MATH. SCAND. 40 (1977), 127-142

THE SPACE OF QUASISYMMETRIC MAPPINGS*

PEKKA TUKIA

Abstract.

The universal Teichmüller space T is an open subset of a Banach space B whose elements are holomorphic mappings of the lower half-plane. It is known that the universal Teichmüller space is contractible. We show that it is homeomorphic to a real Banach space E and that as a real analytic Banach manifold it is equivalent with an open, convex subset of E. The Banach spaces E and B are isomorphic as topological, real linear spaces. The method of proof is to find a set of "moduli" for quasisymmetric mappings.

1. Introduction.

A quasisymmetric mapping is an increasing homeomorphism f of the real line $\mathbf{R}$ satisfying

$$1/\varrho \leqq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leqq \varrho \tag{1}$$

for some $\varrho \geqq 1$ when $x,t \in \mathbf{R}$, $t>0$. A quasisymmetric mapping f is ϱ-quasisymmetric if it satisfies (1) with this particular ϱ. The 1-quasisymmetric maps are just the affine transformations $x \mapsto ax+b$, $x \in \mathbf{R}$ varies, $a,b \in \mathbf{R}$ fixed, $a>0$. The interest of quasisymmetric maps lies in the fact that the restriction of a quasiconformal self-map of the upper half-plane

$$U = \{z \in \mathbf{C} :\ \operatorname{im} z>0\}$$

is quasisymmetric. (Every quasiconformal self-map of U has a unique extension to $\mathbf{R}$ and for simplicity we consider them already extended to $\mathbf{R}$). Conversely, every quasisymmetric mapping is the restriction of such a mapping. (Cf. Beurling–Ahlfors [5].)

It is often convenient to normalize quasiconformal self-maps of U and quasisymmetric maps of $\mathbf{R}$ so that they fix 0 and 1 (∞ is in any case already fixed). A normalized quasiconformal self-map of U is uniquely determined if its complex dilatation $\mu(f)=\bar{\partial} f/\partial f$ is known. Then $\mu(f)$ is an element of

---

* I wish to thank Magnus Ehrnrooth foundation for financial support.
Received May 25, 1976.

$L^\infty(U, \mathbf{C})$, the Banach space of measurable bounded maps $U \to \mathbf{C}$ with supremum norm. It is a basic result in the theory of quasiconformal mappings that the mapping $f \mapsto \mu(f)$ is one-to-one from the set of normalized quasiconformal self-maps of U onto the open unit ball M of $L^\infty(U, \mathbf{C})$ (cf. Lehto–Virtanen [8]). The set M is an open set in a complex Banach space, thus it is in a natural way a complex analytic Banach manifold. Consequently, this is a way to define a complex analytic structure in the set of normalized quasiconformal self-maps of U. We can define an equivalence relation in the set of normalized quasiconformal self-maps of U as follows: let $f \sim g$ if and only if $f \mid \mathbf{R} = g \mid \mathbf{R}$. The quotient set of $\sim$ is the set of normalized quasisymmetric maps of $\mathbf{R}$. The space M is a complex analytic manifold and thus the set of normalized quasisymmetric mappings inherits a complex analytic structure from M. One can also show that it is a Banach manifold. This Banach manifold is called the *universal Teichmüller space.* It can be identified with an open subset in the complex Banach space B of holomorphic functions φ in the lower half-plane L with norm

$$\|\varphi\|_B = \sup\{2(\operatorname{im} z)^2|\varphi(z)| : z \in L\} .$$

(Cf. Bers [2], [3], Earle and Eells [7].)

From the topology of the universal Teichmüller space is known that it is contractible (Earle and Eells [7]). In this paper we show that, more precisely, it is homeomorphic to a Banach space. We also show that as a real analytic manifold it is equivalent with a convex set in a real Banach space. (It is not known whether the above imbedding in B is convex). We prove these results by showing that there is a natural way to associate with a quasisymmetric mapping a sequence (k_i) of real numbers and that the set of such sequences is an open, convex set in a Banach space that is a linear subspace of l^∞, i.e. the set of all bounded sequences (k_i) with supremum norm.

Our method is in a way direct. We define the real analytic structure of normalized quasisymmetric mappings and its topological properties without making essential use of the theory of quasiconformal mappings. It is true that in section 4 some properties of quasiconformal mappings are needed but this is for convenience, add a page and we could do without.

2. Nets and sieves of R.

A family $\mathcal{N}$ of closed, non-empty, non-degenerate intervals of $\mathbf{R}$ is a *net* of $\mathbf{R}$ if

(i) $\bigcup_{I \in \mathcal{N}} I = \mathbf{R}$,
(ii) $I \cap J = \emptyset$ or $=$ a common endpoint, $I, J \in \mathcal{N}$, $I \neq J$, and

(iii) If x is an endpoint of $I \in \mathcal{N}$, then x is also the endpoint of another $J \in \mathcal{N}$, $J \neq I$.

An interval of $\mathcal{N}$ is also called a *mesh* of $\mathcal{N}$. Two meshes are said to be *adjacent* if their intersection is a point. The set of endpoints of meshes of $\mathcal{N}$ is the set of *vertices* of $\mathcal{N}$. It follows that the set of vertices of a net $\mathcal{N}$ can be enumerated as a sequence x_i, $i \in \mathbf{Z}$, so that $x_i < x_j$ if $i < j$ and that the family $\{[x_{i-1}, x_i] \; ; \; i \in \mathbf{Z}\}$ is the net $\mathcal{N}$.

Let us denote the length of an interval I by $l(I)$. A net is said to be *symmetric* if its meshes have the same length. A net is said to be *quasisymmetric* if there is a constant $k \geqq 1$ such that

$$1/k \leqq l(I)/l(J) \leqq k$$

whenever I and J are two adjacent meshes of the net. More precisely, the net $\mathcal{N}$ is *k-quasisymmetric* if the above inequality is true. A 1-quasisymmetric net is symmetric and conversely. If $k > 1$, it need not be true that the ratios of non-adjacent meshes of a k-quasisymmetric net are bounded.

