MATH. SCAND. 40 (1977), 119–126

ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF LINEAR AND ALGEBRAIC DIFFERENTIAL EQUATIONS

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1. Introduction.

It has been shown by Bank in [1] that the growth of a meromorphic solution of a linear differential equation with meromorphic coefficients cannot be estimated uniformly in terms of the growth of the coefficients alone. More precisely, given any increasing real function Φ: $(0, +\infty) \to \mathrm{R}$ there is a meromorphic function h satisfying a first order linear differential equation

$$f_1(z)h' + f_0(z)h = 0$$

with entire coefficients f_0, f_1 of finite order and a sequence $(r_n)_{n \in \mathrm{N}}$ of values of r tending to $+\infty$ such that $T(r_n, h) \geqq \Phi(r_n)$ for all $n \in \mathrm{N}$, where $T(r, h)$ denotes the Nevanlinna characteristic function of h ([1], Theorem 2). Bank further proved that the growth of a meromorphic solution $y = y(z)$ of a linear differential equation

$$(1) \qquad \sum_{j=0}^{n} f_j(z) y^{(j)} = 0$$

with entire coefficients $f_0, \ldots, f_n$ can be estimated uniformly, in the above sense, in terms of the characteristic functions $T(r, f_0), \ldots, T(r, f_n)$ of the coefficients and of the counting functions $N(r, 1/y)$, $\bar{N}(r, y/y')$ for the zeros of y and the distinct zeros of y'/y ([3, Theorem 2]).

The following problems arise quite immediately: Is it possible to dispense with $\bar{N}(r, y/y')$ in the estimate given in [3], and, in addition, is it possible to get a similar estimate in the case of a linear differential equation (1) with meromorphic coefficients? By using the result of J. Miles [6, Theorem on p. 372], on the quotient representation of meromorphic functions, one can easily translate the results of [3] for entire coefficients, into results for meromorphic coefficients. However, both of the above questions can be handled

* The work of this author was done while he was visiting the Department of Mathematics, University of Erlangen-Nürnberg, supported by the Alexander von Humboldt Foundation.

Received May 26, 1976.

simultaneously, and the main purpose of this article, to be realized in Section 3, is to give a positive answer to these questions. These results will be achieved by invoking in the estimate of $T(r, y'/y)$, the original form of a well-known lemma of Clunie [4, Lemma 2]).

The same device will be used in Section 4 to get some new estimates for the growth of a meromorphic solution $y=y(z)$ of an algebraic differential equation

$$\Omega(z, y, y', \ldots, y^{(n)}) = 0 \tag{2}$$

with meromorphic coefficients.

2. Notation and two preliminary lemmas.

All meromorphic functions to be considered here are assumed to be meromorphic in the complex plane. We shall apply the usual notations and basic results of the Nevanlinna theory of the value distribution of meromorphic functions, see e.g. [5].

The equation (2) is a polynomial in the indeterminates $y, y', \ldots, y^{(n)}$, and hence there is a finite set I of multi-indices $\lambda = (i_0, \ldots, i_n)$ such that (2) has a representation of the form

$$\sum_{\lambda \in I} a_\lambda(z) y^{i_0} \ldots (y^{(n)})^{i_n} = 0 \tag{2'}$$

with meromorphic coefficients $a_\lambda(z) = a_{i_0, \ldots, i_n}(z)$. The degree of a single term of multi-index $\lambda \in I$ in Ω is denoted by

$$|\lambda| = i_0 + \ldots + i_n$$

and its weight by

$$\|\lambda\| = i_1 + 2i_2 + \ldots + ni_n .$$

The total degree of Ω is defined by $\max_{\lambda \in I} |\lambda|$.

We state here two preparatory lemmas. The first of them is nothing else than a modification of a lemma of Clunie [4, Lemma 2]. (We remark here that the usual statement of Clunie's lemma involves the characteristics of the coefficients, while our statement involves only the proximity functions of the coefficients. The proof is the same in both cases, and even though the stronger statement is not essential to this paper, it is more useful in the area of differential equations.) The second lemma follows immediately by [2, Lemma 7] and [3, p. 283]. Since $T(r, y) = T(r, 1/y) + O(1)$, the alternate form of Lemma 2 below is an obvious corollary.

