MATH. SCAND. 40 (1977), 183–193

A SIMPLE AXIOMATICS FOR DIFFERENTIATION

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We attempt here to describe a property of ring objects A in a category $\boldsymbol{E}$, which will give A some of the features of the affine line, namely that functions $A \to A$ can be differentiated. The first basic idea is that "the object of infinitesimals in A", D, should exist, as well as, for any other object M, the object of maps M^D from D to M, and that this should have some of the properties of the tangent-bundle of M. In particular, one should have an isomorphism $A^D \cong A \times A$. This approach was suggested by Lawvere in 1967, [5] (unpublished), and some work in this direction has been carried out by Wraith (who constructed a Lie-algebra object out of a monoid object in such a category, [7]). The axiomatics employed by Wraith for D was a bit heavy.

The second basic idea, which seems to be new, is that one can construct D and $A \times A \to A^D$ *canonically*, and that the only axiom needed (except for sufficient categorical properties of the surrounding category $\boldsymbol{E}$) is that this map $A \times A \to A^D$ should be invertible. This axiom is then sufficient to define a map f': $A \to A$ out of any f: $A \to A$, and to prove the rules

$$(f+g)' = f'+g'; \qquad (f \cdot g)' = f' \cdot g + f \cdot g' ,$$

and

$$(g \circ f)' = (g' \circ f) \cdot f'$$

(chain rule).

We finish by giving a model for such A, $\boldsymbol{E}$ which satisfy the axiom; the surrounding category is the topos classifying commutative rings, and A is the generic commutative ring in there.

In a subsequent paper we investigate in the same setting Taylor series theory (in one or several variables), and indicate the relationship of our theory to formal power series.

Received June 17, 1976.

1. Some generalities about rings in cartesian categories.

We consider a commutative ring object A in a cartesian category $\boldsymbol{E}$ (cartesian category = category having finite limits). So there are given maps

$$A \times A \xrightarrow{+} A \qquad A \times A \xrightarrow{m} A$$

("m" for multiplication; in formulas, we sometimes use the usual dot $\cdot$ instead); and

$$1 \xrightarrow{\ulcorner 0 \urcorner} A \qquad 1 \xrightarrow{\ulcorner 1 \urcorner} A ,$$

satisfying the usual equations.

We let D denote the equalizer of the two maps $A \to A$ given by the "descriptions"

$$a \mapsto a \cdot a \qquad a \mapsto 0 .$$

(The reader should have no trouble translating these "descriptions" into actually existing maps in the category.) So D is the set of elements with square zero, in the set case.

We shall assume that the functor $\times D \colon \boldsymbol{E} \to \boldsymbol{E}$ has a right adjoint $(\cdot)^D$, the adjointness being given by the end adjunction, which we denote ev:

$$M^D \times D \xrightarrow{\mathrm{ev}_M} M .$$

We consider the map

$$A \times A \times A \to A$$

given by the description

$$(a_1, a_2, a_3) \mapsto a_1 + (a_2 \cdot a_3) .$$

Since D is a subobject of A, we get by restriction a map

$$A \times A \times D \xrightarrow{\check{\alpha}} A$$

(given by the same description). The exponential adjoint of that is denoted α:

$$A \times A \xrightarrow{\alpha} A^D . \tag{1.0}$$

In section 2, we shall introduce the only axiom we need, namely invertibility of α; in this paragraph, however, we deduce properties of α not depending on that.

We can equip $A \times A$ with the "ring of dual numbers" structure, with addition described coordinatwise, and multiplication

$$(A \times A) \times (A \times A) \xrightarrow{*} A \times A$$

given by the description

$$((a,b),(c,d)) \mapsto (a\cdot c, a\cdot d + b\cdot c)\,.$$

Also, we can equip A^D with a ring structure, induced from that of A; the multiplication is the composite

$$A^D \times A^D \cong (A\times A)^D \xrightarrow{m^D} A^D\,,$$

similarly for addition.

