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ON EDGE-COLOURINGS OF GRAPHS

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Abstract.

Properties of the colour-distribution in edge-colourings of graphs are examined. The results are applied in generalizing two theorems of Vizing and in proving the following for $m=5$ and 7: If G is critical graph with

$$q(G) > \frac{m}{m-1}d(G)+\frac{m-3}{m-1},$$

then G has at most $m-2$ vertices (this has been proved earlier by M. K. Gol'dberg and by Bo Aagaard Sørensen; only Gol'dberg's proof for $m=5$ has been published). Finally $S(d,s)$, the maximal chromatic index of a graph not containing G_s as a subgraph, is determined for all $d \geqq 0$ and all $s \geqq 2$. This answers some questions of Berge and Bosák.

1. Terminology.

We consider finite and infinite graphs without loops. The set of vertices of a graph G will be denoted by $V(G)$, the set of edges by $E(G)$. If X and Y are vertices, $E(X,Y)$ will denote the set of edges joining X and Y; if $E(X,Y)$ contains more than one edge, we call it a *multiple edge*. A graph having no multiple edges is *simple*. The *multiplicity* of a graph G is the supremum $p(G)$ of the cardinalities of all sets $E(X,Y)$.

The *valency* $v_G(X)$ of a vertex X of a graph G is the cardinality of the set of all edges incident with X. The supremum of the valencies of all vertices of G is denoted by $d(G)$. The set of neighbours of X is denoted by $N_G(X)$.

If G is a graph and e an edge of G then $G-e$ is the graph obtained from G by deleting e.

Let G be a graph. If k is a cardinal, by a *k-edge-colouring* or simply *k-colouring* of G we mean a mapping of $E(G)$ into a set of cardinality k such that any two edges incident with the same vertex are assigned different elements, called *colours*. We also say that we *colour* the edges. The *chromatic index* $q(G)$ is the least cardinal k such that G has a k-colouring.

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An edge e of G is said to be *critical* if $q(G-e)<q(G)$. It is easily seen that if G has a critical edge, then $q(G)$ is finite. If e is any critical edge of G we shall call any $(q(G)-1)$-colouring of $G-e$ an *edge-deleted colouring* of G; we shall then refer to e as the *uncoloured edge*. For any such colouring and any vertex X of G we let C_X denote the set of colours colouring edges incident with X. $\bar{C}_X$ denotes the set of the remaining of the $q(G)-1$ colours. We call the colours of C_X *present* at X, those of $\bar{C}_X$ *absent* at X. If E' is any set of edges of G, $C(E')$ is the set of colours colouring the edges of E'.

The graph G is *critical* if it has no isolated vertices, has $q(G)>d(G)$, and every edge is critical. Every critical graph is finite.

Suppose we have an edge-deleted colouring of a graph G, and let a and b be two distinct colours. Then an *(a,b)-chain* is a connected component of the subgraph of G consisting of the edges of colour a or b and all the vertices of G. A chain is a path or an even circuit, hence it can have at most two vertices of valency 1. We shall often perform the process of *interchanging colours* in a chain; obviously this still gives an edge-deleted colouring of G.

If X and Y are the end-vertices of the uncoloured edge and $a \in \bar{C}_X$, $b \in \bar{C}_Y$, then X and Y belong to the same (a,b)-chain. We shall call such a chain (which is obviously a path with end-vertices X and Y) a *critical chain.*

A *fan at X* is a sequence $e_0, e_1, \ldots, e_n$, $(n \geqq 0)$ of distinct edges incident with X with end-vertices $Y=Z_0, Z_1, \ldots, Z_n$ such that e_0 is the uncoloured edge and such that the colour of e_i is absent at Z_{i-1} for $1 \leqq i \leqq n$. Given a fan, we can obtain an edge-deleted colouring of G by uncolouring e_n and giving e_{i-1} the colour previously colouring e_i for $1 \leqq i \leqq n$. We shall refer to this process as *recolouring the fan*; if $n \geqq 1$ it gives an edge-deleted colouring different from the original one.

We define three important sets of vertices corresponding to a fixed edge-deleted colouring where X and Y are the end-vertices of the uncoloured edge: A_X is the set of end-vertices different from X of all edges which are in at least one fan at X. B_X is $N_G(X) \backslash A_X$. A_Y and B_Y are defined analogously. D is the set of all vertices which are in at least one critical chain.

If S is a set then $|S|$ is the cardinality of S. If α is an infinite cardinal we put $\frac{1}{2}\alpha=[\alpha]=\alpha$. For all reals x, $[x]$ is the largest integer not greater than x. Any supremum taken over the empty set is supposed to be zero.

When it is clear to which graph we refer we shall often use d, p and q for $d(G)$, $p(G)$ and $q(G)$.

