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NON-COMMUTATIVE MINIMALLY NON-NOETHERIAN RINGS

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Gilmer and O'Malley [2] proved that, in a ring R, the following conditions are equivalent:

(a) R does not satisfy the maximum condition on two-sided ideals, but each proper subring of R satisfies the maximum condition on two-sided ideals;

(b) R does not satisfy the maximum condition on left ideals, but each proper left ideal of R satisfies the maximum condition on left ideals;

(c) R is the zero ring on a quasi-cyclic group $\mathbf{Z}(p^{\infty})$.

They raised the question as to what can be said if, throughout (b) above, "left" is replaced by "two-sided". In particular, they asked if the multiplication on R would then be trivial. Recently, Hausen and Johnson [4] have shown that the answer to this question is no. Further developments of this topic are suggested by the example in [4] and the paper [2].

The example in [4] was given as an ideal in L, the ring of all linear transformations of a vector space V over a field F with $\dim V = \aleph_{\tau}$, where the ordinal $\tau \geqq \omega$. Specifically the example was the ideal

$$J_{\omega} = \{\alpha \in L : \dim V\alpha < \aleph_{\omega}\} .$$

If U is any infinite dimensional subspace of V with $\dim U < \aleph_{\omega}$, then J_{ω} has a subalgebra isomorphic to $\mathrm{Hom}_F(U,U)$. This is easily seen by selecting a fixed complement, W say, of U in V and extending each transformation of U to a map from V into U by mapping each member of W to 0. It is well-known that the algebra $\mathrm{Hom}_F(U,U)$ satisfies no polynomial identity [5, p. 19] and so J_{ω} will satisfy no polynomial identity. On the other hand for a commutative ring R it follows, from the work of Gilmer and O'Malley [2], that the answer to the question raised in [2] has a positive answer. It is then natural to try to extend this to rings with polynomial identity. Our main result in this direction is Theorem 4, where we show that the question raised in [2] has a positive answer for PI-rings, i.e. there are no non-commutative minimally non-Noetherian PI-rings.

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The example above is a primitive ring and so the question arises as to what happens, at the opposite end of the spectrum, when R is a radical ring (that is, R is equal to its Jacobson radical). Again the paper [2] also points us in this direction, for a ring R may be written as the sum of two left ideals,

$$R = Ra + R(1-a),$$

where $a \in R$ and 1 is being used as a notational convenience without implying that R has a unit element 1, so that if R has condition (b), then $Ra \neq R$ and we must have $R(1-a)=R$. Hence the element a is quasi-regular and R is a radical ring. Unfortunately we are unable to give a complete anwer to this question, but we can show that there are no non-commutative minimally non-Noetherian rings in which the prime radical is the whole of the ring. This result is the content of Corollary 3.

1. Definitions and terminology.

If $f \in \mathbf{Z}[x_1,\ldots,x_d]$, the free ring in the non-commuting variables $x_1,\ldots,x_d$, and if $f(a_1,\ldots,a_d)=0$ for all a_i in the ring A, then we say that f is an identity for A. If in addition some coefficient of f does not annihilate A, then f is called a proper identity for A. A ring A is called a PI-ring if there is a polynomial f which is a proper identity for every non-zero homomorphic image of A. For details, see [5].

We remark that if f is a proper identity for A and if A^* is an extension of A such that A^*/A is commutative, then A^* satisfies a proper identity. If $g \in \mathbf{Z}[y_{11}, y_{21},\ldots, y_{1d}, y_{2d}]$ is the polynomial obtained by the substitution in f of $g_i \equiv y_{1i}y_{2i} - y_{2i}y_{1i}$ for x_i, then it is clear that g is an identity for A^*. Moreover, if $x_{j_1} \ldots x_{j_r}$ is a monomial occurring in f with coefficient $\alpha \in \mathbf{Z}$, then $y_{1j_1}y_{2j_1} \ldots y_{1j_r}y_{2j_r}$ occurs in g with coefficient α. Thus g is a proper identity for A^*.