A net $\mathcal{N}'$ is a *subdivision* of the net $\mathcal{N}$ if the set of vertices of $\mathcal{N}'$ contains the vertices of $\mathcal{N}$. Then a mesh of $\mathcal{N}$ is a finite union of meshes of $\mathcal{N}'$. A *sieve* is a sequence $\Sigma = (\mathcal{N}_i)$, $i \in \mathbf{Z}$, of nets such that $\mathcal{N}_{i+1}$ is a subdivision of $\mathcal{N}_i$ and that each mesh of $\mathcal{N}_i$ is the union of exactly two meshes of $\mathcal{N}_{i+1}$. Thus the set of vertices of $\mathcal{N}_{i+1}$ is obtained from that of $\mathcal{N}_i$ by adding a vertex in every mesh of $\mathcal{N}_i$. A sieve is said to be *fine* if for every sequence $I_i \in \mathcal{N}_i$, $i \in \mathbf{Z}$, such that $I_{i+1} \subset I_i$, the intersection

$$\bigcap_{i \in \mathbf{Z}} I_i = \text{a point} .$$

The words *symmetric*, *quasisymmetric*, *k-quasisymmetric*, *mesh* and *vertex* are applied also to sieves. Thus e.g. a k-quasisymmetric sieve is a sieve containing only k-quasisymmetric nets. In the same manner, a mesh of a sieve Σ is a mesh of some net of Σ.

Proposition 1. *A quasisymmetric sieve is fine.*

Proof. Let Σ be k-quasisymmetric. Let (I_i), $i \in \mathbf{Z}$, be a sequence of meshes of Σ such that $I_{i+1} \subset I_i$, $I_i \in \mathcal{N}_i$, $i \in \mathbf{Z}$. Then we have

$$1/(k+1) \leqq l(I_{i+1})/l(I_i) \leqq k/(k+1) .$$

The conclusion is now obvious.

3. The space of quasisymmetric sieves.

For our purposes it is convenient to assume that the interval $[0,1]$ is a mesh of the net $\mathcal{N}_0$ of a sieve $\Sigma = (\mathcal{N}_i)$, $i \in \mathbf{Z}$. Such a sieve is called *normalized*. Of course any sieve can be normalized by performing a transformation of the form $x \mapsto ax+b$ for $x \in \mathbf{R}$ ($a, b \in \mathbf{R}$, $a>0$). We denote by I_{i0} the mesh of $\mathcal{N}_i$ that has 0 as the smaller endpoint. It is clear that, once I_{i0} is given, there is a unique enumeration $\{I_{ij}\}$, $j \in \mathbf{Z}$, of meshes of $\mathcal{N}_i$ for each $i \in \mathbf{Z}$ such that I_{ij} and $I_{i,j+1}$ are adjacent and that the smaller endpoint of $I_{i,j+1}$ is the bigger endpoint of I_{ij}, $j \in \mathbf{Z}$.

Now we can form the ratios

$$k_{ij} = l(I_{ij})/l(I_{i,j-1}), \quad i,j \in \mathbf{Z}\,. \tag{2}$$

The normalized sieve is uniquely determined once the numbers (k_{ij}), $i,j \in \mathbf{Z}$, are known. For beginning from $I_{00} = [0,1]$ we can determine all other meshes I_{0i}, $i \in \mathbf{Z}$, knowing k_{0i}, $i \in \mathbf{Z}$. The meshes I_{10} and $I_{-1,0}$ are determined from the mesh I_{00} and numbers k_{11} and k_{01}. Thus we can determine step by step all the meshes I_{ij}, $i,j \in \mathbf{Z}$. The sieve Σ is k-quasisymmetric if and only if

$$1/k \leqq k_{ij} \leqq k \quad \text{for all } i,j \in \mathbf{Z}\,.$$

Thus the set of normalized quasisymmetric sieves can be made to a metric space by defining the distance between two sieves Σ and Σ' to be

$$d(\Sigma, \Sigma') = \sup_{i,j \in \mathbf{Z}} |\log k_{ij} - \log k'_{ij}| \tag{3}$$

where k_{ij} and k'_{ij} are the numbers defined by (2) with respect to Σ and Σ'.

The numbers k_{ij}, $i,j \in \mathbf{Z}$, are not independent. A simple calculation shows

$$k_{i,2j} = k_{i-1,j}\frac{1+k_{i,2j-1}^{-1}}{1+k_{i,2j+1}} \quad \text{for } i,j \in \mathbf{Z}\,, \tag{4}$$

i.e. when passing from $\mathcal{N}_{i-1}$ to $\mathcal{N}_i$ only the numbers k_{ij}, j odd, can be given freely, the numbers $k_{i,2j}$, $j \in \mathbf{Z}$, can be computed from these. On the other hand, if the positive real numbers (k_{ij}), $i,j \in \mathbf{Z}$, satisfy the relations (4) there is always a unique normalized sieve Σ such that $k_{ij} = l(I_{ij})/l(I_{i,j-1})$, $i,j \in \mathbf{Z}$, where the meshes I_{ij} are as in (2).

