MATH. SCAND. 40 (1977), 151–160

A NONEXISTENCE TEST FOR BIHARMONIC GREEN'S FUNCTIONS OF CLAMPED BODIES

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The purpose of this paper is to introduce a convenient test for the nonexistence of the biharmonic Green's function β_M on a Riemannian manifold M, with "boundary data" $\beta_M = *d\beta_M = 0$. As an application we exhibit an M whose boundary is harmonically so strong that $M \notin O_{HD}$ but which nevertheless carries no β_M.

1. The class O_β.

Let $\mathbf{\Omega} = \{\Omega\}$ be the directed net of regular subregions of a noncompact Riemannian manifold M of dimension $\mu \geqq 2$. Denote by $G_\Omega(x,y)$ the *harmonic* Green's function on Ω and by $H(\Omega)$ the class of harmonic functions on Ω. The *biharmonic Green's function of the clamped body*, $\beta_\Omega(x,y)$ on Ω is well known to exist and is characterized by the following two conditions:

(β.1) *$\beta_\Omega(\cdot,y) \in C^2(\Omega - y)$ and $\Delta\beta_\Omega(\cdot,y) - G_\Omega(\cdot,y) \in H(\Omega)$ for every y in Ω;*
(β.2) *$\beta_\Omega(\cdot,y) \in C^1(\bar{\Omega} - y)$ and $\beta_\Omega(\cdot,y) = *d\beta_\Omega(\cdot,y) = 0$ on $\partial\Omega$.*

We know that $\beta_{\Omega'} - \beta_\Omega \in C(\Omega \times \Omega)$ for $\Omega \subset \Omega'$ (e.g. Nakai–Sario [4]) and thus we can define $\beta_{\Omega'}(y,y) - \beta_\Omega(y,y)$ as $\lim_{x\to y}(\beta_{\Omega'}(x,y) - \beta_\Omega(x,y))$. If there exists a function $\beta_M(x,y)$ on $M \times M$ with values in $(-\infty,\infty]$, finite on $M \times M$ off the diagonal and such that

$$\lim_{\Omega\to M} (\beta_M(x,y) - \beta_\Omega(x,y)) = 0 \tag{1}$$

on $M \times M$, then we call $\beta_M(x,y)$ the (generalized) *biharmonic Green's function of the clamped body* on M, with "boundary data" $\beta_M = *d\beta_M = 0$. Here we understand (1) for (y,y) as the existence of a finite $\lim_{\Omega'\to M}(\beta_{\Omega'}(y,y) - \beta_\Omega(y,y))$ for one and hence for every $\Omega \in \mathbf{\Omega}$. We denote by O_β the class of noncompact

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This work was sponsored by the U.S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles.

Received January 21, 1976.

Riemannian manifolds M on which there exists no β_M. We observe that $M \notin O_\beta$ is equivalent to

(β.3) *$\lim_{\Omega' \to M} (\beta_{\Omega'}(x,y) - \beta_\Omega(x,y))$ exists and is finite on $\Omega \times \Omega$ for one and hence for every $\Omega \in \boldsymbol{\Omega}$.*

2. The class O_{SH_2}.

We consider the class $H_2(M) = H(M) \cap L_2(M, dV)$ where dV is the volume element of M. The class $H_2(M)$ is known to be a locally bounded Hilbert space and the norm convergence implies the uniform convergence on each compact subset of M (e.g. Nakai–Sario [4]). It is easy to show that $H(\Omega) \cap C(\bar{\Omega})$ is dense in $H_2(\Omega)$ for $\Omega \in \boldsymbol{\Omega}$. We denote by O_{H_2} the class of Riemannian manifolds M with $H_2(M) = \{0\}$ and by O_{SH_2} the class of Riemannian manifolds M such that there exists a subregion $N \neq \emptyset$ of M with $M - \bar{N} \neq \emptyset$ and $N \in O_{H_2}$, that is, $H_2(N) = \{0\}$. We have the strict inclusion relation

$$O_{SH_2} < O_{H_2} . \tag{2}$$

The mere inclusion is trivial and the strictness is seen as follows, by means of the Euclidean space E^μ of dimension $\mu \geqq 2$. First we prove