2. Extensions of two results of Vizing.

In [10] Vizing proved the now classical theorem that for all graphs $q \leqq d+p$. Later he proved ([11]) that in a simple critical graph a vertex adjacent to a vertex of valency v is also adjacent to at least $d-v+1$ other vertices of valency d. Both these theorems will be generalized below.

Results similar to those of Lemma 1 and Theorem 1 have been found independently by Gol'dberg and are essentially the contents of [5].

We first prove a lemma stating that in an edge-deleted colouring certain sets of colours are mutually disjoint. Results of this kind are often in the literature stated for critical graphs only whereas we do not require more than one critical edge present.

LEMMA 1. *Suppose that G is a graph with an edge-deleted colouring, and that X is an end-vertex of the uncoloured edge such that $v_G(X) \leqq q(G)-1$. Then if $A_X = \{Y, V_1, \ldots, V_r\}$ and $B_X = \{Z_1, \ldots, Z_t\}$, $r, t \geqq 0$, the following sets of colours are mutually disjoint:*

$$\bar{C}_X, \bar{C}_Y, \bar{C}_{V_1}, \ldots, \bar{C}_{V_r}, C(E(Z_1, X)), \ldots, C(E(Z_t, X)) .$$

PROOF. We prove five assertions which together imply the lemma.

(i) For every vertex V in A_X, $\bar{C}_V \cap \bar{C}_X = \varnothing$.

PROOF OF (i). Suppose (i) is false and let $Y, V'_1, \ldots, V'_n$, $n \geqq 1$, be the end-vertices of the edges of a fan at X such that the colour a is in $\bar{C}_{V'_n} \cap \bar{C}_X$. Now a $(q-1)$-colouring of G is obtained by recolouring the fan and giving the new uncoloured edge the colour a. This contradiction proves (i).

(ii) For any two distinct vertices V' and V'' in A_X, $\bar{C}_{V'} \cap \bar{C}_{V''} = \varnothing$.

PROOF OF (ii). As $v_G(X) \leqq q-1$, there is a colour b absent at X. If $V \in A_X$ and $a \in \bar{C}_V$ we can prove that V and X are in the same (a, b)-chain. Because suppose this is not true, and suppose that among the vertices V for which it fails we consider one such that the fan with end-vertices $Y = Z_0, Z_1, \ldots, Z_n = V$ has as few edges as possible. If we interchange colours in the (a, b)-chain containing X the same edges do still constitute a fan, for if a does not colour any edge of the fan we have not affected the fan at all, and if a colours an edge of the fan, say

$$a \in \bar{C}_{Z_i} \cap C(E(X, Z_{i+1})), \quad i \leqq n-2 ,$$

then Z_i and X are in the same (a, b)-chain by the choice of V; hence after the interchange we have $b \in \bar{C}_{Z_i} \cap C(E(X, Z_{i+1}))$. So in the colouring obtained by interchanging colours in the (a, b)-chain containing X we still have $V \in A_X$. But we also have $a \in \bar{C}_V \cap \bar{C}_X$, which contradicts (i). This argument shows that V and X are in the same (a, b)-chain. So if $V' \neq V''$ are both in A_X, and

$a \in \bar{C}_{V'} \cap \bar{C}_{V''}$, then the (a,b)-chain containing X must be a path with three end-vertices which is absurd. Now (ii) is proved.

Of the last three statements below (iii) follows from the definition of A_X and B_X, and (iv) and (v) are obvious.

(iii) For any two vertices V in A_X and Z in B_X, $\bar{C}_V \cap C(E(Z,X)) = \emptyset$.
(iv) For all neighbours Z of X, $\bar{C}_X \cap C(E(Z,X)) = \emptyset$.
(v) For any two distinct neighbours Z' and Z'' of X, $C(E(Z',X)) \cap C(E(Z'',X)) = \emptyset$.

Now the lemma is proved.

THEOREM 1. *For any graph G*

$$q(G) \leqq \max\left\{d(G), \sup_{X \in V(G)} \sup_{\substack{A \subseteq N_G(X) \\ |A|=2}} \left[\frac{1}{2} \sum_{V \in A} (v_G(V) + |E(V,X)|)\right]\right\}.$$

PROOF. Suppose first that the theorem fails for a finite graph G'. By deleting edges from G' we can obtain a graph G with $q(G) = q(G')$ such that G has a critical edge e, say, with endvertices X and Y. Consider a $(q-1)$-colouring of $G-e$. As $v_G(X) \leqq d(G) \leqq q-1$ (by hypothesis $q(G) > d(G)$) we can use Lemma 1, and as $v_{G-e}(Y) < q-1$ it follows that $|A_X| \geqq 2$. The total number of colours in the mutually disjoint sets mentioned in the lemma can be at most $q-1$; hence

$$\begin{aligned} q-1 &\geqq |\bar{C}_X| + |\bar{C}_Y| + \sum_{V \in Ax \setminus \{Y\}} |\bar{C}_V| + \sum_{Z \in Bx} |C(E(Z,X))| \\ &= (q-1-(v_G(X)-1)) + (q-1-(v_G(Y)-1)) + \\ &\quad + \sum_{V \in Ax \setminus \{Y\}} (q-1-v_G(V)) + \left(v_G(X) - \sum_{V \in Ax} |E(V,X)|\right). \end{aligned}$$