Following [2] we shall label property:

(C2) R does not satisfy the maximum condition on left ideals, but each proper left ideal of R satisfies the maximum condition on left ideals.

We wish to consider K-algebras, where K is a field, as well as rings. Accordingly, we will label the K-algebra analogue of (C2) by (AC2), in which left ideal means both K-module and left ideal in the ring theoretic sense. The sequence of labels is continued with:

(C3) R does not satisfy the maximum condition on (two-sided) ideals, but each proper ideal of R satisfies the maximum condition on ideals.

Again the K-algebra analogue of (C3) is labelled (AC3).

We have already observed that if R is a ring with property (C2), then $J(R) = R$, where $J(R)$ is the Jacobson radical of R. In such a case we describe R as a radical ring. If R has property (C3), then it is clear that R has no maximal ideals and so, in particular, no homomorphic images among simple rings with unit element. Hence $G(R) = R$ where $G(R)$ is the Brown–McCoy radical of R. We refer to [1] for the definition and properties of this radical and the prime radical, also called the Baer lower radical, which we shall denote by $B(R)$.

2. Algebras over a field K.

The main purpose of this section is to show that in a K-algebra R with property (AC3) the radical $B(R) \neq R$ and to establish, as a corollary, that a K-algebra R, which is a radical PI-ring, will not have property (AC3). The first proposition allows us to refer to results proved for rings with unit element.

PROPOSITION 1. *Let A be a prime radical ring. If A is an algebra over a field K, then the extension A^1 is a prime ring with unit element.*

PROOF. The extension A^1 is the usual extension, by K, to obtain a ring with unit element. We need only show that A^1 is prime.

Suppose $\alpha_1, \alpha_2 \in K$, $a_1, a_2 \in A$ are such that

$$(\alpha_1 + a_1)A^1(\alpha_2 + a_2) = 0 .$$

Then $\alpha_1 K \alpha_2 = 0$ and so $\alpha_1 = 0$ or $\alpha_2 = 0$. Let us take $\alpha_2 = 0$. Then if $\alpha_1 = 0$ we have $a_1 A a_2 = 0$ and so either $a_1 = 0$ or $a_2 = 0$. On the other hand, if $\alpha_1 \neq 0$, then

$$(1 + \alpha_1^{-1} a_1)A^1 a_2 = 0 .$$

Since A is a radical ring, $(1 + \alpha_1^{-1} a_1)A = A$. Hence we have $A a_2 = 0$ and so $a_2 = 0$. Thus we conclude that A^1 is a prime ring with unit element.

To allow us to go from property (AC3) to property (AC2) we appeal to the following theorem of Cauchon:

If R is a prime ring, with unit element, which has a proper identity and satisfies the maximum condition on ideals, then R has the maximum condition on left ideals.

A proof of this theorem is given in [3]. The importance of this result, in the context of this paper, will be seen in the proof of Theorem 1.

PROPOSITION 2. *Let A be a prime radical ring which has a proper identity and suppose that A is an algebra over a field K. If A has the maximum condition on $A^1 - A^1$ bimodules, then A has the maximum condition on left A^1-modules.*

PROOF. From the remarks made in section 1 we see that A^1 satisfies a proper identity, whilst Proposition 1 implies that A^1 is a prime ring with unit element. Furthermore A^1 has the maximum condition on ideals so, by Cauchon's Theorem, A^1 has the maximum condition on left ideals. But in A, any left ideal, which is also a K-subalgebra of A, is a left ideal of A^1. Hence A satisfies the maximum condition on left A^1-modules.

We turn next to K-algebras with property (AC2) or (AC3). The first result in this area is the analogue, for K-algebras, of Gilmer and O'Malley's Theorem [2, Theorem 3.1]. We note that the critical ideals in their proof were Rx, for $0 \neq x \in R$, and the left annihilator of x. Both of these ideals in a K-algebra R are also K-subalgebras. Hence the proof of the next proposition is essentially as in [2] so we omit it here.