$$E^\mu \in O_{H_2} .$$

Take any $h \in H_2(E^\mu)$. Let $(r, \theta) = (r, \theta^1, \ldots, \theta^{\mu-1})$ be the polar coordinates, and $d\theta$ the surface element on Θ: $|x| = 1$. Then

$$g(x) = g(|x|) = \int_\Theta h(|x|, \theta)^2 \, d\theta \geqq 0$$

is subharmonic on E^μ and, by the maximum principle, $g(r)$ is an increasing function on $[0, \infty)$. If $g(r) \not\equiv 0$, then there exist constants $c > 0$ and $\sigma > 0$ such that $g(r) \geqq c$ on $[\sigma, \infty)$. Thus

$$\begin{aligned} \infty = c \int_\sigma^\infty r^{\mu-1} \, dr &\leqq \int_\sigma^\infty g(r) r^{\mu-1} \, dr \\ &= \int_\sigma^\infty \int_\Theta h(r, \theta)^2 r^{\mu-1} \, dr \, d\theta \\ &= \int_{|x| > \sigma} h(x)^2 \, dx^1 \ldots dx^\mu < \infty , \end{aligned}$$

a contradiction. Therefore, $g \equiv 0$ and a fortiori $h \equiv 0$ on every $|x| = \varrho > 0$, that is, $h \equiv 0$, and we conclude that $H_2(E^\mu) = \{0\}$.

Next we show that

$$E^\mu \notin O_{SH_2} .$$

Suppose there exists a subregion $N \neq \emptyset$ of E^μ with $E^\mu - \bar{N} \neq \emptyset$ and $H_2(N) = \{0\}$. Let $x_0 \in E^\mu - \bar{N}$ and $\{|x - x_0| < \varrho\} \subset E^\mu - \bar{N}$. By a parallel translation, if necessary, we obtain $N \subset N_\varrho = \{|x| > \varrho\}$. Since $H_2(N_\varrho) \subset H_2(N) = \{0\}$, we have $H_2(N_\varrho) = \{0\}$, but this is impossible because

$$h(x) = r^{-(n+\mu-2)} S_n(\theta) \in H_2(N_\varrho) \qquad ((r,\theta)=x,\ n>2),$$

with $S_n(\theta)$ any nonzero spherical harmonic of degree n.

The main purpose of the present paper is to prove:

THEOREM 1. *The following inclusion relation is valid*:

(3) $$O_{SH_2} \subset O_\beta .$$

This will give a convenient *test* for $M \in O_\beta$. We only have to find a subregion $N \neq \emptyset$ of M with $M - \bar{N} \neq \emptyset$ and $H_2(N) = \{0\}$ to conclude that $M \in O_\beta$. Note that this is *not* a characterization of O_β, i.e. (3) *is not an equality in general.* In fact,

$$\beta_{\Omega_\varrho}(x,0) = \begin{cases} |x|^2 \log \dfrac{|x|}{\varrho} - \frac{1}{2}(|x|^2 - \varrho^2), & (\mu=2); \\ -|x| + \varrho + \frac{1}{2}\varrho^{-1}(|x|^2 - \varrho^2), & (\mu=3); \\ -\log \dfrac{|x|}{\varrho} + \frac{1}{2}\varrho^{-1}(|x|^2 - \varrho^2), & (\mu=4); \\ |x|^{-\mu+4} - \varrho^{-\mu+4} + \frac{1}{2}(\mu-4)\varrho^{-\mu+2}(|x|^2 - \varrho^2), & (\mu \geqq 5), \end{cases}$$

on $\Omega_\varrho = \{|x| < \varrho\}$ $(0 < \varrho < \infty)$, hence

$$E^\mu \in O_\beta \qquad (\mu = 2, 3, 4) .$$

By this and $E^\mu \notin O_{SH_2}$ we see that the equality does not hold in (3) for the dimensions $\mu = 2, 3$, and 4.

The proof of Theorem 1 will be given in section 7 after we have established, in sections 3–6, three *complete characterizations* of O_β, instead of merely an inclusion as in (3). The significance of (3) lies in its applicability to concrete cases to show the nonexistence of β.