We can use (4) repeatedly to calculate

$$k_{i,2^n j} = k_{i-n,j}\frac{1+k_{i-n+1,2j-1}^{-1}}{1+k_{i-n+1,2j+1}} \cdots \frac{1+k_{i,2^n j-1}^{-1}}{1+k_{i,2^n j+1}} \tag{5}$$

for $i,j,n \in \mathbf{Z}$, If $n<0$ and $2^n j \in \mathbf{Z}$ we have

$$k_{i,2^n j} = k_{i-n,j}\frac{1+k_{i-n,j+1}}{1+k_{i-n,j-1}^{-1}} \cdots \frac{1+k_{i+1,2^{n-1}j+1}}{1+k_{i+1,2^{n-1}j-1}^{-1}}\,.$$

Now we can see that there is a one-to-one correspondence between the set of normalized sieves and indexed families (k_{ij}), $i, j \in \mathbf{Z}$, j odd or $(i,j)=0$, of positive real numbers. From now on we identify the sieve and the family corresponding to it. We write P for $\mathbf{Z} \times (2\mathbf{Z}+1) \cup \{(0,0)\}$, thus we can write the sieve (k_{ij}), $(i,j) \in P$. We can also write the sieve (k_{ij}), $i, j \in \mathbf{Z}$. Then it is understood that the numbers k_{ij}, $(i,j) \notin P$, are determined from equation (5).

We can paraphrase the condition that a sieve is quasisymmetric as follows:

PROPOSITION 2. *A sieve (k_{ij}), $(i,j) \in P$, is k-quasisymmetric if and only if the products* (5) *lie in the interval $[1/k, k]$ for $(i-n, j) \in P$ and $n \geqq 0$ and for $(i-n, j)=0$, $n<0$.*

We shall now try to embed the space of sieves in a Banach space so that the image set is convex. Equation (5) suggests the following transformation

$$\begin{aligned} (6) \qquad h_{ij} &= \log((1+k_{ij})/2) \qquad \text{if } j-1 \in 4\mathbf{Z} \text{ or } (i,j)=0\,, \\ &= \log((1+k_{ij}^{-1})/2) \qquad \text{if } j+1 \in 4\mathbf{Z}\,, \end{aligned}$$

for $(i,j) \in P$. We can express the numbers k_{ij} by means of h_{ij} as follows

$$(7) \qquad k_{ij} = (2\exp h_{ij} - 1)^{\pm 1}$$

for $(i,j) \in P$. Here we choose the exponent -1 if $j+1 \in 4\mathbf{Z}$, $+1$ otherwise. Equation (5) assumes now the form

$$\begin{aligned} (8) \qquad \log k_{i,2^n j} &= \log k_{i-n,j} \\ &\quad + \log(1+k_{i-n+1,2j-1}^{-1}) - \log(1+k_{i-n+1,2j+1}) \\ &\quad + \sum_{k=2}^{n} (h_{i-n+k,2^k j-1} - h_{i-n+k,2^k j+1})] \end{aligned}$$

for $i, j, n \in \mathbf{Z}$, $n \geqq 0$. If $n<0$ the equation is slightly different, cf. equation (5) The reader can easily modify (8) to fit this case.

On the basis of (8) it is reasonable to consider a Banach space E formed of all sequences (h_{ij}), $(i,j) \in P$, of real numbers such that both

$$\begin{aligned} (9) \quad m &= \sup\{|h_{ij}| : (i,j) \in P\} \quad \text{and} \\ M &= \sup\Big\{\Big|\sum_{k=\varepsilon}^{n} (h_{i-n+k,2^k j-1} - h_{i-n+k,2^k j+1})\Big| : \\ &\qquad (i-n,j) \in P \text{ and } n \geqq 0 \text{ or } (i-n,j)=0 \text{ and } n<0;\ \varepsilon = \operatorname{sign}(n)\Big\} \end{aligned}$$

are finite. For such sequences we define the norm to be

$$(10) \qquad \|(h_{ij})\|_E = \max\{m, M\}\,.$$

Finally let E_0 and E_m, $m>0$, be the subsets of E for which

$$E_m = \{(h_{ij}) \in E : h_{ij} \geqq -\log 2 + m \quad \text{for } (i,j) \in P\} \tag{11}$$

$$E_0 = \bigcup_{m>0} E_m .$$

We have identified the set of quasisymmetric sieves with sequences (k_{ij}), $i,j \in \mathbf{Z}$, and these sequences we have mapped by equations (6) to sequences (h_{ij}), $(i,j) \in P$. We can now formulate the main result of this section.

PROPOSITION 3. *The mapping defined by* (6) *from the space of quasisymmetric sieves to the set of sequences* (h_{ij}), $(i,j) \in P$, *is a bijection onto the set* E_0, *which is an open, convex set in the Banach space E. Moreover, if the space of quasisymmetric sieves is topologized by* (3) *then it is a homeomorphism onto* E_0 *as a subspace of E with norm* (10) *and it, as well as its inverse, is locally Lipschitz.*

PROOF. This is mechanical. Use equations (5) and (8) together with the norm (10) and metric (3). It is an immediate consequence of (11) that E_0 is open and convex in E.

The set E_0 is homeomorphic to E. To see this note that for $h = (h_{ij})_{(i,j)\in P} \in E_0$

$$r(h) = \inf\{h_{ij} + \log 2 : (i,j) \in P\} > 0 .$$

Here r is a continuous mapping $E_0 \to \mathbf{R}$. Now it is easy to see that

$$h = (h_{ij})_{(i,j)\in P} \mapsto (h_{ij} - 1/r(h))_{(i,j)\in P}$$

defines a homeomorphism $E_0 \to E$. These observations combined with Proposition 3 yield:

COROLLARY. *The space of normalized quasisymmetric sieves is homeomorphic to a Banach space. This is the space E defined in equations* (9) *and* (10).