PROPOSITION 1. *The map α is a ring-homomorphism with respect to the two ring structures just described.*

PROOF. Proving that α preserves multiplication means that we should prove

$$\begin{array}{ccc}
(A\times A)\times(A\times A) & \xrightarrow{\alpha\times\alpha} & A^D\times A^D \\
 & & \wr\!\!\parallel \\
\Big\downarrow * & & (A\times A)^D \\
 & & \Big\downarrow m^D \\
A\times A & \xrightarrow{\quad\alpha\quad} & A^D
\end{array}$$

commutative. We take the exponential adjoints of the two legs of the diagram. The counterclockwise composite is immediately seen to give $\check{\alpha}\circ(*\times D)$, whereas the clockwise, with a little effort, is seen to give the composite

(1.1) $$(A\times A)\times(A\times A)\times D \xrightarrow{1\times\Delta} (A\times A)\times(A\times A)\times(D\times D)$$
$$\cong (A\times A\times D)\times(A\times A\times D) \xrightarrow{\check{\alpha}\times\check{\alpha}} A\times A \xrightarrow{m} A\,,$$

where the isomorphism indicated is given by shuffling the D's in between the A's, but keeping the relative order of the D's and of the A's. To compare the composite with $\check{\alpha}\circ(*\times D)$, we compare their descriptions:

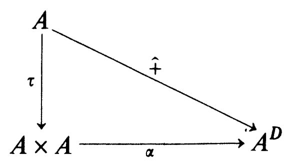
(1.2) $$(a_1,a_2,a_3,a_4,d) \mapsto a_1\cdot a_3 + (a_1\cdot a_4 + a_2\cdot a_3)\cdot d$$

by $\check{\alpha}\circ(*\times D)$, whereas the description for (1.1) is

$$(a_1,a_2,a_3,a_4,d) \mapsto \check{\alpha}(a_1,a_2,d)\cdot\check{\alpha}(a_3,a_4,d)$$
$$= (a_1 + a_2\cdot d)\cdot(a_3 + a_4\cdot d)\,.$$

Multiplying this out, and using $d\cdot d = 0$ gives the right hand side in (1.2). The proof that α preserves addition and 1 is similar (does not use $d\cdot d=0$). (It is well-known that for diagrams of the type considered here, an "elementwise" proof like the one given here is immediately translatable into diagrams which then provide the "real" proof.)

Proposition 2. *Let* $\tau\colon A \to A \times A$ *denote the map with description* $a \mapsto (a, 1)$. *Then the diagram*



commutes.

Proof. Pass to exponential adjoints; the lower map then becomes the composite $(a, d) \mapsto \check{\alpha}(a, 1, d) = a + d$, which is the exponential adjoint of $\hat{+}$.

2. The axiom: invertibility of α.

In this paragraph, we assume

Axiom. *The map* α *in* (1.0) *is invertible.*

One could be more elaborate and include the structure of the surrounding category ***E*** into the axiom; doing that, one would say: a ring object A in a category ***E*** with finite inverse limits *satisfies the Axiom* or is of *line type* iff the equalizer of the squaring map and the zero map $A \to A$ is exponentiable, and the map $\alpha\colon A \times A \to A^D$ is invertible. Doing this, it makes sense to state:

Proposition 3. *The ring object* $\mathbf{Z}[X]$ *in the dual of the category of commutative rings satisfies the Axiom.* (*The ring structure on* $\mathbf{Z}[X]$ *in this dual category is of course the standard coring structure* $\mathbf{Z}[X] \to \mathbf{Z}[X] \otimes \mathbf{Z}[X]$ *in the category of rings.*)

We leave the proof of this Proposition to section 3, and now turn to the derivation of consequences of the Axiom.

We immediately get two maps β and γ from A^D to A:

$$\beta\colon\ A^D \xrightarrow{\alpha^{-1}} A \times A \xrightarrow{\mathrm{proj}_0} A$$

$$\gamma\colon\ A^D \xrightarrow{\alpha^{-1}} A \times A \xrightarrow{\mathrm{proj}_1} A\,.$$

The map β does not really depend on the Axiom:

PROPOSITION 4. *The map* β *equals*

$$A^D \xrightarrow{A^{\ulcorner 0 \urcorner}} A^1 \cong A \,.$$

PROOF. It suffices to prove the diagram

$$(2.1) \qquad \begin{array}{ccc} A \times A & \xrightarrow{\alpha} & A^D \\ {\scriptstyle \mathrm{proj}_0} \downarrow & & \downarrow {\scriptstyle A^{\ulcorner 0 \urcorner}} \\ A & \xrightarrow[\cong]{} & A^1 \end{array}$$

commutative. Passing to exponential adjoints, we get two maps $A \times A \times 1 \to A$ which we should prove equal. One of them (corresponding to the counter-clockwise composite in (2.1)) is just proj_0. The other one is (using extra-ordinary naturality of ev (see e.g. [6], IX,4])):

$$\begin{aligned} \mathrm{ev}_A \circ A^{\ulcorner 0 \urcorner} \times 1 \circ \alpha \times 1 &= \mathrm{ev}_A \circ A^D \times \ulcorner 0 \urcorner \circ \alpha \times 1 \\ &= \mathrm{ev}_A \circ \alpha \times D \circ A \times A \times \ulcorner 0 \urcorner \\ &= \check{\alpha} \circ A \times A \times \ulcorner 0 \urcorner \,, \end{aligned}$$

and this latter map is analyzed by the description $a_1 + (a_2 \cdot 0)$.

This proves the Proposition. On the other hand, the map γ above is new, and should be thought of as "enlarging the infinitesimal into visible size", and is the map which leads to the crucial construction of *derivative* or *differentiation*:

DEFINITION 5. Given a map $f\colon A \to A$, its *derivative* f' is the composite map

$$A \xrightarrow{\hat{+}} A^D \xrightarrow{f^D} A^D \xrightarrow{\gamma} A \,.$$

We can now prove

PROPOSITION 6 (First Taylor Lemma). *For any map* $f\colon A \to A$, *the two maps* $A \times D \to A$ *with descriptions respectively* $(a,d) \to f(a+d)$ *and* $(a,d) \to f(a) + d \cdot f'(a)$ *are equal. In short form*

$$f(a+d) = f(a) + d \cdot f'(a) \,.$$

(The reader will recognize the beginning of a Taylor expansion of f in a in this formula.)

PROOF. It is an easy consequence of Proposition 4 that

$$A \xrightarrow{\hat{+}} A^D \xrightarrow{f^D} A^D \xrightarrow{\beta} A$$

is just f itself. From this, and the definition of f', it follows that

$$\begin{array}{ccc} A & \xrightarrow{\hat{+}} & A^D \\ {\scriptstyle \langle f, f' \rangle}\downarrow & & \downarrow {\scriptstyle f^D} \\ A \times A & \xrightarrow[\alpha]{} & A^D \end{array}$$

commutes. Now pass to exponential adjoints $A \times D \to A$ with the two composite maps of this diagram. The clockwise composite yields

$$\begin{aligned} \mathrm{ev} \circ (f^D \times D) \circ (\hat{+} \times D) &= f \circ \mathrm{ev} \circ (\hat{+} \times D) \\ &= f \circ + \end{aligned}$$

which is the map with description $(a, d) \to f(a+d)$. The counter-clockwise composite yields

$$\check{\alpha} \circ \langle f, f' \rangle \times D$$

whose description is $(a, d) \to \check{\alpha}(f(a), f'(a), d) = f(a) + d \cdot f'(a)$.

Many properties of f' can now be proved from

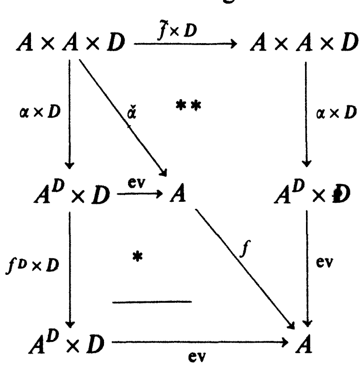
Proposition 7. *For any f: $A \to A$, the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\bar{f}} & A \times A \\ {\scriptstyle \alpha}\downarrow & & \downarrow {\scriptstyle \alpha} \\ A^D & \xrightarrow[f^D]{} & A^D \end{array}$$

commutes, where $\bar{f}$: $A \times A \to A \times A$ is the map with description

$$(a_0, a_1) \to (f(a_0), a_1 \cdot f'(a_0)) .$$

Proof. It suffices to prove that the exponential adjoint diagram commutes. The exponential diagram is the outer diagram in