3. The β-density $H_\Omega(\cdot, y)$.

As a consequence of (β.1) and (β.2), $\beta_\Omega(\cdot, y)$ is a Green potential with the density $H_\Omega(\cdot, y) = \Delta\beta_\Omega(\cdot, y)$, which we call the *$\beta$-density* on Ω:

$$\beta_\Omega(\cdot, y) = \int_\Omega G_\Omega(\cdot, \xi) H_\Omega(\xi, y)\, dV_\xi . \tag{4}$$

Since $H_\Omega(\cdot, y) \in C(\bar{\Omega} - y)$, a property of the Green kernel (e.g. Miranda [1]) gives

$$*d\beta_\Omega(\cdot, y) = \int_\Omega *dG_\Omega(\cdot, \xi) H_\Omega(\xi, y)\, dV_\xi$$

on $\partial\Omega$. Again by (β.2),

$$\int_\Omega *d_x G_\Omega(x, \xi) H_\Omega(\xi, y)\, dV_\xi = 0$$

for every $x \in \partial\Omega$. On multiplying both sides by an arbitrary $h \in H(\Omega) \cap C(\bar{\Omega})$ and integrating over $\partial\Omega$, we obtain by Fubini's theorem

$$\int_\Omega \left(\int_{\partial\Omega} h(x) *d_x G_\Omega(x, \xi) \right) H_\Omega(\xi, y)\, dV_\xi = 0 .$$

By the reproducing property of G_Ω, we conclude that

$$\int_\Omega h(\xi) H_\Omega(\xi, y)\, dV_\xi = 0 \tag{5}$$

for every $h \in H(\Omega) \cap C(\bar{\Omega})$, and since $H(\Omega) \cap C(\bar{\Omega})$ is dense in $H_2(\Omega)$, for every $h \in H_2(\Omega)$. We have obtained for the β-density the following *orthogonality property* which plays an important role in the study of O_β:

$$H_\Omega(\cdot, y) \perp H_2(\Omega) . \tag{6}$$

4. The β-span S_β.

We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and the norm in $L_2(M, dV)$. We consider $\beta_\Omega(x, y)$ and $H_\Omega(x, y)$ as defined on all of $M \times M$ by giving values zero outside of their original domains of definition. First observe that, by (6) and $H_\Omega(\cdot, y) - G_\Omega(\cdot, y) \in H_2(\Omega)$,

$$\beta_\Omega(x, y) = \int_\Omega H_\Omega(\xi, x) H_\Omega(\xi, y)\, dV_\xi . \tag{7}$$

Similarly by (6) and $H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y) \in H_2(\Omega)$, we have for $\Omega \subset \Omega'$,

$$\beta_{\Omega'}(x, y) - \beta_\Omega(x, y) = \int_{\Omega'} (H_{\Omega'}(\xi, x) - H_\Omega(\xi, x))(H_{\Omega'}(\xi, y) - H_\Omega(\xi, y))\, dV_\xi . \tag{8}$$

In particular,

$$\beta_{\Omega'}(y,y)-\beta_{\Omega}(y,y) = \|H_{\Omega'}(\cdot,y)-H_{\Omega}(\cdot,y)\|^2 . \tag{9}$$

Again by (6),

$$\begin{aligned} &\|H_{\Omega''}(\cdot,y)-H_{\Omega'}(\cdot,y)\|^2 \\ &\qquad = \|H_{\Omega''}(\cdot,y)-H_{\Omega}(\cdot,y)\|^2-\|H_{\Omega'}(\cdot,y)-H_{\Omega}(\cdot,y)\|^2 \end{aligned} \tag{10}$$

for $\Omega\subset\Omega'\subset\Omega''$. From (9) and (10) it follows that $\{\beta_{\Omega'}(y,y)-\beta_{\Omega}(y,y)\}$ $(\Omega'\supset\Omega, \Omega'\in\boldsymbol{\Omega})$ is an increasing net. Therefore, we can define for $y\in M$ and $\Omega\in\boldsymbol{\Omega}$ with $y\in\Omega$,

$$\begin{aligned} S_\beta(y) &= S_\beta(y;M) = S_\beta(y;\Omega,M) \\ &= \lim_{\Omega'\to M}(\beta_{\Omega'}(y,y)-\beta_{\Omega}(y,y)) \\ &= \lim_{\Omega'\to M}\|H_{\Omega'}(\cdot,y)-H_{\Omega}(\cdot,y)\|^2 \in (0,\infty] , \end{aligned} \tag{11}$$

which we will call the *β-span* of M at $y\in M$ with respect to Ω. The property $S_\beta(y)<\infty$ is clearly independent of the choice of Ω and is thus a property of (M,y). We maintain:

THEOREM 2. *The manifold M does not belong to O_β if and only if the β-span $S_\beta(y)$ of M is finite at every point $y\in M$.*

If $M\notin O_\beta$, or (β.3) is valid, then we trivially have $S_\beta(y)<\infty$ for every $y\in M$. Conversely, assume that $S_\beta(y)<\infty$ for every $y\in M$. Then, by (8) and (9), the Schwarz inequality implies that

$$|\beta_{\Omega''}(x,y)-\beta_{\Omega'}(x,y)| \leqq \|H_{\Omega''}(\cdot,x)-H_{\Omega'}(\cdot,x)\|\cdot\|H_{\Omega''}(\cdot,y)-H_{\Omega'}(\cdot,y)\| \tag{12}$$

on $\Omega'\times\Omega'$ for $\Omega'\subset\Omega''$. By (10), (11), $S_\beta(x)<\infty$, and $S_\beta(y)<\infty$, we see that the right-hand side of (12) converges to zero on $\Omega\times\Omega$ as $\Omega'\to M$ for any $\Omega\subset\Omega'\subset\Omega''$, and, since Ω is arbitrary,

$$\lim_{\Omega'\subset\Omega'',\Omega'\to M}(\beta_{\Omega''}(x,y)-\beta_{\Omega'}(x,y)) = 0$$

on $M\times M$,

$$\lim_{\Omega''\to M}|\beta_{\Omega''}(x,y)-\beta_{\Omega}(x,y)| \leqq S_\beta(x)^{\frac{1}{2}}S_\beta(y)^{\frac{1}{2}} ,$$

and

$$S_\beta(y) = \lim_{\Omega'' \to M} (\beta_{\Omega''}(y,y) - \beta_\Omega(y,y)) < \infty .$$

Thus (β.3) is fulfilled for every $\Omega \in \boldsymbol{\Omega}$ and, therefore, $M \notin O_\beta$.

5. The β-density $H_M(\cdot, y)$.

Assume $M \notin O_\beta$. By Theorem 2 and relations (10) and (11), we conclude that $\{H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)\}$ ($\Omega' \supset \Omega$, $\Omega' \in \boldsymbol{\Omega}$) is a Cauchy net in $L_2(M, dV)$ and has a limit $H_{M\Omega}(\cdot, y) \in L_2(M, dV)$. Set

$$H_M(\cdot, y) = H_{M\Omega}(\cdot, y) + H_\Omega(\cdot, y) .$$

Then since

$$H_M(\cdot, y) - H_{\Omega'}(\cdot, y) = H_{M\Omega}(\cdot, y) - (H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)) ,$$

the net $\{H_M(\cdot, y) - H_{\Omega'}(\cdot, y)\}$ is convergent to zero in $L_2(M, dV)$. Fix an arbitrary $\Omega_0 \in \boldsymbol{\Omega}$ with $y \notin \Omega_0$. Then $\{H_{\Omega'}(\cdot, y)\}$ is a Cauchy net in $H_2(\Omega_0)$, $H_M(\cdot, y)$ is its limit, and a fortiori $H_M(\cdot, y) \in H_2(\Omega_0)$. Therefore,

$$H_M(\cdot, y) \in H(M - y) . \tag{13}$$

Fix an arbitrary $\Omega \in \boldsymbol{\Omega}$ with $y \in \Omega$. Observe that $\{H_{\Omega'}(\cdot, y) - G_\Omega(\cdot, y)\}$ ($\Omega' \supset \Omega$, $\Omega' \in \boldsymbol{\Omega}$) is also a Cauchy net in $H_2(\Omega)$, convergent to $H_M(\cdot, y) - G_\Omega(\cdot, y)$, which is again in $\mathrm{H}_2(\Omega)$. Thus we have