4. Sieves and quasisymmetric mappings.

Let Σ and Σ' be two normalized, fine sieves. Then, as we have seen, we can enumerate the meshes of Σ and Σ' as

$$\mathcal{N}_i = \{I_{ij}, j \in \mathbf{Z}\} \quad \text{and} \quad \mathcal{N}'_i = \{I'_{ij}, j \in \mathbf{Z}\};$$

$i \in \mathbf{Z}$, beginning from the interval $[0,1] = I_{00} = I'_{00}$. Using this enumeration we can define an increasing homeomorphism of $\mathbf{R}$ as follows. If $x \in \mathbf{R}$ there is a sequence $\{I_{nk_n} \in \mathcal{N}_n\}$, $n \in \mathbf{Z}$, such that

$$\bigcap_{n \in \mathbf{Z}} I_{nk_n} = \{x\} .$$

This sequence of meshes of Σ is unique if x is not a vertex of Σ. In case x is a vertex of Σ then there are just two such sequences. Since Σ' is fine,

$$\bigcap_{n \in \mathbf{Z}} I'_{nk_n} = \{y\}$$

for some $y \in \mathbf{R}$. It is clear that y depends only on x and not on the sequence (I_{nk_n}), $n \in \mathbf{Z}$, if there is any choice. Thus there is a unique mapping $\mathbf{R} \to \mathbf{R}$, denoted $f_{\Sigma\Sigma'}$, such that

$$f_{\Sigma\Sigma'}(x) = y ,$$

x and y as above. The mapping $f_{\Sigma\Sigma'}$ is obviously a homeomorphism. If Σ, Σ' and Σ'' are fine, normalized sieves then we have

$$f_{\Sigma'\Sigma''} \circ f_{\Sigma\Sigma'} = f_{\Sigma\Sigma''} \quad \text{and} \quad (f_{\Sigma\Sigma'})^{-1} = f_{\Sigma'\Sigma} .$$

Conversely, given an increasing homeomorphism $f\colon \mathbf{R} \to \mathbf{R}$, we can construct from it a fine sieve, denoted Σ_f. We can start from the normalized symmetric sieve Σ_0 i.e. from the sieve whose vertices are the numbers $k/2^m$, $k, m \in \mathbf{Z}$. Then we define Σ_f to be the image of the symmetric sieve under f. Thus the ith net of Σ_f consists of the intervals $[f((k-1)/2^i), f(k/2^i)]$, $k \in \mathbf{Z}$. If f fixes 0 and 1 then the sieve Σ_f is normalized. It is easy to see that the map $f \mapsto \Sigma_f$ is a bijection from the set of normalized increasing homeomorphisms of $\mathbf{R}$ to the set of normalized, fine sieves and that the inverse of $f \mapsto \Sigma_f$ is $\Sigma \mapsto f_{\Sigma_0\Sigma}$. As a rule, when we form mappings from sieves, the initial sieve is the symmetric sieve Σ_0 and we shall write

$$f_\Sigma = f_{\Sigma_0\Sigma} .$$

Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is quasisymmetric. Then, by (1), the sieve Σ_f is also quasisymmetric. We shall prove that the converse is also true. We shall also show that this mapping is a homeomorphism. Before we do that we must define a topology in the set of normalized quasisymmetric mappings. (The set of normalized quasisymmetric sieves is topologized by (3)).

Let f be quasisymmetric. The *quasisymmetry* $q(f)$ of f is the smallest number ϱ for which (1) is valid. The *dilatation* $d(f)$ of f is the smallest number k for which $M(Q)/k \leq M(f(Q)) \leq kM(Q)$ for all quadrilaterals Q with interior U, $M(Q)$ is the modulus of Q. Let f and g be quasisymmetric and let

$$d(f,g) = \log d(f \circ g^{-1}) , \qquad \text{(12a)}$$

(12b) $$d_r(f,g) = \log q(f\circ g^{-1}),$$

(12c) $$d_l(f,g) = \log q(g\circ f^{-1}).$$

Now d is a metric in the set of normalized quasisymmetric mappings. This is true neither for d_r nor d_l but we can use also them to define a topology in this set (they are more convenient in practice), and we also call d_r and d_l metrics. Then bounded sets are the same in each metric, and they are equivalent if restricted to some bounded set. This follows from the fact that the Beurling–Ahlfors extension of a ϱ-quasisymmetric mapping is ϱ^2-quasiconformal ([5]) and that the function λ in [8, II.6.3] is continuously differentiable (cf. also [8, I.2.4] and [5, p. 131]).

The composition $(f,g)\mapsto f\circ g$ defines a group structure in the set of normalized quasisymmetric functions. Then the above topology is not a group topology, but the right translations $g\mapsto g\circ f$, g varies, f fixed, are continuous (in fact, they are isometries in each of the metrics d, d_r and d_l).

We denote by Q the set of normalized quasisymmetric functions and by S the set of normalized quasisymmetric sieves.

PROPOSITION 4. *The mapping $f\mapsto \Sigma_f$ is a homeomorphism $Q\to S$ with inverse $\Sigma\mapsto f_\Sigma$. Moreover, these mappings satisfy a Lipschitz condition in every bounded set of Q and S.*

PROOF. It is immediate that Σ_f is k-quasisymmetric if f is a k-quasisymmetric mapping. It is also easy to see that $f\mapsto\Sigma_f$ is an injection with left inverse $\Sigma\mapsto f_\Sigma$. Thus our Proposition is proved if we can prove the following three assertions. We shall make free use of the fact that instead of d in (12a) we can use d_r or d_l in (12b) and (12c).

(i) The mapping $f\mapsto\Sigma_f$ is a continuous mapping $Q\to S$ and satisfies a Lipschitz condition in every bounded set of Q.

(ii) If $\Sigma\in S$, then $f_\Sigma\in Q$ and the mapping $\Sigma\mapsto f_\Sigma$ maps bounded sets onto bounded sets.

(iii) The mapping $\Sigma\mapsto f_\Sigma$ is a continuous mapping $S\to Q$ and satisfies a Lipschitz condition in every bounded set of S.