This gives

$$|A_X| \cdot q \leqq \sum_{V \in Ax} (v_G(V) + |E(V,X)|) + |A_X| - 2$$

implying

$$\begin{aligned} q &\leqq \left[\frac{\sum_{V \in A_X} (v_G(V) + |E(V,X)|)}{|A_X|} + 1 - \frac{2}{|A_X|}\right] \\ &\leqq \max_{\substack{|A|=2 \\ A \subseteq N_G(X)}} \left[\frac{1}{2} \sum_{V \in A} (v_G(V) + |E(V,X)|)\right] \end{aligned}$$

contradicting the assumption that the theorem is false. So the theorem is true for finite graphs.

For infinite graphs the theorem now follows by the techniques of Bosák [2]. He observed that if d is infinite then $q=d$; and if k is a finite cardinal, then an infinite graph has a k-colouring if and only if every finite subgraph of it has.

Theorem 1 is better than the extension of Vizing's result that

$$q \leqq \sup_{X\in V(G)} \left(v_G(X) + \sup_{Y\in V(G)} |E(Y,X)| \right)$$

(implicit in Vizing's paper [12], stated in Ore [8]). It also generalizes Ore's result [8], that

$$q \leqq \max \{d(G), \sup [\tfrac{1}{2}(v_G(X)+v_G(Y)+v_G(Z))]\}$$

where the supremum is taken over all paths with exactly three vertices X, Y and Z. From this the theorem of Shannon [9], that $q \leqq [\frac{3}{2}d]$, follows.

THEOREM 2. *Let G be a graph for which*

$$q(G) = \max_{X\in V(G)} \left(v_G(X) + \max_{Y\in V(G)} |E(Y,X)| \right)$$

and which has a critical edge e with end-vertices X and Y. Put $t=v_G(Y)+|E(Y,X)|$.

Then X is adjacent to at least $q-t+1$ vertices V different from Y and such that $v_G(V)+|E(V,X)|=q$.

PROOF. Consider a $(q-1)$-colouring of $G-e$. The proof will show that the $q-t+1$ vertices mentioned all belong to A_X. Using Lemma 1 (obviously $v_G(X) \leqq q-1$) we get (as in the proof of Theorem 1)

$$q-1 \geqq \sum_{V\in A_X\setminus\{Y\}} (q-1-v_G(V)) + q - v_G(Y) + q - v_G(X) + v_G(X) - \sum_{V\in A_X} |E(V,X)|$$

giving

$$\sum_{V\in A_X\setminus\{Y\}} (q-v_G(V)-|E(V,X)|) \leqq t-q-1+(|A_X|-1)\,.$$

Every term of the sum on the left-hand side of this inequality is non-negative; as the sum is less than or equal to $t-q+|A_X|-2$ at most that many terms can be non-zero — the remaining $(|A_X|-1)-(t-q+|A_X|-2)$ terms must all be zero. But this number is exactly $q-t+1$, and the corresponding $q-t+1$ vertices V of $A_X\setminus\{Y\}$ must satisfy $v_G(V)+|E(V,X)|=q$.

Corollary 1. *Let G be a critical graph for which*

$$q(G) = \max_{X \in V(G)} \left(v_G(X) + \max_{Y \in V(G)} |E(Y,X)| \right).$$

Put $m(G) = \min_{X \in V(G)} (v_G(X) + \min_{Y \in N_G(X)} |E(Y,X)|)$.

(i) *Every vertex X of G is adjacent to at least two vertices V such that* $v_G(V) + |E(V,X)| = q$.

(ii) *G contains at least* $\max\{3, q - m(G) + 1\}$ *vertices W such that* $v_G(W) + \max_{U \in V(G)} |E(U,W)| = q$.

Corollary 2. (Vizing [11]). *If G is a simple critical graph and the vertex X is adjacent to the vertex Y, then X is adjacent to at least $d - v_G(Y) + 1$ vertices other than Y and of maximum valency in G.*

3. The colour-distribution in edge-deleted colourings of graphs with $q > d+1$.

Lemma 1 states that in an edge-deleted colouring, X being an end-vertex of the deleted edge, the sets $\bar{C}_V$, $V \in A_X \cup \{X\}$, are mutually disjoint. In this section we show that for graphs with $q > d+1$ this disjointness property can be extended to a considerably larger set than $A_X \cup \{X\}$.

We first prove a lemma very similar to Lemma 1.