$$H_M(\cdot, y) - G_\Omega(\cdot, y) \in H(\Omega) \tag{14}$$

for one and hence for every $\Omega \in \boldsymbol{\Omega}$ with $y \in \Omega$. It is also clear that

$$H_M(\cdot, y) \in H_2(M - \Omega) \tag{15}$$

for any $\Omega \in \boldsymbol{\Omega}$ with $y \in \Omega$. Besides properties (13)–(15) of $H_M(\cdot, y)$, the following *orthogonality relation* is of fundamental importance:

$$H_M(\cdot, y) \perp H_2(M) , \tag{16}$$

or, equivalently,

$$\int_M h(\xi) H_M(\xi, y) \, dV_\xi = 0 \tag{17}$$

for every $h \in H_2(M)$. Here the integral on the right is well defined because of (14) and (15). For the proof, observe that the inequality

$$|(h, H_M(\cdot, y) - H_\Omega(\cdot, y))| \leqq \|h\| \cdot \|H_M(\cdot, y) - H_\Omega(\cdot, y)\|$$

implies

$$\int_M h(\xi)H_M(\xi, y)\,dV_\xi = \lim_{\Omega\to M}\int_\Omega h(\xi)H_\Omega(\xi, y)\,dV_\xi\,.$$

Since $h \in H_2(M) \subset H_2(\Omega)$, (5) yields (17).

We shall call a function $H_M(\cdot, y)$ on $M-y$ with properties (13)–(16) the *β-density on M* for $y \in M$. It is *unique*. In fact, if $K(\cdot, y)$ satisfies (13)–(16), then $h = H_M(\cdot, y) - K(\cdot, y) \in H_2(M)$, and by (16) for $H_M(\cdot, y)$ and $K(\cdot, y)$ we obtain $(h, h) = \|h\|^2 = 0$, that is, $h \equiv 0$ on M.

We claim:

THEOREM 3. *The manifold M does not belong to O_β if and only if the β-density $H_M(\cdot, y)$ exists on M for every $y \in M$.*

We only have to show that the existence of the β-density $H_M(\cdot, y)$ on M for every $y \in M$ implies $M \notin O_\beta$. Let $\Omega \subset \Omega'$. Since $H_M(\cdot, y) - H_{\Omega'}(\cdot, y) \in H_2(\Omega') \subset H_2(\Omega)$, we have

$$\begin{aligned}((H_M(\cdot, y) - H_\Omega(\cdot, y)) - (H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)),\, H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y))\\ = (H_M(\cdot, y) - H_{\Omega'}(\cdot, y),\, H_{\Omega'}(\cdot, y))\\ - (H_M(\cdot, y) - H_{\Omega'}(\cdot, y), H_\Omega(\cdot, y)) = 0\end{aligned}$$

and a fortiori

$$(H_M(\cdot, y) - H_\Omega(\cdot, y), H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)) = \|H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)\|^2\,.$$

By the Schwarz inequality,

$$\|H_{\Omega'}(\cdot, y) - H_\Omega(\cdot, y)\| \leqq \|H_M(\cdot, y) - H_\Omega(\cdot, y)\|\,.$$

In view of (11) it follows that $S_\beta(y) < \infty$ for every $y \in M$, that is, $M \notin O_\beta$.

COROLLARY. *The β-span $S_\beta(y)$ is finite if and only if the β-density $H_M(\cdot, y)$ exists on M at y, and in this case,*

$$S_\beta(y; \Omega, M) = \|H_M(\cdot, y) - H_\Omega(\cdot, y)\|^2\,. \tag{18}$$

6. An extremal property of $H_M(\cdot, y)$.

Assume the existence of β_M. Then by (1), (8), and $\lim_{\Omega\to M}\|H_M(\cdot, y) - H_\Omega(\cdot, y)\| = 0$, we have

$$\beta_M(x, y) - \beta_\Omega(x, y) = (H_M(\cdot, x) - H_\Omega(\cdot, x), H_M(\cdot, y) - H_\Omega(\cdot, y))$$

on $\Omega \times \Omega$. By (6) and (7),

(19) $$\beta_M(x,y) = \int_M H_M(\xi,x)H_M(\xi,y)\,dV_\xi$$

on $\Omega\times\Omega$ for every $\Omega\in\boldsymbol{\Omega}$, and a fortiori on $M\times M$. Instead of (1) we can take (19) as the definition of β_M (cf. Nakai–Sario [4]) starting from β-densities $H_M(\cdot,y)$ for all $y\in M$.