The "square" * commutes by naturality of ev, and the triangle commutes by definition. So it suffices to prove ** commutative. The two composite maps here have the descriptions

$$(2.2)\qquad (a_0, a_1, d) \to f(a_0 + d \cdot a_1)$$

$$(a_0, a_1, d) \to \check{\alpha}(f(a_0), a_1 \cdot f'(a_0), d) = f(a_0) + d \cdot a_1 \cdot f'(a_0)$$

respectively. But $d \cdot a_1$ has square 0 since d has (formally "$A \times D \to A \times A \to A$ factors through D"), so that the Taylor Lemma can be applied to $f(a_0 + d \cdot a_1)$ to yield the equality of the right hand sides in (2.2).

We can now prove

THEOREM 8. *Given maps f, g: $A \to A$, then*

$$(2.3)\qquad (f+g)' = f' + g'$$

$$(2.4)\qquad (f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(2.5)\qquad (g \circ f)' = (g' \circ f) \cdot f' \quad \text{(chain rule)}.$$

(Here $+$ and $\cdot$ denotes the ring structure on the set hom (A, A) derived from the ring object structure on the co-domain ring object. Similarly for $+$ and $\cdot$ on hom (A, A^D) in (2.6) below.)

PROOF. Since, with the "ring-of-dual-numbers" ring structure on $A \times A$, $\text{proj}_1: A \times A \to A$ is a derivation with respect to $\text{proj}_0: A \times A \to A$, and since α is a ring isomorphism (Proposition 1), we get that $\gamma: A^D \to A$ is a derivation with respect to $\beta: A^D \to A$. Since

$$(2.6)\qquad (f+g)^D = f^D + g^D \quad \text{and} \quad (f \cdot g)^D = f^D \cdot g^D,$$

we get

$$\begin{aligned}(f+g)' &= \gamma \circ (f+g)^D \circ \hat{+} \\ &= \gamma \circ (f^D + g^D) \circ \hat{+} \\ &= (\gamma \circ f^D \circ \hat{+}) + (\gamma \circ g^D \circ \hat{+}) \\ &= f' + g'\end{aligned}$$

(the third equality sign using that γ is an additive homomorphism). Similarly

$$\begin{aligned}(f \cdot g)' &= \gamma \circ (f \cdot g)^D \circ \hat{+} \\ &= \gamma \circ (f^D \cdot g^D) \circ \hat{+} \\ &= ((\gamma \circ f^D) \cdot (\beta \circ g^D) + (\beta \circ f^D) \cdot (\gamma \circ g^D)) \circ \hat{+}\end{aligned}$$

$$= (\gamma \circ f^D \circ \hat{+}) \cdot (\beta \circ g^D \circ \hat{+}) + (\beta \circ f^D \circ \hat{+}) \cdot (\gamma \circ g^D \circ \hat{+})$$
$$= f' \cdot g + f \cdot g' .$$

Again, the third equality sign uses that γ is a derivation with respect to β, and the last equality follows from the definition of f' and g', and from $\beta \circ g^D \circ \hat{+} = g$ (this equality we noted in the proof of Proposition 6).

This proves (2.3) and (2.4). To see the chain rule (2.5), we observe that (with the notation of Proposition 7), $(g \circ f)^{\sim} = \tilde{g} \circ \tilde{f}$ (because of invertibility of α and because $(g \circ f)^D = g^D \circ f^D$).

Now $(g \circ f)^{\sim}$ and $\tilde{g} \circ \tilde{f}$ have descriptions

$$(a_0, a_1) \to (g(f(a_0)), a_1 \cdot (g \circ f)'(a_0))$$

and

$$(a_0, a_1) \to \tilde{g}(f(a_0), a_1 \cdot f'(a_0))$$
$$= (g(f(a_0)), a_1 \cdot f'(a_0) \cdot g'(f(a_0))) .$$

Comparing the second coordinates, we see that the two maps

$$A \times A \to A \tag{2.8}$$

with descriptions

$$(a_0, a_1) \to a_1 \cdot (g \circ f)'(a_0)$$
$$(a_0, a_1) \to a_1 \cdot f'(a_0) \cdot g'(f(a_0))$$

are equal. If we now "put $a_1 = 1$", that is, compose (2.8) on the left with τ: $A \to A \times A$ (given by $a_0 \to (a_0, 1)$), we get equality of two maps $A \to A$, these two maps being precisely those of (2.5). This proves the Theorem.