Lemma 2. *Suppose that G is a graph with an edge-deleted colouring, and that X and Y are the end-vertices of the uncoloured edge, and suppose that $v_G(X) < q-1$ and $v_G(Y) \leqq q-1$. Then the sets $\{\bar{C}_V \mid V \in A_X \cup A_Y\}$ are mutually disjoint.*

Proof. Lemma 1 gives $\bar{C}_U \cap \bar{C}_V = \emptyset$ for two distinct vertices U and V where $\{U,V\} \subseteqq A_X$ or $\{U,V\} \subseteqq A_Y$ or $\{U,V\} \cap \{X,Y\} \neq \emptyset$. So suppose that $U \in A_X \setminus (A_Y \cup \{Y\})$ and $V \in A_Y \setminus (A_X \cup \{X\})$. Let $X = Y_0, Y_1, \ldots, Y_n = V$, $n \geqq 1$, be the end-vertices of the edges of a fan at Y, b a colour of $\bar{C}_X \cap C(E(Y_1, Y))$. Assume that the theorem fails for U and V and let $a \in \bar{C}_U \cap \bar{C}_V$. As $v_G(X) < q-1$ there is a colour c different from b absent at X. It follows from the proof of Lemma 1 that both U and V must be the other end-vertex of the path which is the (a,c)-chain containing X. This contradiction proves the lemma.

In Lemma 2 it is not assumed that $q > d+1$, but on the other hand $q > d+1$ would imply that $v_G(X) < q-1$ and $v_G(Y) < q-1$.

In [3], Gol'dberg proved the following:

LEMMA 3. *Let G be a graph with $q>d+1$ and consider an edge-deleted colouring of G. Then for any critical chain H the sets $\{\bar{C}_V \mid V \in V(H)\}$ are mutually disjoint.*

PROOF. Suppose the lemma is false and let U and V be two distinct vertices of a critical (a,b)-chain such that $c \in \bar{C}_U \cap \bar{C}_V$. Obviously $c \neq a$ and $c \neq b$. If U and V are not neighbours in the chain, we select a vertex Z between them. If $c \in C_Z$ we choose a colour $f \in \bar{C}_Z$ (by assumption, $q-1$ exceeds the valency of Z) and interchange colours in the (c,f)-chain containing Z. Then the colour c is absent at Z and at one (or both) of U and W. We can therefore assume that U and V are neighbours of the (a,b)-chain.

So there is an edge of the critical chain joining U and V. Recolouring this edge with colour c gives an edge-deleted colouring such that at neither end-vertex of the uncoloured edge do a and b both occur and at the same time these end-vertices are not in the same (a,b)-chain. This is a contradiction and so the lemma is proved.

The next theorem shows that Lemmas 2 and 3 can be combined and extended. We prove that the sets $\bar{C}_V$ are mutually disjoint for all V in the union of A_X, A_Y, and the set D of vertices of any critical chain.

THEOREM 3. *Let G be a graph with $q(G)>d(G)+1$ and X and Y two vertices of G such that the edge $e \in E(X,Y)$ is critical. Then in any $(q(G)-1)$-colouring of $G-e$ the sets $\{\bar{C}_V \mid V \in A_X \cup A_Y \cup D\}$ are mutually disjoint.*

PROOF. We first prove that for any two distinct vertices U and V of D, $\bar{C}_U \cap \bar{C}_V = \emptyset$. If they are in the same critical chain this follows from Lemma 3. So suppose that they are not, say U is in a critical (a,b)-chain and V in a critical (c,d)-chain, where a and c are in $\bar{C}_X$, b and d in $\bar{C}_Y$. a and c *or* b and d may coincide, but not both. Now assume that a colour h is in $\bar{C}_U \cap \bar{C}_V$. Obviously $h \neq a, b, c, d$. Let the successive vertices of the (a,b)-chain be $X, Z_1, \ldots, Z_t, Y$, $t \geqq 1$, and of the (c,d)-chain $X, W_1, \ldots, W_r, Y$, $r \geqq 1$. Now either $Z_1 \neq W_1$, $Z_1 \neq W_r$, or $Z_t \neq W_1$, because otherwise we would have $r=t=1$ and $U=V$. Let $\{W, Z\}$ be one of these pairs of vertices such that $Z \neq W$. If $Z \neq U$ we choose $k \in \bar{C}_Z$, and interchange colours in the (h,k)-chain having Z and U as end-vertices. Similarly if $W \neq V$ we choose $l \in \bar{C}_W$, and interchange colours in the (h,l)-chain containing W and V. After these interchanges we have $h \in \bar{C}_Z \cap \bar{C}_W$, and as we still have W and Z in $A_X \cup A_Y$ this contradicts Lemma 2. So the disjointness is established for two vertices both in D.