In this connection, we consider the family $F(M,y)$ of functions $K(\cdot,y)$ on $M-y$ satisfying (13)–(15), with K replacing H_M. If $H_M(\cdot,y)$ exists, then it is in the class $F(M,y)$ and thus

(20) $$F(M,y) \neq \varnothing\,.$$

Since $H_M(\cdot,y)-K(\cdot,y)\in H_2(M)$, we have

$$(H_M(\cdot,y)-K(\cdot,y), H_M(\cdot,y)-H_\Omega(\cdot,y)) = 0$$

for every $\Omega\in\boldsymbol{\Omega}$. By the Schwarz inequality applied to

$$\|H_M(\cdot,y)-H_\Omega(\cdot,y)\|^2 = (K(\cdot,y)-H_\Omega(\cdot,y), H_M(\cdot,y)-H_\Omega(\cdot,y))$$

we obtain the following *extremal property* of $H_M(\cdot,y)$:

(21) $$\|H_M(\cdot,y)-H_\Omega(\cdot,y)\| = \lim_{K\in F(M,y)} \|K(\cdot,y)-H_\Omega(\cdot,y)\|$$

for any $\Omega\in\boldsymbol{\Omega}$ with $y\in\Omega$.

This property actually characterizes $H_M(\cdot,y)$ in the class $F(M,y)$ if (20) is valid. In fact, suppose $F(M,y)\neq 0$. Fix an arbitrary $\Omega\in\boldsymbol{\Omega}$. Then the family

$$X_\Omega = \{K(\cdot,y)-H_\Omega(\cdot,y)\,;\; K(\cdot,y)\in F(M,y)\}$$

is clearly a *nonempty convex set* in $L_2(M,dV)$. It is also *closed*. To see this, let $\{K_n(\cdot,y)-H_\Omega(\cdot,y)\}$ $(n=1,2,\ldots)$ be a sequence in X_Ω converging to a $\bar{K}\in L^2(M,dV)$. Set $K=\bar{K}+H_\Omega(\cdot,y)$. Then $\{K_n-K\}$ is a Cauchy sequence in $L^2(M,dV)$, and therefore, $\{K_n\}$ is Cauchy in $H_2(\Omega)$ for every $\Omega\in\boldsymbol{\Omega}$ with $y\notin\Omega$, and $\{K_n-G_\Omega(\cdot,y)\}$ is Cauchy in $H_2(\Omega)$ for every $\Omega\in\boldsymbol{\Omega}$ with $y\in\Omega$. Thus K enjoys properties (13)–(15), i.e.,

$$K-H_\Omega(\cdot,y) = \lim_{n\to\infty}(K_n(\cdot,y)-H_\Omega(\cdot,y))\in X_\Omega\,,$$

and X_Ω is closed.

Since any nonempty closed convex subset of a Hilbert space contains a unique element of minimum norm, there exists a unique element $K_0-H_\Omega(\cdot,y)\in X_\Omega$ such that

$$\|K_0-H_\Omega(\cdot,y)\| = \min_{K\in F(M,y)}\|K(\cdot,y)-H_\Omega(\cdot,y)\|\,.$$

Let h be any element in $H_2(\Omega)$, and $t>0$. In view of $K_0+th\in F(M,y)$, we have

$$\|K_0-H_\Omega(\cdot,y)+th\|^2 \geqq \|K_0-H_\Omega(\cdot,y)\|^2$$

or

$$2t(K_0 - H_\Omega(\cdot, y), h) + t^2 \|h\|^2 \geqq 0 .$$

Since this is true for every $t > 0$,

$$(K_0 - H_\Omega(\cdot, y), h) = 0$$

for every $h \in H_2(M)$. From this and (5) we deduce the validity of (16) or (17) with K_0 replacing $H_M(\cdot, y)$. Thus K_0 satisfies (13)–(16), i.e., K_0 is the β-density $H_M(\cdot, y)$ on M for y. We have shown:

The β-density $H_M(\cdot, y)$ on M for $y \in M$ exists if and only if $F(y, M) \neq \varnothing$.

