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EXTREME BOUNDARIES AND CONTINUOUS AFFINE FUNCTIONS

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1. Introduction.

In this paper we shall use the term *continuous function space* to denote a pair (X,A) of a Hausdorff space X and a subspace A of the linear space $C_b(X)$ of all continuous and bounded real-valued functions on X. Two continuous function spaces (X,A) and (Y,B) are said to be isomorphic if there exists a homeomorphism $\varrho: X \to Y$ such that the adjoint map $\varrho^*: B \to A$ is a (surjective) norm- and order preserving linear isomorphism. We will study the following problem:

Under which conditions on X and A does there exist a compact convex K of a locally convex Hausdorff vector space over the reals such that (X,A) is isomorphic to $(\partial_e K,\ A(K) \mid \partial_e K)$?

Here $\partial_e K$ denotes the extreme boundary of K (i.e. the set of extreme points), $A(K)$ is the space of continuous and affine real-valued functions on K, and $A(K) \mid \partial_e K$ is the space of all restrictions $f \mid \partial_e K$, $f \in A(K)$. We are also interested in additional conditions on X and A ensuring that (X,A) is isomorphic to $(\partial_e K, A(K) \mid \partial_e K)$ with K a simplex.

Part 2 of this paper contains necessary and sufficient conditions for the existence of isomorphisms of the abovementioned type. In part 3 we consider some examples, and finally we give a slight extension of a recent theorem of Haydon [4].

Unless explicitly stated we use the terminology of Alfsen [1] in convexity theory and Willard [7] in topology. The author would like to express here his gratitude to E. M. Alfsen for enlightening and helpful discussions on the subject.

2. The main results.

We start by finding some necessary conditions for (X,A) to be isomorphic to some $(\partial_e K, A(K) \mid \partial_e K)$. We will say that A^+ (the set of non-negative functions

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in A) *separates points and closed sets* if for any given $\varepsilon>0$ and for any closed set $F\subset X$ and any point $x\in X\setminus F$, there exists $f\in A^+$ satisfying:

$$(2.1)\qquad f(x)<\varepsilon,\quad f\geqq 1 \text{ on } F.$$

Let K be a non-empty compact convex subset of a locally convex Hausdorff vector space E over the reals. We put $A=A(K)\,|\,\partial_e K$. Clearly, A is a norm-closed linear subspace of $C_b(\partial_e K)$ containing the constants. Furthermore, we have the following important property of A:

2.1. Proposition. *A^+ separates points and closed sets in $\partial_e K$.*

Proof. Let $\varepsilon>0$ be given, assume that $F\subset\partial_e K$ is closed (in the relative topology), and let x be a point in $\partial_e K\setminus F$. Denote by F_1 the closed convex hull of F. By Milman's theorem we have $x\notin F_1$. χ_{F_1} is upper semi-continuous, so by proposition I.4.1 of [1] we may choose $f\in A(K)$ such that $f\geqq\chi_{F_1}$ and $\varepsilon>f(x)\geqq\hat{\chi}_{F_1}(x)=\chi_{F_1}(x)=0$. This f clearly meets the requirements of the proposition.

We conclude from proposition 2.1 that if X is a Hausdorff topological space and A a linear subspace of $C_b(X)$, then the following conditions (2.2), (2.3), and (2.4) are necessary for (X,A) to be isomorphic to some $(\partial_e K, A(K)\,|\,\partial_e K)$:

(2.2) A contains the constant functions.
(2.3) A is norm-closed.
(2.4) A^+ separates points and closed sets.

The following lemma is derived from the important property (2.4) and will be needed later.

2.2. Lemma. *Let (X,A) be a continuous function space such that A contains the constant functions and such that A^+ separates points and closed sets. For each upper semi-continuous real-valued function g bounded above on X, we have*

$$g(x)=\inf\{f(x) : f\in A,\ g\leqq f\}$$

for all $x\in X$.