Lemma 2 takes care of the case of two vertices both in $A_X \cup A_Y$. So assume finally that $U \in A_X \backslash D$ and $V \in D \backslash (A_X \cup A_Y)$. Then U and V are both different from X and from Y. Let V belong to a critical (a,b)-chain, $a \in \bar{C}_X$ and $b \in \bar{C}_Y$. Let $c \in \bar{C}_X$, $c \neq a$. We must prove that $\bar{C}_U \cap \bar{C}_V = \emptyset$. Suppose on the contrary that $d \in \bar{C}_U \cap \bar{C}_V$. Then $d \neq a$, $d \neq b$, and $d \neq c$. By Lemma 1, U and X belong to the same (d,c)-chain, and it follows from Lemma 3 that V and X belong to the same (d,c)-chain. But this is impossible, and therefore $\bar{C}_U \cap \bar{C}_V = \emptyset$.

Theorem 3 is now proved.

COROLLARY 3. *Let G be a graph with $q > d+1$ and with an edge-deleted colouring, and let X and Y be the end-vertices of the uncoloured edge. Then*

$$|A_X \cup A_Y \cup D| \leqq \frac{q-3}{q-d-1}\,.$$

PROOF. Counting colours in the disjoint sets of Theorem 3 gives

$$q-1 \geqq (|A_X \cup A_Y \cup D|-2)(q-1-d)+2(q-d)\,,$$

which implies the statement of the corollary.

To help the discussion in the next section we also state:

COROLLARY 4. *Let G be a graph with*

$$q > \frac{m}{m-1}d+\frac{m-3}{m-1} \qquad (m>2)$$

with an edge-deleted colouring, and let X and Y be the end-vertices of the uncoloured edge. Then $|A_X \cup A_Y \cup D| < m$.

PROOF. The assumption gives $q > d+1$, then we must have $d \geqq 3$, and from Corollary 3 we get

$$|A_X \cup A_Y \cup D| \leqq \frac{q-3}{q-d-1} < \frac{\dfrac{m}{m-1}d+\dfrac{m-3}{m-1}-3}{\dfrac{m}{m-1}d+\dfrac{m-3}{m-1}-d-1} = m\,.$$

4. The number of vertices in critical graphs with $q > d+1$.

In the following, a *ringgraph* will mean a circuit or any graph obtained from a circuit by substituting one or more edges by multiple edges.

LEMMA 4. *The critical graphs in which every critical chain in every edge-deleted colouring has exactly 3 vertices, are precisely the ringgraphs with three vertices.*

PROOF. Let G have the property stated. Consider a $(q-1)$-colouring of $G-e$, where e is an edge of G with end-vertices X and Y. G being critical, there are colours absent at X and Y, say $a \in \bar{C}_X$ and $b \in \bar{C}_Y$. Let Z be the third vertex of the critical (a,b)-chain. Now we first prove:

(i) $\bar{C}_Z \subseteqq C(E(X,Y))$, $\bar{C}_X \subseteqq C(E(Y,Z))$ and $\bar{C}_Y \subseteqq C(E(X,Z))$.

PROOF OF (i). Suppose $c \in \bar{C}_Z$. Uncolouring the edge of $E(X,Z)$ coloured b and giving e the colour b we obtain another edge-deleted colouring. As the critical (a,c)-chain must have only three vertices, c must colour an edge of $E(X,Y)$. So $\bar{C}_Z \subseteqq C(E(X,Y))$. The other two inclusions are obvious from considering critical chains with b respectively a as one colour (in fact we see that X, Y, and Z are symmetrical as the uncoloured edge can be "shifted" among them). Now (i) is proved.

(ii) $C_Z \subseteqq C(E(X,Y)) \cup C(E(Y,Z)) \cup C(E(X,Z))$.

PROOF OF (ii). Suppose not. Let $c \in C_Z$ and assume that c is not colouring any edge joining two of X, Y, and Z. Colour c is present at X and Y, and the (b,c)-chain containing Y must contain the edge of $E(X,Z)$ of colour b (otherwise an interchange of colours in the chain would give a critical (a,c)-chain with more than three vertices contrary to hypothesis). Following this chain from Y we can assume that X is met before Z. Now we uncolour the edge of $E(X,Z)$ coloured b and give e the colour b; the (b,c)-chain containing Y is now a circuit. Interchanging colours in this circuit gives a critical (a,b)-chain with more than three vertices contrary to hypothesis. This contradiction proves (ii).

(i) and (ii) together imply that *all* colours $(\bar{C}_Z \cup C_Z)$ colour an edge of the subgraph spanned by X, Y, and Z. This subgraph must then have $q-1$ edges in addition to e; hence it has chromatic index q. G being critical, the subgraph is the whole of G. So G is a ringgraph with 3 vertices, and Lemma 4 is proved.

In [2], Bosák made the following conjecture:

Let G be a graph with finite d and let t denote the maximal number of edges of a subgraph of G with at most 3 vertices. Then

$$\text{If } t \geqq [\tfrac{5}{4}d+\tfrac{1}{2}], \text{ then } q = t\,.$$

$$\text{If } t < [\tfrac{5}{4}d+\tfrac{1}{2}], \text{ then } q \leqq [\tfrac{5}{4}d+\tfrac{1}{2}]\,.$$

A proof of this has first been published by Gol'dberg [4], unpublished proofs have been found by Bo Aagaard Sørensen and by the author. Theorem 4 below implies the validity of the conjecture; the proof given is based on Lemma 4 which is adapted from Gol'dberg's proof.