3. Ring objects of line type.

Let $\boldsymbol{R}$ denote the category of finitely presented commutative rings, and let $\mathscr{X}$ denote the category of (covariant) functors from $\boldsymbol{R}$ to the category of sets, $\mathscr{S}$. (Then $\mathscr{X}$ is the classifying topos for the notion of commutative ring, see [2] or [4].) We have the Yoneda embedding

$$y \colon \boldsymbol{R}^{\mathrm{op}} \to \mathscr{X}$$
$$B \mapsto [B, -]$$

(square brackets denoting the hom-functor $\boldsymbol{R}^{\mathrm{op}} \times \boldsymbol{R} \to \mathscr{S}$). The forgetful-functor $A \colon \boldsymbol{R} \to \mathscr{S}$ is a ring-object in $\mathscr{X}$, and may be identified with $y(\mathbf{Z}[X])$. (It is the *generic* commutative ring object in the sense of classifying topos.)

Now $\mathscr{X}$ is a cartesian closed category, and has finite inverse limits, so D exists and is exponentiable. The inclusion $D \to A$ may be identified with y applied to the map in $\boldsymbol{R}$

$$\mathsf{Z}[X] \to \mathsf{Z}[\varepsilon] = \mathsf{Z}[X]/(X^2) \tag{3.1}$$

which sends X to ε.

If $M \in \mathscr{X}$ and $C \in \boldsymbol{R}$, it is easy to describe the object $M^{y(C)}$ in $\mathscr{X}$. It is the functor $\boldsymbol{R} \to \mathscr{S}$, which to $B \in \boldsymbol{R}$ associates $M(B \otimes C)$. And the evaluation map

$$M^{y(C)} \times y(C) \xrightarrow{\text{ev}} M \tag{3.2}$$

is the natural transformation whose Bth component ev^B ($B \in \boldsymbol{R}$) is the map (in $\mathscr{S}$)

$$M(B \otimes C) \times [C, B] \to M(B)$$

which to the pair $\langle t, h \rangle$ with $t \in M(B \otimes C)$ and $h\colon C \to B$ associates $M(\bar{h})(t)$, where $\bar{h}$ is the composite ring map

$$B \otimes C \xrightarrow[B \otimes h]{} B \otimes B \to B\,.$$

With this, we can now describe the map $\alpha\colon A \times A \to A^D$ in explicit set-theoretic terms:

PROPOSITION 9. *Let $B \in |\boldsymbol{R}|$. Then the map α_B*

$$\alpha_B\colon A(B) \times A(B) \to A(B \otimes \mathsf{Z}[\varepsilon])$$

is given by

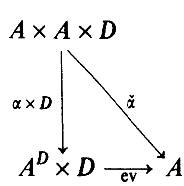
$$\langle b_1, b_2 \rangle \mapsto (b_1 \otimes 1) + (b_2 \otimes \varepsilon)\,.$$

Identifying $B \otimes \mathsf{Z}[\varepsilon]$ with $B[\varepsilon]$ (the "ring-of-dual-numbers" over B), the map described is

$$\langle b_1, b_2 \rangle \mapsto b_1 + \varepsilon \cdot b_2\,.$$

PROOF. We have to see that the map α as described in the Proposition has the property that it makes the following diagram commutative:

(3.3)



where ev is as described in (3.2) ($D=y(\mathbf{Z}[\varepsilon])$), and the Bth component of $\check{\alpha}$ ($B \in |\boldsymbol{R}|$) is given by

$$B \times B \times \{b \in B \mid b^2=0\} \to B$$

$$(b_1, b_2, b_3) \mapsto b_1 + b_2 \cdot b_3 \ .$$

Now the Bth component of the composite map in (3.3) is given by the composite

$$A(B) \times A(B) \times D(B) \to A(B \otimes \mathbf{Z}[\varepsilon]) \times D(B)$$
$$= A(B \otimes \mathbf{Z}[\varepsilon]) \times [\mathbf{Z}[\varepsilon], B] \xrightarrow{\text{ev}^B} A(B)$$