PROOF OF (i). Let us fix some $\varrho>1$. We must show that there is a constant $c>0$ depending only on ϱ such that if f and g are normalized quasisymmetric functions, f and $g\circ f$ ϱ-quasisymmetric, g λ-quasisymmetric, then

$$d(\Sigma_{g\circ f},\Sigma_f)\leq c\log\lambda.$$

Now, by the definition of the metric d

$$d(\Sigma_{g\circ f}, \Sigma_f) = \sup_{i,j\in \mathbf{Z}} |\log k'_{ij} - \log k_{ij}| ,$$

where

$$k_{ij} = \frac{f((j+1)/2^i) - f(j/2^i)}{f(j/2^i) - f((j-1)/2^i)}$$

and

$$k'_{ij} = \frac{g(f((j+1)/2^i)) - g(f(j/2^i))}{g(f(j/2^i)) - g(f((j-1)/2^i))} .$$

Since g is λ-quasisymmetric, it has a quasiconformal extension G to the upper half plane U so that the dilatation of $G \leqq \lambda^2$ (cf. Beurling–Ahlfors [5]). If $M(a,b,c,\infty)$ denotes the modulus of the quadrilateral with vertices a, b, c and ∞ $(a,b,c \in \mathbf{R},\ a<b<c)$ with interior U, we have

$$(1/\lambda^2)M(a,b,c,\infty) \leqq M(g(a),g(b),g(c),\infty) \leqq \lambda^2 M(a,b,c,\infty) .$$

Now it is easy to see (cf. Lehto–Virtanen [8, pp. 15–16]) that $M(a,b,c,\infty)$ depends only on $t=(b-a)/(c-b)$ so that it is a continuously differentiable, increasing function of t, and tends to ∞ as t tends to ∞, and tends to 0 as t tends to 0.

On the basis of the above discussion it is easy to see that (i) is true.

PROOF OF (ii). Let $\Sigma = (\mathcal{N}_i)$, $i \in \mathbf{Z}$, be ϱ-quasisymmetric. We must show that for some $\sigma \geqq 1$ that depends only on ϱ

$$1/\sigma \leqq (f_\Sigma(x+t) - f_\Sigma(x))/(f_\Sigma(x) - f_\Sigma(x-t)) \leqq \sigma$$

for $x, t \in \mathbf{R}$, $t>0$. It is no restriction to assume that $0 \leqq x \leqq \frac{1}{2}$ and $1 \leqq t < 2$. For now we can consider the net $\mathcal{N}_1$ of Σ, otherwise we simply replace it by $\mathcal{N}_i$ for some $i \in \mathbf{Z}$. As usual, we denote the meshes of $\mathcal{N}_i$ by I_{ij}, $j \in \mathbf{Z}$. Let Σ_0 be the symmetric sieve and its nets $\mathcal{N}_i^0 = (I_{ij}^0)$, $j \in \mathbf{Z}$, for $i \in \mathbf{Z}$. Now $x \in I_{10}^0$ and $x+t \in I_{12}^0 \cup I_{13}^0 \cup I_{14}^0$. Let $a = l(I_{10}) = l(f(I_{10}^0))$. Then

$$\begin{aligned} a/\varrho \leqq l(I_{11}) \leqq f_\Sigma(x+t) - f_\Sigma(x) &\leqq l(I_{10} \cup I_{11} \cup I_{12} \cup I_{13} \cup I_{14}) \\ &\leqq a(1+\varrho+\varrho^2+\varrho^3+\varrho^4) . \end{aligned}$$

In the same manner one sees that

$$a/\varrho \leqq f_\Sigma(x) - f_\Sigma(x-t) \leqq a(1+\varrho+\varrho^2+\varrho^3+\varrho^4) .$$

Thus, if $\sigma = \varrho(1+\varrho+\varrho^2+\varrho^3+\varrho^4)$, then f_Σ is σ-quasisymmetric.

PROOF OF (iii). We must show that for some $c>0$ and $\varepsilon>0$

$$d_l(f_\Sigma, f_{\Sigma'}) \leqq cd(\Sigma, \Sigma')$$

for all $\Sigma, \Sigma' \in S$ with $d(\Sigma, \Sigma')<\varepsilon$ and $d(\Sigma, \Sigma_0)<K$ (Σ_0 is the symmetric sieve $\in S$) where the constants c and ε depend only on K. This and (ii) imply (iii).

Let us fix $K>0$ and $\Sigma \in S$, $\Sigma=(k_{ij})$, $i, j \in \mathbf{Z}$, such that

$$(13) \qquad |\log k_{ij}| \leqq K \text{ for all } i, j \in \mathbf{Z} .$$

We must estimate the quasisymmetry of $f_{\Sigma'} \circ f_\Sigma^{-1} = f_{\Sigma\Sigma'}$. For this we fix $x, t \in \mathbf{R}$, $t>0$, and estimate the quotient in (1) for $f=f_{\Sigma\Sigma'}$. Let us, as usual, denote the nets of Σ and Σ' by

$$\mathcal{N}_i = \{I_{ij}, j \in \mathbf{Z}\} \quad \text{and} \quad \mathcal{N}'_i = \{I'_{ij}, j \in \mathbf{Z}\} \text{ for } i \in \mathbf{Z} .$$

Then there is some $i \in \mathbf{Z}$ such that x and $x+t$ are in adjacent meshes I_{ij} and $I_{i,j+1}$ of $\mathcal{N}_i$, and that this is not true for $i'>i$. We can also assume that $x-t \in I_{ij}$, otherwise we reverse in the following the roles of $x-t$ and $x+t$. For simplicity we assume that $i=0$, $j=-1$. Thus

$$x+t \in I_{00}, \quad x \text{ and } x-t \in I_{0,-1} .$$

There is a sequence $n(0), n(1), \ldots$ of integers such that

$$\{x+t\} = \bigcap_{i \geqq 0} I_{in(i)}$$

where $n(0)=0$ and each $I_{in(i)}$ is a subinterval of $I_{i-1,n(i-1)}$. Now it is easy to see that

$$(14) \qquad x+t = \alpha_1 c_1 + \alpha_2 c_1 c_2 + \ldots$$

where each α_i, $i>1$, is either 0 or 1 (depending on $n(i)$) and

$$(15) \qquad \begin{aligned} c_i &= k_{in(i)}/(1+k_{in(i)}) \quad \text{if } \alpha_i=1 , \\ &= 1/(1+\mathrm{k}_{i,n(i)+1}) \quad \text{if } \alpha_i=0 . \end{aligned}$$