THEOREM 4. *If G is a critical graph with $q(G)>\frac{5}{4}d(G)+\frac{1}{2}$, then G is a ringgraph with 3 vertices.*

PROOF. By Corollary 4 with $m=5$, in every edge-deleted colouring of G D has at most 4 vertices. As a critical chain contains an odd number of vertices, any such chain must have exactly 3 vertices. It follows from Lemma 4 that G is a ringgraph with 3 vertices.

Another question, extending that of Bosák, can be formulated in the following way (Jakobsen [7]):

Is (*) below true for all odd integers $m \geqq 3$?

(*) If G is a critical graph with

$$q > \frac{m}{m-1}d + \frac{m-3}{m-1},$$

then G has at most $m-2$ vertices.

If the answer is yes, then there are only a finite number of different critical graphs with $q>d+1$ for each d (and these all have less than d vertices).

For $m=3$ the statement is true by the theorem of Shannon [9]. For $m=5$ it is true by Theorem 4, and Theorem 5 below shows that it is also true for $m=7$. In [7] Jakobsen reports that Theorem 5 has been proved by Bo Aagaard Sørensen, and recently he informed the author that M. K. Gol'dberg has found a proof as well (both proofs unpublished).

We may note that (*) is true for ringgraphs for all odd $m \geqq 3$.

THEOREM 5. *If G is a critical graph with $q(G)>\frac{7}{6}d(G)+\frac{2}{3}$, then G has at most 5 vertices.*

PROOF. By Corollary 4 with $m=7$, in every edge-deleted colouring of G $|A_X \cup A_Y \cup D|<7$, X and Y being the end-vertices of the uncoloured edge. Hence every critical chain must have 3 or 5 vertices. If every critical chain in every edge-deleted colouring has 3 vertices, by Lemma 4, G has exactly 3 vertices. So assume in the following that G has an edge-deleted colouring with a critical chain having 5 vertices; let the deleted edge e have end-vertices X and Y, $a \in \bar{C}_X$, $b \in \bar{C}_Y$, such that the critical (a,b)-chain has successive vertices X, Z_1, Z_2, Z_3, Y. Let the subgraph of G spanned by these 5 vertices be denoted by H. Observe that there is symmetry among the vertices of H as we can get other edge-deleted colourings having critical (a,b)-chains with the same 5 vertices by shifting the colour b from the edge of $E(X,Z_1)$ to e etc. — Now we prove:

(i) $D=V(H)$.

Proof of (i). Suppose that D contains a vertex W not in $V(H)$. In the critical chain to which W belongs it cannot be joined to X or Y, because if $c \in \bar{C}_Y \cap C(E(W, X))$ then the critical (a, c)-chain contains X, W, Y, Z_3 and necessarily another vertex not in H contradicting $|D| \leqq 6$. So in the critical chain to which W belongs it is not joined to X or Y; its neighbours must be in H, hence we can assume that one is Z_3. Let $c \in C(E(Z_3, W))$, c being a colour of the critical chain, i.e., $c \in \bar{C}_X$ or $c \in \bar{C}_Y$. Now if $c \in \bar{C}_Y$ then we can uncolour the edge of $E(Z_3, Y)$ coloured a and give e the colour a; afterwards W is joined to an end-vertex of the critical (a, b)-chain which we have just seen is impossible. If $c \in \bar{C}_X$ then the critical (b, c)-chain cannot contain W (for then it would contain at least seven vertices). Hence an interchange of colours in the (b, c)-chain containing W gives a colouring where the critical (a, b)-chain has at least 7 vertices. This contradiction shows that D cannot contain a vertex which is not in $V(H)$, and (i) is proved.

We proceed to prove

(ii) Every colour of $\bar{C}_X \cup \bar{C}_{Z_1} \cup \bar{C}_{Z_2} \cup \bar{C}_{Z_3} \cup \bar{C}_Y$ colours two edges of H.

Proof of (ii). It is true for a and b. Let c be a colour of the union different from a and b; by the symmetry we can assume that $c \in \bar{C}_X$. Then c is present at Y and Z_1, and it follows from (i) that in both cases it colours an edge of H. So either c colours two edges of H, or it colours an edge of $E(Z_1, Y)$. Suppose the latter. Interchange colours in the critical (b, c)-chain. By considering the critical (a, c)-chain we see that c must colour an edge of $E(Z_2, Z_3)$, so also in this case c colours two edges of H. This proves (ii). (iii) goes further:

(iii) Each of the $q-1$ colours colours two edges of H.