Proof. Let $g\colon X\to\mathbf{R}$ be upper semicontinuous and bounded above, and let $x\in X$ be arbitrary. We put $g(x)=\lambda$. Given $\varepsilon>0$ we choose an open neighbourhood U of x such that $g<\lambda+\frac{1}{2}\varepsilon$ everywhere on U. We also choose $M\geqq 0$ such that $g<M$ everywhere on X. Since A^+ separates points and closed sets and since A contains the constants, we can find $h\in A$ such that $h\geqq\lambda+\frac{1}{2}\varepsilon$ on X, $h(x)<\lambda+\varepsilon$, and $h\geqq M$ on $X\setminus U$. Now $g\leqq h$ on X and $h(x)<g(x)+\varepsilon$. (In fact, we have $g<h$, but this strict inequality is not needed.) This proves the lemma.

An immediate consequence of the above lemma is the following corollary which will be needed in the proof of our main theorem.

2.3. COROLLARY. *Let (X, A) be as in lemma 2.2. Then for any real number α and any $x \in X$ the set $A_{x,\alpha} = \{h \in A : h(x) > \alpha\}$ is directed downwards and for any $y \in X$, $y \neq x$,*

$$\inf\{h(y) : h \in A_{x,\alpha}\} \leqq 0 .$$

PROOF. Let $h_1, h_2 \in A_{x,\alpha}$ be given and let $y \in X$, $y \neq x$, be arbitrary. Define $g: X \to \mathbf{R}$ by

$$g(z) = \begin{cases} (h_1 \wedge h_2)(z), & \text{for } z \neq y \\ (h_1 \wedge h_2 \wedge 0)(y), & \text{for } z = y \end{cases}$$

Then g is lower semi-continuous. Hence we may apply lemma 2.2 to $-g$ to obtain $h \in A$ such that $h(x) > \alpha$ and such that $h \leqq g$. Then $h \in A_{x,\alpha}$ and $h \leqq h_1 \wedge h_2$. Furthermore $h(y) \leqq 0$, and our claim is proved.

2.4. DEFINITION. A continuous function space (X, A) has the *Dini property* if for every descending net $\{f_i\} \subset A$ with $\inf f_i = 0$ and for every $\varepsilon > 0$, there exists an index i such that $f_i < \varepsilon$. (X, A) is said to have the *strong Dini property* if the same conclusion holds for every descending net $\{f_i\} \subset A$ such that $\inf f_i \leqq 0$.

It is evident that the strong Dini property implies the Dini property. On the other hand, let A consist of all functions f on $X = [0, 1)$ of the form

$$f(x) = ax + b; \quad a, b \in \mathbf{R},\ x \in X .$$

Then it is easily checked that (X, A) has the Dini property. But it will not have the strong Dini property, consider f.ex. the sequence $\{f_n\}$ given by

$$f_n(x) = \left(n + 1 + \frac{1}{n}\right)x - n, \quad x \in [0, 1) .$$

There is, however, an important special case in which the Dini property implies the strong Dini property, as we shall see in proposition 2.5 below. Recall that (X, A) is said to have the *Riesz interpolation property* if for any $f_1, f_2, g_1, g_2 \in A$ such that $f_1 \vee f_2 \leqq g_1 \wedge g_2$ there exists $h \in A$ such that $f_1 \vee f_2 \leqq h \leqq g_1 \wedge g_2$.

2.5. PROPOSITION. *Let (X, A) be a continuous function space and assume that A contains the constants and that A^+ separates points and closed sets. Assume further that (X, A) has the Riesz interpolation property. Then for (X, A) the Dini property will imply the strong Dini property.*

PROOF. Let $\{f_i\} \subset A$ be a descending net such that $\inf f_i \leqq 0$. Let B consist of all $h \in A$ such that $f_i \vee 0 \leqq h$ for some index i. We claim that B is directed downwards. To verify this claim, we consider $g, h \in B$. Let i and j be indices such that $f_i \vee 0 \leqq g$, $f_j \vee 0 \leqq h$. Since $\{f_i\}$ is descending, there exists an index k such that $f_k \leqq f_i$, $f_k \leqq f_j$. Then $f_k \vee 0 \leqq g \wedge h$, so by the Riesz interpolation property there exists $h' \in A$ such that $f_k \vee 0 \leqq h' \leqq g \wedge h$. Then, by definition, we have $h' \in B$, thus B is directed downward and may be considered as a descending net. By lemma 2.2 we have $\inf_{h \in B} h = 0$. Hence, for every $\varepsilon > 0$, there exists by the Dini property a function $h \in B$ such that $h < \varepsilon$. By definition we have $f_i \vee 0 \leqq h$ for some index i, then clearly also $f_i < \varepsilon$, and the proof is complete.