LEMMA 1. *Let $w=w(z)$ be a meromorphic solution of the equation*

$$w^n P(w) = Q(w), \tag{3}$$

where $P(w)$ and $Q(w)$ are polynomials in w and its derivatives with meromorphic coefficients $\{a_\lambda \mid \lambda \in I\}$. Let us denote

$$\Psi(r) = \max_{\lambda \in I} (\log r, m(r, a_\lambda)).$$

If the total degree of Q is at most n, then there exists a positive constant K such that

$$m(r, P(w)) \leqq K\Psi(r) + o(T(r, w))$$

outside of a possible exceptional set of finite linear measure.

LEMMA 2. *Let $y=y(z)$ be any nonconstant meromorphic function and denote $w=y'/y$. Then for any $\alpha>1$, there exist positive constants A, B and $r_1 \geqq 1$ such that for all $r \geqq r_1$*

$$T(r, y) \leqq A\big(rN(\alpha r, y) + r^2 \exp\big(BT(\alpha r, w) \log (rT(\alpha r, w))\big)\big)$$

and

$$T(r, y) \leqq A\big(rN(\alpha r, 1/y) + r^2 \exp\big(BT(\alpha r, w) \log (rT(\alpha r, w))\big)\big).$$

3. Linear differential equations.

THEOREM 3. *Let $y=y(z)$ be a meromorphic solution of a linear differential equation (1) with meromorphic coefficients, and denote $w=y'/y$. If*

$$\Phi(r) = \max (\log r, T(r, f_0), \ldots, T(r, f_n)),$$

then for any $\sigma>1$, there exist positive constants C, C_1 and $r_0 \geqq 1$ such that for all $r \geqq r_0$,

$$T(r, y) \leqq C\big(rN(\sigma r, 1/y) + r^2 \exp\big(C_1 H(\sigma r) \log (rH(\sigma r))\big)\big),$$

where

$$H(r) = \bar{N}(r, 1/y) + \Phi(r).$$

Therefore, the growth of the solution $y(z)$ can be estimated uniformly in terms of the growth of the coefficients and the counting function for the zeros of y.

PROOF. We may assume that y is nonconstant and that f_n does not vanish identically. Clearly w satisfies an algebraic differential equation

$$(4) \qquad \Lambda(z, w, w', \ldots, w^{(n-1)}) = 0\,,$$

where Λ is a polynomial in $w, w', \ldots, w^{(n-1)}$, whose homogeneous part of maximum total degree in $w, w', \ldots, w^{(n-1)}$ equals $f_n w^n$, and where the coefficients of Λ are linear combinations of $f_0, \ldots, f_n$. Clearly the equation (4) can be written in the form

$$(5) \qquad w \cdot w^{n-1} = \Lambda_1(z, w, w', \ldots, w^{(n-1)})\,,$$

where the total degree of Λ_1 is at most $n-1$ and where the coefficients of Λ_1 are linear combinations of the meromorphic functions $f_0/f_n, \ldots, f_{n-1}/f_n, 1$. An obvious application of Lemma 1 gives a constant $K_1 > 0$ such that outside a set of finite measure,

$$m(r, w) \leqq K_1 \Phi(r) + o(T(r, w))\,.$$

Let us consider now a point z_0 where all the coefficients $f_0, \ldots, f_n$ take finite, non-zero values. Clearly the solution $y(z)$ of (1) does not have a pole at z_0. Therefore we may find two constants $K_2 > 0$ and $r_2 \geqq r_1$ such that

$$(6) \qquad \bar{N}(r, y) \leqq K_2 \Phi(r)$$

for all $r \geqq r_2$. We get the obvious estimate

$$N(r, w) = \bar{N}(r, w) = \bar{N}(r, 1/y) + \bar{N}(r, y) \leqq \bar{N}(r, 1/y) + K_2 \Phi(r)$$

for all $r \geqq r_2$, hence

$$T(r, w) \leqq \bar{N}(r, 1/y) + (K_1 + K_2)\Phi(r) + o(T(r, w))$$

holds outside of a possible exceptional set of finite linear measure. An obvious application of [2, § 2], gives two constants $K > 0$ and $r_3 \geqq r_1$ such that, given $\beta > 1$,

$$T(r, w) \leqq K(\bar{N}(\beta r, 1/y) + \Phi(\beta r))$$

holds for all $r \geqq r_3$. The assertion follows now by a straightforward combination of this result with the estimate of Lemma 2, if we choose $C = A$, $C_1 = 2BK$, $r_0 \geqq r_3$ and select α in Lemma 2 and β above to satisfy $\alpha\beta \leqq \sigma$.