Proof of (iii). It is true for any colour absent at some vertex of H, by (ii). Suppose c is a colour present at every vertex of H. We want to prove that c colours two edges of H, so suppose it does not, say c colours edges joining X and Y to vertices not in H. Then we can assume that the (b, c)-chain containing Y contains another vertex of H (by symmetry. For otherwise we can interchange colours in the (b, c)-chain containing Y without affecting the edge incident with X of colour c, and in the resulting colouring the critical (a, c)-chain has exactly four edges and contains Z_3. As none of these four edges has been affected by the interchange we see that also in the original colouring does the (a, c)-chain containing X contain another vertex of H.) Following the chain from Y we let V denote the vertex of H different from Y first met. By symmetry, it is sufficient to consider two cases, $V=X$ and $V=Z_1$.

If $V=X$ we uncolour the edge of $E(X, Z_1)$ coloured b, and give this colour to e. We then interchange colours in the (b, c)-chain containing X (which is now a circuit) and obtain a colouring where the critical (a, b)-chain has at least 7 vertices, which is a contradiction.

So we have $V = Z_1$. Let U and W be the neighbours of Y and Z_1 respectively such that $c \in C(E(Y, U))$ and $c \in C(E(Z_1, W))$. We investigate two possibilities, namely whether c colours an edge of $E(Z_2, Z_3)$ or not.

Assume $c \in C(E(Z_2, Z_3))$. Interchanging colours in the critical (a, b)-chain we obtain a (b, c)-chain which is a circuit containing Y; hence it does not contain X. By interchanging colours in the (b, c)-chain containing X and considering the critical (a, c)-chain we see that $a \in C(E(U, W))$. Now let $d \in \bar{C}_X$, $d \neq c$. d is present at Y, and by (ii), it colours an edge joining Y to a vertex both in the original critical (a, b)-chain and in the critical (a, c)-chain containing U and W; hence it colours an edge of $E(Y, Z_1)$. By (ii), d also colours an edge of $E(Z_2, Z_3)$, and so there is a (b, d)-chain which is a circuit with vertices Z_1, Z_2, Z_3, and Y. Now we interchange colours in the (b, d)-chain containing X. In the colouring that we have finally obtained D contains all of X, Y, Z_1, Z_2, Z_3, U, W, which is a contradiction. So $c \notin C(E(Z_2, Z_3))$.

This implies that Z_2 and Z_3 are in the (b, c)-chain containing Y (otherwise interchanging colours on this chain would result in a critical (a, c)-chain with more than five vertices, a contradiction). By symmetry we can assume that Z_2 is met before Z_3 (but after X). Interchanging colours in the critical (a, b)-chain, uncolouring the edge of $E(Z_3, Y)$ now coloured b, and giving e the colour b we obtain a colouring where there is a (b, c)-chain which is a circuit not containing Z_3. We interchange colours in the (b, c)-chain containing Z_3, and by considering the critical (a, c)-chain it is seen that Z_2 and U have a common neighbour $Z \neq W$ outside H. By choosing $d \in \bar{C}_{Z_3}$, $d \neq c$, we get a contradiction in the same way as above. This finally proves (iii).

From (iii) we see that H consists of a circuit with 5 vertices to which $(q-3)$ sets of two independent edges are added. But such a graph is easily seen to be critical with chromatic index q. As G is critical $G = H$, and the theorem is proved.

The proof given above for the case $m = 7$ of the statement (*) mentioned before the theorem does not generalize to greater m. We formulate instead the following

CONJECTURE. *If G is a critical graph with $q(G) > d(G) + 1$, then in every edge-deleted colouring of G all sets $\{\bar{C}_V \mid V \in V(G)\}$ are mutually disjoint.*

One can prove that this conjecture is equivalent to a conjecture by Gol'dberg in [4]. It implies the "critical graph conjecture" (see [6]) that every critical graph has an odd number of vertices, in the special case $q > d + 1$; for if $a \in \bar{C}_V$ and $a \in C_U$ for any $U \neq V$, then obviously the graph under consideration has an odd number of vertices. The conjecture also implies (*). For if we consider an edge-deleted colouring of the critical graph G with n vertices, then by the conjecture

$$(n-2)(q-d-1)+2(q-d) \leqq q-1 .$$

Combining this with

$$q > \frac{m}{m-1}d+\frac{m-3}{m-1}$$

(the assumption of (*)) we get $n<m$. Since n and m are both odd, the conclusion of (*) follows.

In connection with the critical graph conjecture we should mention that Lemma 4 has the following corollary:

COROLLARY 5. *There are no critical graphs with precisely four vertices.*

5. Application to questions of Berge and Bosák.

For all $s \geqq 2$, let G_s denote the ringgraph with three vertices X, Y, and Z, such that $|E(X,Z)|=|E(Y,Z)|=[s/2]$ and $|E(X,Y)|=[(s+1)/2]$. Then $d(G_s)=s$ and $q(G_s)=[\frac{3}{2}s]$. Vizing [12] has shown that for all $d \geqq 4$, G_d is the only critical graph with $q=[\frac{3}{2}d]$. This leads to the question of finding a better upper bound for the chromatic index of a graph *not* containing G_s as a subgraph. In [1] Berge made the following conjecture:

If G is a graph not containing G_s as a subgraph, where $4 \leqq s \leqq d(G)$, then $q(G) \leqq [\frac{3}{2}d(G)]-[d(G)/s]$.