We are now able to state our main theorem.

2.6. THEOREM. *Let (X, A) be a continuous function space where A is norm-closed and contains the constant functions and where A^+ separates points and closed sets. Then (X, A) is isomorphic to $(\partial_e K, A(K) \mid \partial_e K)$ for some compact convex set K if and only if (X, A) has the strong Dini property.*

REMARK. Professor Alfsen has informed me that the idea to use Dini properties to characterize extreme boundaries has also been considered by St. Raymond (unpublished).

The proof of the above theorem will be broken down into several lemmas. The "only if"-part of the theorem follows from our next result.

2.7. PROPOSITION. *Let K be a compact convex set. Then $(\partial_e K, A(K) \mid \partial_e K)$ has the strong Dini property.*

PROOF. It suffices to consider a descending net $\{f_i\}$ in $A(K)$ such that

$$\inf f_i(x) \leqq 0 \quad \text{for all } x \in \partial_e K .$$

(Note that if $\{f_i \mid \partial_e K\}$ is descending, so is $\{f_i\}$.) The function $f = \inf f_i$ is clearly upper semi-continuous and affine, hence it attains its supremum value at a point in $\partial_e K$. (The set $\{x \in K : f(x) = \sup_{y \in K} f(y)\}$ is a closed face of K and thus must contain an extreme point of K.) Thus we have $f \leqq 0$ on K, and the proposition follows by compactness of K as in the proof of the classical Dini lemma.

We turn to the proof of sufficiency in theorem 2.6. Let (X, A) be a continuous function space where A is norm-closed and contains the constant functions and where A^+ separates points and closed sets. We denote as usual the space of continuous linear functionals on A by A^* and define

$$K = \{\mu \in A^* : \mu(1) = 1 = \|\mu\|\} .$$

Then K is a w^*-compact convex subset of A^*. Furthermore, we define a map $\delta: X \to K$ by putting $\delta(x)(f) = f(x)$ for each $f \in A$, that is, $\delta(x)$ is the Dirac measure at x restricted to A.

2.8. LEMMA. *The map δ is a homeomorphism of X onto its image $\delta(X) \subset K$.*

PROOF. Continuity and injectivity of δ are obvious. An easy argument using (2.4) shows that δ is an open map onto its image.

2.9. LEMMA. $\delta(X) \subset \partial_e K$.

PROOF. Assume $\delta(x) = t\mu + (1-t)\nu$ with $\mu, \nu \in K$ and $0 < t < 1$. We will show $\delta(x) = \mu$.

Let $g \in A$ be arbitrary and put $\lambda = g(x)$. By lemma 2.2 there exists a sequence $\{g_n\}$ from A such that $g_n \geqq g \vee \lambda$ on X and $g_n(x) < \lambda + 1/n$ for $n = 1, 2, \ldots$. Now the functions $f_n = g_n - \lambda \in A$ are all non-negative and

$$t\mu(f_n) + (1-t)\nu(f_n) = f_n(x) \to 0$$

as $n \to \infty$. Hence $\mu(f_n) \to 0$, and so $\mu(g_n) \to \lambda$ as $n \to \infty$. It follows from the inequality $g_n \geqq g$ that $\lambda \geqq \mu(g)$. By symmetry also $\lambda \leqq \mu(g)$. Hence $\delta(x)(g) = \mu(g)$, and the proof is complete.

2.10. LEMMA. *The map $h \to \tilde{h}$ defined by*

$$\tilde{h}(\mu) = \mu(h), \quad \mu \in K \tag{2.5}$$

is a norm-preserving linear order isomorphism of A onto $A(K)$.