PROPOSITION 3. *Let S be a K-algebra with property* (AC2). *Then S has the trivial multiplication.*

COROLLARY 1. *There are no K-algebras, where K is a field, with property* (AC2).

PROOF. Consider a K-algebra S with property (AC2). As a vector space over K, S cannot be finite dimensional, but if S is infinite dimensional there are proper ideals which are not finitely generated.

The above Corollary 1 may also be derived directly from the results in [2].

THEOREM 1. *Let R be a radical* PI-*ring which is an algebra over a field K. If R has property* (AC3), *then R is the union of a chain of nilpotent ideals.*

PROOF. The prime radical $B(R)$ is a K-subalgebra of R. If $B(R) \neq R$ then R/B is a semi-prime K-algebra and so R has a non-zero image S which is a prime K-algebra. The algebra S has property (AC3). Now if A is a proper ideal of S (as K-algebras), then A has the maximum condition on A^1-A^1 bimodules by (AC3). In addition A inherits the radical and PI properties from R and primeness from S. Therefore, by Proposition 2, A has the maximum condition on left A^1-modules. This means that, as a K-algebra, A has the maximum condition on left ideals.

Since the K-algebra S does not satisfy the maximum condition on ideals, it certainly does not satisfy the maximum condition on left ideals. Thus S has property (AC2) and we have a contradiction to Corollary 1. We conclude that $B(R)=R$.

Finally, the construction of $B(R)$ as the lower radical determined by the class of all nilpotent rings [1, Chapter 3] and property (AC3) imply that either R is nilpotent or there is a strictly increasing chain of nilpotent ideals $A_1 \subset A_2 \subset \ldots$ with $R = \bigcup_{i=1}^{\infty} A_i$. This completes the proof of the theorem.

The last theorem leads us to consider further the prime radical $B(R)$. As a consequence of the next theorem we have that there are no rings satisfying the hypotheses of Theorem 1. Thus Theorem 1, though necessary for our argument, turns out to be redundant.

THEOREM 2. *Let R be a K-algebra with property* (AC3). *Then* $B(R) \neq R$.

PROOF. Let us suppose that $B(R) = R$. Then R is a nil algebra and contains a non-zero left ideal I_0 with $I_0^2 = 0$. Using these properties we show that the ideal

$$J = \{a \in R : Ra = 0\}$$

is non-zero. Certainly this is so if $I_0 = R$, so we take $I_0 \neq R$.

Suppose that in I_0 we have a strictly descending chain of left ideals

$$I_0 \supset I_1 \supset \ldots \supset I_k$$

with $I_j = Ra_j$ for $j \geqq 1$, $a_j \in R$. If $I_k \neq 0$ then pick $0 \neq a_{k+1} \in I_k$ and put $I_{k+1} = Ra_{k+1}$. Hence $I_k \supseteqq I_{k+1}$. Now $a_{k+1} = xa_{k+1}$ and x nilpotent imply that $a_{k+1} = 0$, so $a_{k+1} \in I_k \backslash I_{k+1}$ and $I_k \supset I_{k+1}$.

Now $I_0 \neq R$ so I_0 has the maximum condition on ideals by (AC3). But the trivial multiplication on I_0 means then that I_0 has the minimum condition on ideals. Thus we conclude that, for some $k \geqq 1$, $I_k = 0$ and $Ra_k = 0$ for some $0 \neq a_k \in I_{k-1}$. Therefore $J \neq 0$.

If $J = R$ then R is a K-algebra with property (AC2) and so, by Corollary 1, we must have $J \neq R$.

Let us factor out R by the non-zero ideal J and consider R/J which, being non-zero, satisfies exactly the same hypotheses as R. Hence $\exists\, 0 \neq b \in R \backslash J$ such that $Rb \subseteqq J$. If $R^2 = R$, then

$$Rb = R(Rb) \subseteqq RJ = 0 .$$

But $b \notin J$ and this contradiction means that $R^2 \neq R$.