We restate this in the following form:

THEOREM 4. *The manifold M does not belong to O_β if and only if there exists a harmonic function $K(\cdot, y)$ on $M - y$ which has the harmonic fundamental singularity at y and is square integrable on M off any neighborhood of y.*

7. Proof of Theorem 1.

Inclusion (3) can now be established using Theorem 3 or 4. For example, let $M \in O_{SH_2}$. Then there exists a subregion $N \neq \varnothing$ of M with $M - \bar{N} \neq \varnothing$ and $H_2(N) = \{0\}$. If $M \notin O_\beta$, choose a point $y \in M - \bar{N}$. By Theorem 3, the β-density $H_M(\cdot, y)$ exists on M for y, and by taking $\Omega \in \boldsymbol{\Omega}$ with $y \in \Omega$ and $\bar{\Omega} \subset M - \bar{N}$, we infer by (15) that $H_M(\cdot, y) | N \in H_2(N) = \{0\}$. Thus $H_M(\cdot, y) \equiv 0$ on N. By the unique continuation property of harmonic functions, $H_\Omega(\cdot, y) \equiv 0$ on $M - y$, which contradicts (14). Therefore, $M \in O_\beta$, and we have proved Theorem 1.

8. An application.

As an illustration of the use of our test (3), we exhibit a manifold M which shows that

(22) $$O_\beta - O_{HD} \neq \varnothing .$$

Let E_λ^2 be the plane with the metric

$$ds = \sqrt{\lambda(x)}\, |dx|, \qquad \lambda(x) = \exp(|x|^2) .$$

Choose $M = \{|x| > 1\}$ in E_λ^2. Since $HD(M, ds) = HD(M, |dx|)$, we clearly have $(M, ds) \notin O_{HD}$. Let $N_\varrho = \{|x| > \varrho\}$ $(1 < \varrho < \infty)$. We assert that $H_2(N_\varrho, ds) = \{0\}$. It suffices to show that if h is harmonic on $|x| > \varrho$ and

$$A = \int_{N_\varrho} h(r,\theta)^2 \lambda(r) r \, dr \, d\theta < \infty ,$$

then $h \equiv 0$. The expansion

$$h(r,\theta) = a \log r + \sum_{n=-\infty}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

on N_ϱ gives

$$L(r) = \int_0^{2\pi} h(r,\theta)^2 \, d\theta$$

$$= 2\pi (a_0 + a \log r)^2 + \pi \sum_{n=-\infty, n\neq 0}^{\infty} (a_n^2 + b_n^2) r^{2n} ,$$

and we have

$$A = \int_\varrho^\infty L(r) \lambda(r) r \, dr$$

$$= 2\pi \int_\varrho^\infty (a_0 + a \log r)^2 e^{r^2} r \, dr + \pi \sum_{n=-\infty, n\neq 0}^{\infty} (a_n^2 + b_n^2) \int_\varrho^\infty r^{2n+1} e^{r^2} \, dr .$$

From $A < \infty$ we infer that $a = a_n = b_n = 0$ for $n = 0, \pm 1, \ldots$, that is, $h \equiv 0$ on N_ϱ. Therefore,

$$(M, ds) \in O_{SH_2} ,$$

and (3) yields $(M, ds) \in O_\beta$.

BIBLIOGRAPHY

1. C. Miranda, *Partial Differential Equations of Elliptic Type*, 2nd ed. (Ergebnisse Math. 2) Springer-Verlag, New York · Heidelberg · Berlin, 1970.
2. M. Nakai and L. Sario, *Parabolic Riemannian planes carrying biharmonic Green's functions of the clamped plate*, J. Analyse Math. 30 (1976), 372–389.
3. M. Nakai and L. Sario, *A strict inclusion related to biharmonic Green's functions of clamped and simply supported bodies*, Ann. Acad. Sci. Fenn. (to appear).
4. M. Nakai and L. Sario, *Existence of biharmonic Green's functions*, Proc. London Math. Soc. (to appear).
5. J. Ralston and L. Sario, *A relation between biharmonic Green's functions of simply supported and clamped bodies*, Nagoya Math. J. 61 (1976), 59–71.

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