PROOF. It is straightforward to check that the map defined in (2.5) is linear and norm-preserving. Hence the map is an isometry, and therefore it is a uniform isomorphism of A onto its image. A is complete, hence $\{\tilde{h} : h \in A\}$ is complete and therefore closed in $A(K)$. To prove surjectivity it thus suffices to prove that $\{\tilde{h} : h \in A\}$ is dense in $A(K)$. But this is an easy consequence of corollary I.1.5 in [1] and the fact that the w^*-continuous linear functionals on A^* are precisely the evaluations at elements of A. Finally one easily checks the equivalence

$$h_1 \leqq h_2 \Leftrightarrow \tilde{h}_1 \leqq \tilde{h}_2 \quad (h_1, h_2 \in A) ,$$

and the proof is complete.

To establish the above three lemmas we have only used the fact that (X, A) satisfies (2.2), (2.3), and (2.4). We now bring in the strong Dini property to finish the proof of theorem 2.6.

PROOF OF THEOREM 2.6. Only the "if"-part of the proof remains. By the above lemmas it is clear that the proof will be complete if we can prove the inclusion $\partial_e K \subset \delta(X)$. Assume then that this inclusion does not hold and select $\mu \in \partial_e K \setminus \delta(X)$. The continuous functions space $(\partial_e K, A(K) \mid \partial_e K)$ satisfies (2.2), (2.3), and (2.4), therefore, using corollary 2.3, we see that

$$\tilde{B} = \{H \mid \partial_e K : H \in A(K),\ H(\mu) > 1\}$$

is directed downwards and that

$$\inf_{H \mid \partial_e K \in \tilde{B}} H(\nu) \leqq 0 \qquad \text{for } \nu \in \partial_e K,\ \nu \neq \mu .$$

From lemma 2.10 it follows that

$$\begin{aligned} B &= \{h \in A : \tilde{h} \mid \partial_e K \in \tilde{B}\} \\ &= \{h \in A : \mu(h) > 1\} \end{aligned}$$

is directed downwards, and thus it may be considered as a descending net $\{h_i\}$. Since $\delta(X) \subset \partial_e K$ by lemma 2.9 and $\mu \notin \delta(X)$ by assumption, we have

$$\inf_i h_i(x) = \inf_i \tilde{h}_i(\delta(x)) \leqq 0$$

for all $x \in X$. By the strong Dini property there then exists an index i such that $h_i < 1$. This means $\mu(h_i) \leqq 1$ contradicting the inequality $\mu(h_i) > 1$. Thus we have proved $\delta(X) = \partial_e K$. Finally we note that by lemma 2.10 the map $\delta^* \colon A(K) \mid \partial_e K \to A$ defined by $\delta^*(H \mid \partial_e K)(x) = H(\delta(x))$ is a (surjective) isomorphism preserving all relevant structure, the inverse isomorphism being $h \to \tilde{h} \mid \partial_e K$. The proof is complete.

It is well known (corollary II. 3.11 in [1]) that a compact convex set K is a simplex if and only if $(K, A(K))$ has the Riesz interpolation property. From theorem 2.6 together with proposition 2.5 and lemma 2.10 the following result will follow.

2.11. COROLLARY. *Let (X, A) be a continuous function space where A is norm-closed and contains the constant functions and where A^+ separates points and closed sets. Then (X, A) is isomorphic to $(\partial_e K, A(K) \mid \partial_e K)$ for some simplex K if and only if (X, A) has the Dini property and the Riesz interpolation property.*

REMARK. In the above corollary the Dini property may be replaced by the following equivalent condition:

(2.6) If $\{f_i\} \subset A$ is a descending net converging pointwise to 0, then $f_i \to 0$ weakly (with respect to the duality (A, A^*)).

Obviously the Dini property implies (2.6), and conversely if (2.6) holds, then we have in particular (in view of lemma 2.10) that $\tilde{f}_i(\mu) = \mu(f_i) \searrow 0$ for each $\mu \in K$. But K is w^*-compact so the classical Dini lemma gives that $\tilde{f}_i \to 0$ uniformly, and the desired conclusion follows.