(Notice the analogy of expressing $x+t$ in binary notation as $x+t= a_1/2+a_2/2^2+\ldots$, $a_i=0$ or $=1$). By (13),

$$1/(e^K+1) \leqq c_i \leqq e^K/(e^K+1), \quad i \geqq 1 .$$

Let us consider then another $\Sigma' \in S$, $\Sigma'=(k'_{ij})$, $i, j \in \mathbf{Z}$. Then, by the definition of $f_{\Sigma\Sigma'}$, $\{f_{\Sigma\Sigma'}(x+t)\} = \bigcap_{i \geqq 0} I'_{in(i)}$. Thus

$$f_{\Sigma\Sigma'}(x+t) = \alpha_1 c'_1 + \alpha_2 c'_1 c'_2 + \ldots ,$$

with the same α_i's as in (14) and c'_i's are defined as in (15) if we furnish the c_i's and $k_{in(i)}$'s of (15) with dots.

We define real numbers u_i, $i \geqq 1$, as follows:

$$1+u_i = c_i'/c_i = \frac{k_{in(i)}'}{1+k_{in(i)}'}\,\frac{1+k_{in(i)}}{k_{in(i)}} \quad \text{if } \alpha_i = 1\,,$$
$$= \frac{1+k_{i,\,n(i)+1}}{1+k_{i,\,n(i)+1}'} \quad \text{if } \alpha_i = 0\,.$$

Let us denote $u = \sup_{i \geqq 1} |u_i|$. Then $u \leqq C_1 d(\Sigma, \Sigma')$ if $d(\Sigma, \Sigma') < \varepsilon_1$ where $C_1 > 0$ and $\varepsilon_1 > 0$ depend only on K. We have

$$\begin{aligned} (16) \qquad f_{\Sigma\Sigma'}(x+t) &= \alpha_1(1+u_1)c_1 + \alpha_2(1+u_1)(1+u_2)c_1c_2 + \ldots \\ &\leqq \alpha_1(1+u)c_1 + \alpha_2(1+u)^2 c_1 c_2 + \ldots \\ &= (\alpha_1 c_1 + \alpha_2 c_1 c_2 + \ldots) + A_1 u + A_2 u^2 + \ldots \\ &= x + t + A_1 u + A_2 u^2 + \ldots\,. \end{aligned}$$

It is clear that the power series $A_1 u + A_2 u^2 + \ldots$ converges in a neighbourhood of the origin which depends only on K. Thus, if $T = d(\Sigma, \Sigma')$,

$$f_{\Sigma\Sigma'}(x+t) \leqq x + t + C_2 T$$

for $T < \varepsilon_2$ where C_2 and ε_2 are positive numbers depending only on K. From (16) we also obtain

$$f_{\Sigma\Sigma'}(x+t) \geqq x + t - C_2 T$$

for $T < \varepsilon_2$ (clearly we can assume that C_2 is big enough and ε_2 small enough to be used once more). In the same manner we can calculate estimates for $f_{\Sigma\Sigma'}(x)$ and $f_{\Sigma\Sigma'}(x-t)$ and get

$$x - C_3 T \leqq f_{\Sigma\Sigma'}(x) \leqq x + C_3 T$$

and

$$x - t - C_4 T \leqq f_{\Sigma\Sigma'}(x-t) \leqq x - t + C_4 T$$

for $T < \varepsilon_3$ where ε_3, C_3 and C_4 depend only on K. Thus

$$\frac{t - (C_2 + C_3)T}{t + (C_3 + C_4)T} \leqq \frac{f_{\Sigma\Sigma'}(x+t) - f_{\Sigma\Sigma'}(x)}{f_{\Sigma\Sigma'}(x) - f_{\Sigma\Sigma'}(x-t)} \leqq \frac{t + (C_2 + C_3)T}{t - (C_3 + C_4)T}\,.$$

This is valid if $T < \varepsilon$ where ε depends only on K. Now our choice of notation (replacing the net $\mathcal{N}_i$ by $\mathcal{N}_0$) implies that $t \geqq c_K$ where $c_K > 0$ depends only on K. This concludes the proof of (iii).

5. The analytic structure.

We have shown (Propositions 3 and 4) that we can identify the set of normalized quasisymmetric functions with a convex set in a real Banach space. Thus it is naturally a real analytic Banach manifold. We have indicated in the Introduction that this is not the only way to endow it with the structure of a Banach manifold. It can be embedded in a complex Banach space of holomorphic functions (the space B in the Introduction) or, yielding the same structure, it can be identified with a quotient manifold of the open unit ball M in the Banach space $L^\infty(U, \mathbf{C})$ (cf. Introduction). The question arises whether these structures are in some sense equivalent. The answer is that they are equivalent as real analytic Banach manifolds. Since our Banach space E is only a real linear space this is the best result we can hope for.

We need not go into details but it is useful to resume some results in Earle and Eells [7]. One can define a mapping $\Phi\colon M \to B$ as follows. Let $\mu \in M$. Then

$$\Phi(\mu) = [w^\mu] .$$

Here w^μ is the unique quasiconformal mapping that fixes 0, 1 and ∞, whose complex dilatation is μ in U and that is holomorphic in L. Brackets $[w^\mu]$ denote the Schwarzian derivative of w^μ (or more precisely, its restriction to L). Then one can show that $\Phi(\mu) = \Phi(\nu)$, $\mu, \nu \in M$, if and only if $w_\mu \mid \mathbf{R} = w_\nu \mid \mathbf{R}$ where w_λ denotes the unique quasiconformal self-map of U with dilatation $\lambda \in M$ and that fixes 0, 1 and ∞. The image $\Phi(M)$ is the universal Teichmüller space T. The mapping Φ is continuous and holomorphic and at every point $\mu \in M$ the differential $d\Phi(\mu)\colon L^\infty(U, \mathbf{C}) \to B$ has a right inverse $B \to L^\infty(U, \mathbf{C})$. ([7, section 6]). From this it follows that the complex analytic structure of $T = \Phi(M)$ is inherited from that of M and that a mapping $f\colon T \to X$, X a manifold with a complex analytic (real analytic, differentiable) structure, is holomorphic (real analytic, differentiable) if and only if $f \circ \Phi\colon M \to X$ is holomorphic (real analytic, differentiable). For this, and other results we need from the theory of infinite-dimensional Banach manifolds and analytic mappings in Banach spaces we refer to Bourbaki [6].