which takes (a_1, a_2, a_3) (with $a_3^2=0$) to $A(\bar{h})(t)$ where

$$t = a_1 \otimes 1 + a_2 \otimes \varepsilon \in A(B \otimes \mathbf{Z}[\varepsilon]) = B \otimes \mathbf{Z}[\varepsilon] \ ,$$

and where $\bar{h}$ is the composite map

$$B \otimes \mathbf{Z}[\varepsilon] \xrightarrow[B \otimes h]{} B \otimes B \to B \ , \tag{3.4}$$

where h is the ring map $\mathbf{Z}[\varepsilon] \to B$ given by $h(\varepsilon)=a_3$. Now the value of (3.4) at $a_1 \otimes 1 + a_2 \otimes \varepsilon$ is $a_1 + a_2 \cdot a_3$. This proves the Proposition.

THEOREM 10. *The ring object A in $\mathscr{X}$ (i.e. the generic comutative ring object) is of line type.*

PROOF. We just analyzed that α is the natural transformation with B-component given in Proposition 9, but that is a bijective map (the canonical bijection $B \times B \cong B[\varepsilon]$), so α is invertible in $\mathscr{X}$.

REMARK 11. It follows from well-known commutative algebra that $\mathbf{Z}[\varepsilon]$ is exponentiable in the category $\boldsymbol{R}^{\mathrm{op}}$ itself (for a $C \in \boldsymbol{R}$, the exponential is the symmetric C-algebra on the Kähler differentials of C, see e.g. [3, Proposition T.9]). Since also $\boldsymbol{R}^{\mathrm{op}}$ has finite inverse limits, and $y\colon \boldsymbol{R}^{\mathrm{op}} \to \mathscr{X}$, like any Yoneda embedding, preserves those finite inverse limits and exponentials that exist, the analysis in Proposition 9 and the proof of the theorem also give that the ring-object $\mathbf{Z}[X]$ in $\boldsymbol{R}^{\mathrm{op}}$ is of line type.

Returning to the notation of Theorem 10, we thus have, for $A=y(\mathbf{Z}[X])$ in $\mathscr{X}$ a differentiation process which to a map $f\colon A \to A$ yields a new map $f'\colon A \to A$. Now, since y is full and faithful (or by Remark 11), we also have a process, which to a map $\varphi\colon \mathbf{Z}[X] \to \mathbf{Z}[X]$ yields another map $\varphi'\colon \mathbf{Z}[X] \to \mathbf{Z}[X]$. Now a ring map $\varphi\colon \mathbf{Z}[X] \to \mathbf{Z}[X]$ is completely given by $\varphi(X) \in \mathbf{Z}[X]$. We can now prove that the element $\varphi'(X) \in \mathbf{Z}[X]$ actually is the derived polynomial, so that the Definition 5 is correct. To see this, we note that multiplication and addition of maps f correspond to multiplication and addition of the corresponding polynomials $\varphi(X)$. So by Theorem 10, it suffices to prove the assertion for one single polynomial, namely X, which corresponds to the identity map id: $A \to A$. So we should prove that id': $A \to A$ equals $A \to 1 \xrightarrow{\ulcorner 1 \urcorner} A$. But id^D is the identity map of A^D, so that id' is given by

$$\gamma \circ \mathrm{id}^D \circ \hat{+} = \gamma \circ \hat{+} = \gamma \circ \alpha \circ \alpha^{-1} \circ \hat{+}$$

$$= \mathrm{proj}_1 \circ \tau ,$$

using the definition of γ, and Proposition 2. But from the description of τ in Proposition 2, we see that

$$\mathrm{proj}_1 \circ \tau = (A \to 1 \xrightarrow{\ulcorner 1 \urcorner} A) .$$

This establishes what we may briefly express

PROPOSITION 12. *For the ring af line type of Theorem* 10 *or Remark* 11, *the differentiation process considered here coincides with usual formal differentiation of polynomials.*

Let us finally remark that in any cartesian category, the terminal object 1 carries a unique (commutative) ring structure, and, with this, is of line type. I believe that in the category of sets, this trivial case in the *only* ring of line type. At least, I can prove that if A is a ring of line type which is not the null ring, then D must contain infinitely many elements, and therefore the isomorphism $A \times A \cong A^D$ implies that A is uncountable.

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