3. Examples and comments.

The results of the preceding paragraph are clearly related to the following question: Which topological spaces are homeomorphic to extreme boundaries? In view of theorem 2.6 this question is equivalent to finding an intrinsic topological characterization of the Tychonoff spaces X allowing subspaces A of $C_b(X)$ satisfying (2.2), (2.3), and (2.4) and having the strong Dini property.

It is known that all locally compact Hausdorff spaces are homeomorphic to extreme boundaries, and recently Haydon [4] has proved that every Polish space is homeomorphic to the extreme boundary of a simplex. In view of the above remark it is of some interest to give direct constructions of the relevant subspaces A in these cases, thus furnishing alternative proofs that the spaces in question are homeomorphic to extreme boundaries and at the same time giving examples of continuous function spaces (X, A) of a general character satisfying the conditions of theorem 2.6.

Let X be a non-compact, locally compact Hausdorff space. We fix $x_0, y_0 \in X$, $x_0 \neq y_0$, and let A_0 consist of the continuous functions $f\colon X \to \mathbf{R}$ with compact support satisfying $f(x_0)+f(y_0)=0$. We define A to be the norm-closure of the linear space of functions of the form $f+r$ where $f \in A_0$, $r \in \mathbf{R}$. Thus A consists of precisely those continuous $f\colon X \to \mathbf{R}$ satisfying

$$\lim_{x \to \infty} f(x) = \tfrac{1}{2}(f(x_0)+f(y_0)) .$$

3.1. PROPOSITION. *Let X be a non-compact, locally compact Hausdorff space and let A be constructed as above. Then (X, A) satisfies the conditions of theorem 2.6, hence X is homeomorphic to an extreme boundary.*

PROOF. It is trivial to verify that (2.2), (2.3), and (2.4) are satisfied. So let $\{f_i\} \subset A$ be a descending net with $\inf f_i \leqq 0$. Let $\varepsilon > 0$ be given. We put

$$F_i = \{x \in X : f_i(x) \geqq \varepsilon\}$$

and assume $F_i \neq \emptyset$ for all indices i. $\{F_i\}$ is then a filter base. If one of the F_i's is compact, then clearly $\bigcap_i F_i \neq \emptyset$. If no F_i is compact, we have $\lim_{x\to\infty} f_i(x) \geqq \varepsilon$ for all i. On the other hand we have

$$f_i(x_0)+f_i(y_0) = 2 \lim_{x\to\infty} f_i(x) .$$

Thus $f_i(x_0)+f_i(y_0) \geqq 2\varepsilon$ for all i, hence either x_0 or y_0 lies in $\bigcap_i F_i$. It follows that $\bigcap_i F_i$ is nonempty in any case. But for a point x in this intersection we must have $\lim_i f_i(x) \geqq \varepsilon$, a contradiction. Hence $F_i = \emptyset$ for some i, and the proof is complete.

REMARKS. Results similar to the above proposition are proved in [3], [5], and [6].

In case X is a compact Hausdorff space the above construction of A simply gives $A = C(X)$, so in this case we arrive at the pair $(X, C(X))$, and corollary 2.11 is evidently applicable. (In this connection it should be noted that the compact convex set K corresponding to (X, A) of proposition 3.1 is, in fact, also a simplex.)

Our next task will be to give a direct construction of the subspace A when X is a Polish (=completely meterizable and separable) space. Our proof leans heavily on Haydon's construction in [4]: Let X be a Polish space and T a metric compactification of X. X is a G_δ-set in T, so there exist open sets $T = G_0 \supset G_1 \supset \ldots \supset G_n \supset \ldots$ such that $X = \bigcap_{n=0}^{\infty} G_n$. Now there exists for each $n \geqq 0$ a sequence of non-negative functions $\{h_k^n\} \subset C(T)$ such that $\sum_{k=0}^{\infty} h_k^n = \chi_{G_n}$ and

$$\operatorname{diam}(\operatorname{supp} h_k^n) \leqq 2^{-n}\varepsilon_k^n, \qquad k \geqq 0 ,$$

where $\{\varepsilon_k^n\}$ is a double sequence of positive numbers satisfying $\lim_{k\to\infty} \varepsilon_k^n = 0$ for each n. We put

$$p_k^n = h_k^n \chi_{T\setminus G_{n+1}}, \qquad n \geqq 0,\ k \geqq 0 .$$

It is furthermore possible to choose distinct points x_k^n, y_k^n, $n \geqq 0$, $k \geqq 0$, in X such that if $p_k^n \neq 0$ we have $x_k^n, y_k^n \in \operatorname{supp} h_k^n$.