The proof of our main result in this section is based on the fact that in

$$M_{\mathbf{C}} = \{\mu \in L^\infty(\mathbf{C}, \mathbf{C}) : \|\mu\|_\infty < 1\}$$

w^μ depends analytically on μ if w^μ is the quasiconformal mapping of $\mathbf{C}$ with complex dilatation μ fixing 0, 1 and ∞. We formulate this precisely in the following Lemma.

Let A be some topological space and B a subset of A. Then we denote by $F_B(A, \mathbf{C})$ the complex Banach space of continuous mappings $A \to \mathbf{C}$ with norm

$$\|f\|_B = \sup\{|f(x)| : x \in B\}.$$

LEMMA. (a) *The mapping* $\mu \mapsto w^\mu$ *is a holomorphic mapping* $M_C \to F_K(\mathbf{C},\mathbf{C})$ *for each compact* $K \subset \mathbf{C}$.

(b) *The mapping* $\mu \mapsto w_\mu$ *is a real analytic mapping* $M \to F_K(U,\mathbf{C})$ *for each* $K < U$ *compact.*

PROOF. This has been proved by Ahlfors and Bers [1]. We must only make apparent what is latent in [1, Theorem 11, p. 403]. First note that a mapping betweeen two open sets of complex Banach spaces is holomorphic if and only if it is locally bounded and has at every point partial derivatives in every direction. Now Theorem 11 of Ahlfors and Bers [1] implies that the mapping $\mu \mapsto w^\mu$ is a holomorphic mapping $M_k \mapsto B_{R,p}$ where

$$M_k = \{\mu \in L^\infty(\mathbf{C},\mathbf{C}) : \|\mu\|_\infty < k\}$$

and $B_{R,p}$ is the Banach space of continuous mappings $\mathbf{C} \to \mathbf{C}$ that vanish at the origin with norm [1, p. 397]

$$\|f\|_{B_{R,p}} = \sup_{|x|,|y| \leqq R} \frac{|f(x)-f(y)|}{|x-y|^{1-2/p}} + \left(\iint_{|z| \leqq R} |f_z|^p \, dx dy\right)^{1/p}$$

Here $k < 1$, $R > 0$ is arbitrary and $p > 2$ satisfies the conditions (8) of [1, p. 387]. Now let $K \subset \mathbf{C}$ be compact and $R \in \mathbf{R}$ be a constant such that $\sup_{x \in K} |x| \leqq R$. Then it is obvious that there is some $c > 0$ such that

$$\|f\|_K \leqq c\|f\|_{B_{R,p}}$$

for every f for which these norms are defined

What has been said above proves (a). We can embed M in $M_{\mathbf{C}}$ as follows: Let $\mu \in M$. Denote by $\mu^*: \mathbf{C} \to \mathbf{C}$ the element of $M_{\mathbf{C}}$ that coincides with μ in U and, if $z \in L$,

$$\mu^*(z) = \overline{\mu(\bar{z})}$$

the bar $^-$ denoting complex conjugation. Then $\mu \mapsto \mu^*$ embeds M as a convex subset in $M_{\mathbf{C}}$ and $w_\mu = w^{\mu^*} | U$. This proves (b).

We can define a map Ψ: $M \to S$, S the space of normalized quasisymmetric sieves, as follows: If $\mu \in M$, let

$$k_{ij}(\mu) = \frac{w_\mu((j+1)2^{-i}) - w_\mu(j2^{-1})}{w_\mu(j2^{-i}) - w_\mu((j-1)2^{-i})}.$$

Then $(k_{ij}(\mu))$, $i, j \in \mathbf{Z}$, defines a sieve $\Sigma = \Psi(\mu) \in S$. Since Σ depends only on $w_\mu \mid \mathbf{R}$ it can be factorized uniquely as $\Psi = \varphi \circ \Phi$ where φ is a map $T \to S$. As we have indicated above, φ is real analytic if and only if Ψ is real analytic. The real analytic structure of S was defined by embedding it by a transformation $(k_{ij}) \mapsto (h_{ij})$, $i, j \in \mathbf{Z}$, (cf. equation (6)) as a convex subset in a real Banach space E (equation (9)). An inspection of (6), (9) and (10) shows that φ is real analytic if

$$\mu \mapsto (k_{ij}(\mu)), \quad i, j \in \mathbf{Z} ,$$

is a real analytic map from M to the l^∞-space of all sequences (a_{ij}), $i, j \in \mathbf{Z}$, of real numbers. This is indeed so as is seen by the above Lemma (b) since then

$$((1 - w_\mu(x))/w_\mu(x)), \quad x \in K ,$$

is, for compact $K < (0, 1)$, a real analytic map from M to the l^∞-space of all sequences of real numbers (y_t), $t \in K$.