Berge himself proved this for s and d even (ibid.), and Bosák verified the conjecture for the cases $d \leqq 2s$ and d even ([2]). The truth of the conjecture follows from Theorem 4 of this paper (Gol'dberg also observed this in his paper [4]). Bosák denoted by $S(d,s)$ $(d \geqq 0, s \geqq 2)$ the maximal chromatic index of a graph not containing G_s as a subgraph, and gave some bounds for this number. In Theorem 6 below, $S(d,s)$ is determined for all $d \geqq 0$ and all $s \geqq 2$. It is seen that the actual value of $S(d,s)$ is much less than the upper bound of Berge's conjecture.

THEOREM 6. *Let d and s be integers such that $d \geqq 0$ and $s \geqq 2$, and let $S(d,s)$ denote the maximal chromatic index of a graph not containing G_s as a subgraph. Then*

(1) $S(d,s)=[\frac{3}{2}d]$ *for* $d \leqq s-1$.

(2) $S(d,s)=d-1+[s/2]$ *for* $s \leqq d \leqq 2s-6$.

(3) $S(d,s)=[\frac{5}{4}d+\frac{1}{2}]$ *for* $d \geqq 2s-5$.

PROOF. (1). $S(d,s) \leqq [\frac{3}{2}d]$ by Shannon's theorem, and for $d \leqq s-1$ this bound is obtained for graphs not containing G_s, namely if $d=0$ or $d=1$ by the complete graph with 1 or 2 vertices, and if $d \geqq 2$ by the graph G_d.

(2). Let $s \leqq d \leqq 2s-6$ (this implies $s \geqq 6$) and suppose (reductio ad absurdum) that G is a graph such that $G \not\supseteq G_s$ and such that $q > d-1+[s/2]$. We can assume that G is critical. Then

$$q \geqq d+\left[\frac{s}{2}\right] \geqq d+\left[\frac{d+6}{4}\right] > \left[\frac{5}{4}d+\frac{1}{2}\right]$$

and so by Theorem 4, G is a ringgraph with three vertices. Since $q \geqq d+[s/2]$ and q equals the number of edges of G, it follows that each pair of vertices of G is joined by at least $[s/2]$ edges, so $G \supseteq G_{2[s/2]}$. Since $G \not\supseteq G_s$, we conclude that s is odd and $G=G_{s-1}$. But this contradicts the assumption $d \geqq s$. So we have that $S(d,s) \leqq d-1+[s/2]$.

The upper bound is obtained for the ringgraph with three vertices X, Y, and Z, such that

$$|E(X,Y)| = \left[\frac{d}{2}\right], \quad |E(X,Z)| = \left[\frac{d+1}{2}\right], \quad |E(Y,Z)| = \left[\frac{s}{2}\right]-1 .$$

Obviously it does not contain G_s as a subgraph.

(3) Let $d \geqq 2s-5$ and suppose (reductio ad absurdum) that G is a critical graph such that $G \not\supseteq G_s$ and such that $q > [\frac{5}{4}d+\frac{1}{2}]$. By Theorem 4 G is a ringgraph with three vertices. Now

$$q \geqq \left[\frac{5}{4}d+\frac{1}{2}\right]+1 = d+\left[\frac{d}{4}+\frac{3}{2}\right] \geqq d+\left[\frac{s}{2}-\frac{5}{4}+\frac{3}{2}\right] \geqq d+\left[\frac{s}{2}\right].$$

Thus as $G \not\supseteq G_s$, G must have three multiple edges each consisting of $[s/2]$ edges, with s odd. Such a graph has $d=s-1$, and $s-1 \geqq 2s-5$ implies $s \leqq 4$. As s is odd we get $s=3$. But then G is a circuit with 3 vertices and has $q=[\frac{5}{4}d+\frac{1}{2}]$ which is a contradiction. Thus $S(d,s) \leqq [\frac{5}{4}d+\frac{1}{2}]$.

The bound is obtained for all d and s. If $d=0$ or $d=1$ take the same graphs as in the proof of (1). If $d \geqq 2$ take the ringgraph with 5 vertices X_1, X_2, X_3, X_4, X_5 such that

$$|E(X_1,X_2)| = |E(X_2,X_3)| = |E(X_4,X_5)| = \left[\frac{d}{2}\right],$$

and

$$|E(X_3, X_4)| = |E(X_5, X_1)| = \left[\frac{d+1}{2}\right].$$

This graph has $q=[\frac{5}{4}d+\frac{1}{2}]$ (see [1] or [2]). Note that it is not always critical.

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