We now define $A(T)$ as the linear space of all $f \in C(T)$ satisfying

$$(3.1) \qquad f(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_k^n(t) \frac{f(x_k^n)+f(y_k^n)}{2} = \sum_{k=0}^{\infty} p_k^{n_0}(t) \frac{f(x_k^{n_0})+f(y_k^{n_0})}{2}$$

for each $t \in T\setminus X$. Here n_0 is the unique integer such that $t \in G_{n_0}\setminus G_{n_0+1}$. We then put

$$(3.2) \qquad A = \{f \mid X : f \in A(T)\} .$$

For any $\mu \in C(T)^*$ we define

$$\gamma(\mu) = \mu - \tfrac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\int p_k^n \, d\mu \right) (\delta(x_k^n) + \delta(y_k^n)) . \tag{3.3}$$

Putting

$$M = \{\gamma(\mu) : \mu \in C(T)^*, \ |\mu|(X) = 0\} ,$$

it is easy to see that $f \in A(T)$ if and only if $\gamma(\mu)(f) = 0$ for all $\gamma(\mu) \in M$, that is, $A(T)$ is the annihilator of M.

Clearly A is a norm-closed subspace of $C_b(X)$ containing the constant functions. The main difficulty is to establish that A^+ separates points and closed sets. The proof rests upon the following crucial result in [4]:

3.2. Lemma (Haydon). *M is w^*-closed.*

For $\varepsilon > 0$, $x \in X$ and an open subset V of T with $x \in V$ we put

$$B(x, V, \varepsilon) = \{f \in C(T) : f < 1, \ f(x) > 1 - \varepsilon, \ f \mid (T \setminus V) < 0\} .$$

Then we have to prove that $B(x, V, \varepsilon) \cap A(T) \neq \emptyset$. We suppose that there exists a triple (x, V, ε) such that $B(x, V, \varepsilon) \cap A(T) = \emptyset$. $B(x, V, \varepsilon)$ is (norm-) open and convex, so by Hahn–Banach separation there exists $\mu \in C(T)^*$ with $\mu(f) > 0$ for all $f \in B(x, V, \varepsilon)$ and $\mu(f) = 0$ for all $f \in A(T)$. This means that μ annihilates $A(T)$, that is, $\mu \in M^{\perp\perp}$. By lemma 3.2, $M^{\perp\perp} = M$, hence $\mu \in M$, i.e. we have $\mu = \gamma(\nu)$ for some $\nu \in C(T)^*$ with $|\nu|(X) = 0$. Thus, for each $f \in C(T)$ we have

$$\mu(f) = \int_{T \setminus X} f \, d\nu - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\int p_k^n \, d\nu \right) \frac{f(x_k^n) + f(y_k^n)}{2} . \tag{3.4}$$

We omit the proof of the following simple lemma.

3.3. Lemma. $\inf\{\mu(f) : f \in B(x, V, \tfrac{1}{2}\varepsilon)\} > 0$.

Assume now that $x \notin \{x_k^n, y_k^n\}$; $n, k \geqq 0$, and choose some ε_1 satisfying

$$0 < \varepsilon_1 \leqq \inf\{\mu(f) : f \in B(x, V, \tfrac{1}{2}\varepsilon)\} .$$