Thus the mapping φ: $T \to S$,

$$\varphi(\Phi(\mu)) = (k_{ij}(\mu)), \quad i, j \in \mathbf{Z} ,$$

is real analytic. It is easily seen to be a bijection (this follows from the fact that every quasisymmetric map has a quasiconformal extension to U). We wish to show that it is a real analytic isomorphism, i.e. also φ^{-1} is real analytic. For this it is sufficient to show that the derivative $d\varphi(x)$ has a continuous inverse at every point $x \in T$. And for this it is sufficient that for each $x \in T$ there is a neighbourhood U of x and a constant $c > 0$ such that

$$(17) \qquad c\|y - x\|_B \leqq \|\varphi(y) - \varphi(x)\|_E \quad \text{for } y \in U$$

where $\|\cdot\|_B$ and $\|\cdot\|_E$ denote the norms in the Banach spaces B and E (instead of the sequences (k_{ij}), $i, j \in \mathbf{Z}$, we consider here the sequences (h_{ij}), $(i, j) \in P$, defined by (6)). By Propositions 3 and 4 we can instead of (17) show that for each $x \in T$ there is a neighbourhood U of x and $c > 0$ such that

$$(18) \qquad c\|x - y\|_B \leqq d(f_{\varphi(y)}, f_{\varphi(x)}) \quad \text{for } y \in U$$

where d is the metric of (12a).

Equation (18) is valid if $f_{\varphi(x)} = \mathrm{id} \in Q$. This is seen from the fact that the Beurling–Ahlfors extension $\sigma(f)$ of a ϱ-quasisymmetric function f is ϱ^2-quasiconformal (cf. [5]) and from [5, section 4.4]. Thus for some $c' > 0$

$$c'\|\mu(\sigma(f))\|_\infty \leqq d(f, \mathrm{id})$$

in a neighbourhood of id $\in Q$. Since for a sieve $\Sigma \in S$, $\varphi^{-1}(\Sigma) = \Phi(\mu(\sigma(f_\Sigma)))$, and Φ is holomorphic, (18) is valid if $f_{\varphi(x)} = \mathrm{id}$. For arbitrary $x \in T$ the validity of (18) follows from the facts that the right translations are isometries of Q and that if in M a natural group structure is defined by setting $\mu \cdot \lambda = \nu$ if and only if $w_\mu \circ w_\lambda = w_\nu$, then the right translations of M are holomorphic self-equivalences of M (cf. Earle and Eells [7]).

This together with Corollary to Proposition 3 proves:

THEOREM. *The universal Teichmüller space T is real analytically equivalent to an open, convex set in a real Banach space and it is homeomorphic to this Banach space.*

REMARKS. The Banach space of the above Theorem is the space E defined in (9) and (10), and the convex set is the set E_0 of (11). Since there is a real analytic isomorphism $T \to E_0$, the tangent spaces, i.e. the spaces B and E, are isomorphic as topological (real) linear spaces. Hence the universal Teichmüller space is homeomorphic to the space B of holomorphic mappings of the lower half-plane of which it is a subset.

The above homeomorphism f of T onto a Banach space E is not diffeomorphism since the map r used in Corollary to Proposition 3 is not differentiable. However, there are some natural metrics for T and we can ask whether f fulfils some Lipschitz conditions with respect to these metrics and the norm metric of E. We consider on T, which we also identify with the set Q of normalized quasisymmetric mappings, the metric induced by the norm $\|\cdot\|_B$ if T is regarded as a subset of B (cf. Introduction), the metric d of (12a) and the Teichmüller metric $d_T(f, g) = \inf_{F,G} K(F \circ G^{-1})$ where F and G are quasiconformal extensions to U of the quasisymmetric mappings f and respectively g. $K(h)$ is the dilatation of a map h. Then $d \leqq d_T$ and they are equivalent if restricted to some d-bounded set (which is also d_T-bounded) (cf. the discussion in the paragraph containing formula (12a)). Now it follows from Propositions 3 and 4 and from the properties of the map used to prove Corollary to Proposition 3 that f and f^{-1} are Lipschitz if restricted to some bounded set of T or E and if the metric of T is d or d_T. Also it is seen that f and f^{-1} are locally Lipschitz if we consider T with the norm metric $\|\cdot\|_B$. We can also consider T with the metric d of (3) ($\neq d$ of (12a)), if we identify T with S. As above, f and f^{-1} are Lipschitz if restricted to some bounded set if T is provided with this metric. This is perhaps the most natural metric of T in the sense that it can be expressed directly in terms of quotients of form (1).

In recent years there has been a rapid advance in the topology of infinite dimensional manifolds. In particular, it has been shown that in many cases two infinite dimensional manifolds with the same model space are homeomorphic if and only if they are homotopically equivalent. This is true at least if the model space Y is a linear metric space homeomorphic to $Y^N = Y \times Y \times \ldots$, the countable product with the product topology. It is always so if Y is separable.

The Banach space B is not separable but whether it is homeomorphic to B^N, I do not know. But if this were the case it would follow from the contractibility of T, as proved by Earle and Eells [7], that T is homeomorphic to B.

For these and other results in infinite dimensional topology cf. Bessaga and Pełczyński [4].

REFERENCES

1. L. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. 72 (1960), 385–404.
2. L. Bers, *On moduli of Riemann surfaces*, Lecture notes, Eidgenössische Technische Hochschule, Zürich, 1964, (mimeographed).
3. L. Bers, *Universal Teichmüller space*, Published in *Analytic methods in mathematical physics*, ed. R. P. Gilbert and R. G. Newton, 65–83, Gordon and Breach, New York, 1968.
4. C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, Monografie Matematyczne 58, Polish Scientific Publishers, Warszawa, 1975.
5. A. Beurling and L. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125–142.
6. N. Bourbaki, *Variétés differentielles et analytiques*, Fascicule de résultats, paragraphs 1 à 7, (Act. Sci. Ind. 1333, Eléments de mathématique no. 33), Hermann, Paris, 1967.
7. C. Earle and J. Eells, *On the differential geometry of Teichmüller spaces*, J. Analyse Math. 19 (1967), 35–52.
8. O. Lehto and K. I. Virtanen, *Quasikonforme Abbildungen* (Grundlehren Math. Wissensch. 126), Springer-Verlag, Berlin · Heidelberg · New York, 1965.

UNIVERSITY OF HELSINKI
FINLAND