REMARK. We note in this connection that the coefficients of (4) are actually some integral multiples of the original coefficients of (1). This fact could be possibly used for a more detailed study of the value distribution of $y(z)$.

4. Algebraic differential equations.

We consider in this section an algebraic differential equation (2′) with meromorphic coefficients. Let us denote

$$\Phi(r) = \max_{\lambda \in I} \left(\log r, T(r, a_\lambda(z))\right)$$

and let Ω_q denote the homogeneous part of Ω of total degree q, i.e.

$$\Omega_q = \sum_{\substack{\lambda \in I \\ |\lambda| = q}} a_\lambda(z) y^{i_0} \dots (y^{(n)})^{i_n} .$$

If $y = y(z)$ is a meromorphic solution of (2′), which for some integer q does not satisfy the equation $\Omega_q = 0$, then the growth of $y(z)$ can be estimated uniformly in terms of the growth of the coefficients and the counting functions for the distinct zeros and distinct poles of y, (see [3, Lemma 4]).

If $y = y(z)$ is simultaneously a solution of all of the homogeneous equations $\Omega_q = 0$, then some estimates for the growth of y can be found in [2, Theorem 9]. For a homogeneous part Ω_q of Ω, let us denote by $A_q(z)$ the sum of all coefficients $a_\lambda(z)$ in Ω_q having multi-indices of maximal weight, i.e. for $k = \max_{|\lambda| = q} \|\lambda\|$ we have

$$A_q(z) = \sum_{\substack{|\mu| = q \\ \|\mu\| = k}} a_\mu(z) .$$

The significance of the functions $A_q(z)$ for the possible growth of $y = y(z)$ has been noticed earlier, see e.g. the results of Yang [9, Theorem 3] and of Mohon'ko–Mohon'ko [7, Theorems 7–10]. This section is devoted to give an improvement of these results as a further application of Lemmas 1 and 2.

THEOREM 4. *Let $y = y(z)$ be a meromorphic solution of (2′) which also satisfies all homogeneous equations $\Omega_q = 0$. If for some q, for which $\Omega_q \not\equiv 0$, we have $A_q \not\equiv 0$, then for any $\sigma > 1$, there exist positive constants C, C_1 and $r_0 \geqq 1$ such that for all $r \geqq r_0$,*

$$T(r, y) \leqq C\left(rN(\sigma r, y) + r^2 \exp\left(C_1 H(\sigma r) \log\left(rH(\sigma r)\right)\right)\right), \tag{7}$$

where

$$H(r) = \bar{N}(r, 1/y) + \bar{N}(r, y) + \Phi(r) .$$

Therefore, the growth of the solution $y(z)$ can be estimated in this case uniformly in terms of the growth of the coefficients and the counting functions for the poles and distinct zeros of y.

PROOF. It is well-known [5, Lemma 3.5], that $w = y'/y$ satisfies

$$y^{(n)} = (w^n + P_{n-1}(w))y , \tag{8}$$

where $P_{n-1}(w)$ is a polynomial in w and its derivatives, of total degree at most