Finally we consider the ring $\bar{R} = R/R^2 \neq 0$. Since $\bar{R}$ is a zero ring with property (AC3) it has property (AC2). Thus $\bar{R}$ is a K-algebra with property (AC2), which contradicts Corollary 1. Therefore $B(R) \neq R$ as required.

COROLLARY 2. *Let R be a radical* PI-*ring which is an algebra over a field K. Then R cannot have property* (AC3).

PROOF. Immediate from Theorems 1 and 2.

We conclude this section with two formulations of the same question:

1. Are there any radical K-algebras with property (AC3)?
2. Are there any prime radical K-algebras with property (AC3)?

The equivalence of these two questions follows from Theorem 2.

3. Rings with (C3).

The aim here is to show that the case of rings with property (C3) reduces to that of algebras with property (AC3), dealt with in the preceding section.

THEOREM 3. *Let R be a ring with property* (C3). *Then either R is the zero ring on a $\mathbf{Z}(p^\infty)$ group or R has a homomorphic image S such that*

(i) $S^2 = S$;
(ii) *S is an algebra over a field K;*
(iii) *S has property* (AC3) *as a K-algebra.*

PROOF. If $qR \neq R$ for some prime q, then $S = R/qR$ is an algebra over $K = \mathrm{GF}(q)$ with property (AC3) as a K-algebra. Now either $S^2 = S$ or S/S^2 is a zero ring with property (C3), in which case, by [2, Theorem 3.2], $(S/S^2)^+ \cong \mathbf{Z}(p^\infty)$ for some prime p. But S is an algebra over GF (q) so the only allowable possibility is that $S^2 = S$. Thus S is as given in the statement.

We may now assume that R^+ is a divisible group and, since the divisible torsion subgroup is an ideal with trivial multiplication, either R^+ is torsion-free or $R^+ \cong \mathbf{Z}(p^\infty)$ for some prime p. This reduces our considerations to the case when R^+ is a torsion-free divisible group and so R is an algebra over the field $K = \mathbf{Q}$.

Since the additive group of R/R^2 is a direct sum of copies of $\mathbf{Q}^+$, the ring R/R^2 cannot have property (C3). Hence $R^2 = R$.

If I is an ideal of R and $pI \neq I$ for some prime p, then $\bar{R} = R/pI$ has, as additive group, a divisible group with non-zero p-component. The property (C3) then forces $\bar{R}$ to have a divisible p-group as its additive group. But then $R^2 \subseteqq pI \subset I \subseteqq R = R^2$ is a contradiction which means that $pI = I$ for every ideal I of R and prime p. Thus each ideal I of R is also a K-algebra and the condition (C3) on R now means that R has property (AC3), as a K-algebra, and the image S of the statement can be taken to be R itself.

COROLLARY 3. *Let R be a ring with property* (C3) *and $B(R) = R$. Then R is the zero ring on a $\mathbf{Z}(p^\infty)$ group.*

PROOF. Immediate from Theorems 2 and 3.

This corollary leaves us with another form of the question already posed in section 2.

3. Are there any radical rings with property (C3) other than the zero rings on the $\mathbf{Z}(p^\infty)$ groups?

Finally we return to PI-rings and obtain the generalisation of the result, in the commutative case, on rings with property (C3), which is an immediate consequence of the work in [2].

THEOREM 4. *Let R be a* PI-*ring with property* (C3). *Then R is the zero ring on a $\mathbf{Z}(p^\infty)$ group.*

PROOF. If R is not the zero ring on a $\mathbf{Z}(p^\infty)$ group, then we have an image S as in Theorem 3. If S contains a primitive ideal P, then S/P is a primitive PI-ring. Kaplansky's Theorem [5, p. 30] then tells us that S/P is simple and P is a maximal ideal. Since rings with property (C3) cannot have maximal ideals, S cannot contain any primitive ideals. But then S is a radical PI-ring and an algebra over a field K, with property (AC3) as a K-algebra. This contradicts Corollary 2 and so completes the proof of the theorem.

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