zeta - z - h| \geqq \frac{|h|}{2}\right\},$$

$$E = \left\{\zeta : \ |\zeta - z| \geqq |\zeta - z + h|, |\zeta - z + h| \geqq \frac{|h|}{2}\right\},$$

$$F = \left\{\zeta : \ |\zeta - z| \leqq |\zeta - z - h|, |\zeta - z| \leqq |\zeta - z + h|, |\zeta - z| \geqq \frac{|h|}{2}\right\}.$$

We obtain

$$\begin{aligned}\left|2h^2 \iint_A \frac{g(\zeta)\, d\xi d\eta}{(\zeta-z-h)(\zeta-z+h)(\zeta-z)}\right| &\leqq 8\|g\|_\infty \iint_A \frac{d\xi d\eta}{|\zeta - z - h|} \\ &\leqq 16\pi\|g\|_\infty \int_0^{\frac{1}{2}|h|} dr \\ &\leqq 8\pi\|g\|_\infty |h| .\end{aligned}$$

This estimate also holds for integrals over B and C. Similarly,

$$\left|2h^2 \iint_D \frac{g(\zeta)\,d\xi d\eta}{(\zeta-z-h)(\zeta-z+h)(\zeta-z)}\right| \leqq 2\|g\|_\infty |h|^2 \iint_D \frac{d\xi d\eta}{|\zeta-z-h|^3}$$

$$\leqq 4\pi \|g\|_\infty |h|^2 \int_{\frac{1}{2}|h|}^{\infty} \frac{dr}{r^2}$$

$$\leqq 8\pi \|g\|_\infty |h| \,.$$

Since this estimate also holds for integrals over E and F, f satisfies a Zygmund condition.

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