$n-1$ with constant coefficients. Substituting (8) into the equation $\Omega_q(z, y, \ldots, y^{(n)}) = 0$ we get

$$\Omega_q(z, y, y', \ldots, y^{(n)}) = (A_q(z)w^k + Q_{k-1}(w))y^q = 0,$$

where $k = \max_{|\lambda|=q} \|\lambda\|$ and $Q_{k-1}(w)$ is a polynomial in w and its derivatives, of total degree at most $k-1$ with coefficients which are linear combinations of the original coefficients $a_\lambda(z)$, $|\lambda| = q$. Clearly we may assume

$$A_q(z)w^k + Q_{k-1}(w) = 0,$$

hence

$$m(r, A_q w) \leqq K_1 \Phi(r) + o(T(r, w))$$

for some constant $K_1 > 0$ outside of a possible exceptional set of finite linear measure as an application of Lemma 1. Obviously

$$N(r, A_q w) \leqq N(r, A_q) + N(r, w) \leqq K_2 \Phi(r) + \bar{N}(r, 1/y) + \bar{N}(r, y)$$

for some constant $K_2 > 0$, hence

$$\begin{aligned} T(r, w) &\leqq T(r, 1/A_q) + (K_1 + K_2)\Phi(r) + \bar{N}(r, 1/y) + \bar{N}(r, y) + o(T(r, w)) \\ &\leqq K_3 \Phi(r) + \bar{N}(r, 1/y) + \bar{N}(r, y) + o(T(r, w)) \end{aligned}$$

for some constant $K_3 > 0$ outside of an exceptional set of finite linear measure. Hence there exist two constants $K > 0$ and $r_2 \geqq r_1$ such that, given $\beta > 1$,

$$T(r, w) \leqq K\Phi(\beta r) + \bar{N}(\beta r, 1/y) + \bar{N}(\beta r, y)$$

for all $r \geqq r_2$. The same conclusion as in the proof of Theorem 3 gives the assertion, if we choose $C = A$, $C_1 = 2BK$, $r_0 \geqq r_2$ and if $\alpha\beta \leqq \sigma$.

COROLLARY 5. *Let $y = y(z)$ be a meromorphic solution of* (2′) *which also satisfies all homogeneous equations $\Omega_q = 0$. If for some q, for which $\Omega_q \not\equiv 0$, we have $A_q \not\equiv 0$, then the growth of the solution $y(z)$ can be estimated uniformly in terms of the growth of the coefficients and the counting functions for the zeros and distinct poles of y.*

PROOF. This corollary follows immediately, if we apply the alternate form of Lemma 2 in the final conclusion of the preceding proof.

REMARKS. 1) If all of the functions $A_q(z)$ vanish identically, the estimate given in Theorem 4 (or equivalently in Corollary 5) can fail. The example (given in [8, p. 70])

$$(w''w)^2 - 2w''(w')^2 w + (w')^4 + (w'w)^2 - w^4 = 0$$

possessing an entire solution $w = \exp(\sin z)$ can be used to demonstrate this fact.

2) Using Lemma 4 of [3] and Theorem 4 above (or Corollary 5 as well) we can determine the quantities which are needed to get a uniform estimate for the growth of a meromorphic solution of (2′). We list the following three possibilities:

(a) there is a q such that y does not satisfy $\Omega_q = 0$,
(b) y satisfies all $\Omega_q = 0$ and there is an $A_q \not\equiv 0$,
(c) y satisfies all $\Omega_q = 0$ and all $A_q \equiv 0$.

To determine now what quantities enter into the growth estimate for a solution y of (2′), we can use a refinement of the reasoning in [2, Theorem 9], based on Theorem 4 (or Corollary 5) above. If (a) or (b) holds, then Lemma 4 of [3] or Theorem 4 above, respectively, can be used to determine these quantities. If (c) holds, then $w = y'/y$ solves an equation $\Lambda_1 = 0$ of order $n-1$. If (a) or (b) holds for w and Λ_1, then Lemma 4 of [3] or Theorem 4 above can again be used to determine the quantities which enter into the growth estimate for $T(r, w)$, and then Lemma 2 above can be used to estimate $T(r, y)$. If (c) holds for w and Λ_1, then w'/w solves an equation $\Lambda_2 = 0$ of order $n-2$, and the process can be repeated. If (c) continues to hold, then eventually we obtain a first-order equation $\Lambda_{n-1} = 0$. For Λ_{n-1}, (c) obviously cannot hold.

3) Theorem 3 can be also proved as a corollary to Theorem 4. One must only combine the estimates (7) and (6) and observe, that the equation (1) is a homogeneous equation of type (2′) of total degree one such that $A_1(z) = f_n(z) \not\equiv 0$.

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