We then select N and K such that (cf. (3.4))

$$\mu(f) = \int_{T \setminus G_{N+1}} f \, d\nu - \sum_{n=0}^{N} \sum_{k=0}^{K} \left(\int p_k^n \, d\nu \right) \frac{f(x_k^n) + f(y_k^n)}{2} + \varepsilon_1(f) \tag{3.5}$$

where $|\varepsilon_1(f)| < \varepsilon_1$ for $f \in C(T)$, $\|f\| \leqq 1$. We choose an open set $U \subset V \cap G_{N+1}$ such that $x \in U$ and

$$U \cap \{x_k^n, y_k^n\}_{\substack{n \leqq N \\ k \leqq K}} = \emptyset .$$

Let $f_0: T \to [0,1]$ be continuous and such that $f_0(x)=1$, $f_0 \mid (T\backslash U)=0$. Then $f_0-\frac{1}{4}\varepsilon \in B(x, V, \frac{1}{2}\varepsilon)$, thus

$$\mu(f_0) = \mu(f_0-\tfrac{1}{4}\varepsilon) \geqq \inf\{\mu(f) : f \in B(x, V, \tfrac{1}{2}\varepsilon)\} \geqq \varepsilon_1 .$$

On the other hand, because of our choice of U and f_0 (3.5) gives $\mu(f_0)=\varepsilon_1(f) < \varepsilon_1$. This contradiction forces us to conclude that, in fact, $B(x, V, \varepsilon) \cap A(T) \neq \emptyset$. Only a minor modification is needed in the above argument if $x \in \{x_k^n, y_k^n\}_{n,k \geqq 0}$, and the details are omitted.

3.4. Proposition. *Let X be a Polish space and let A be defined as in* (3.2). *Then (X, A) satisfies the conditions of theorem* 2.6, *hence X is homeomorphic to an extreme boundary.*

Proof. Only the verification of the strong Dini property remains. Let $\{f_i\}$ be a descending net from A with $\inf f_i \leqq 0$. For each index i let $\bar{f}_i$ be the unique extension of f_i to a member of $A(T)$. Let $\varepsilon > 0$ be arbitrary. Assuming

$$\{x \in X : f_i(x) \geqq \varepsilon\} \neq \emptyset$$

for all indices i, we have by compactness of T that

$$\bigcap_i \{t \in T : \bar{f}_i(t) \geqq \varepsilon\} \neq \emptyset .$$

A point t in this intersection must lie in $T\backslash X$, $t \in G_m \backslash G_{m+1}$, say, since $\inf f_i(x) \leqq 0$ for all $x \in X$. In view of (3.1) we then have

$$\bar{f}_i(t) = \tfrac{1}{2} \sum_{k=0}^{\infty} p_k^m(t)(f_i(x_k^m)+f_i(y_k^m))$$

for each i. We fix an index i_1. Since f_{i_1} is bounded there exists k_1 such that

$$\sum_{k>k_1} p_k^m(t)(f_{i_1}(x_k^m)+f_{i_1}(y_k^m)) < \tfrac{1}{2}\varepsilon .$$

Since $\{f_i\}$ is descending the last inequality subsists for indices $i \geqq i_1$. Thus, for $i \geqq i_1$, we have

$$\bar{f}_i(t) < \sum_{k=0}^{k_1} p_k^m(t)(f_i(x_k^m)+f_i(y_k^m))+\tfrac{1}{2}\varepsilon .$$

However, the finite sum in the last inequality must also be less than $\varepsilon/2$ for i sufficiently large since $\lim f_i \leqq 0$. Hence we have $\bar{f}_i(t) < \varepsilon$ for i sufficiently large, contradicting the fact that $\bar{f}_i(t) \geqq \varepsilon$ for all indices i. The result follows.

REMARK. Let X be Polish and let A be defined by (3.2). Let K be constructed as in the proof of theorem 2.6. Then K is a simplex. The easiest way to prove this seems to be to establish the following: If $\bar{\mu}$ is a boundary measure on K such that $\int_K \tilde{h}\, d\bar{\mu}=0$ for all $\tilde{h} \in A(K)$, then $\bar{\mu}=0$, i.e. we will have no boundary affine dependences on K different from 0. The proof of this fact follows the pattern of the proof of the corresponding statement in [4] and is therefore omitted.

We conclude this paper by sketching the proof of a slight extension of Haydon's result. Let $\{(X_i, A_i)\}_{i\in I}$ be a family of continuous function spaces satisfying the conditions of theorem 2.6. Let $\sum_{i\in I} X_i$ denote the disjoint union of the X_i's. We may assume without loss of generality that the X_i's are mutually disjoint and that at least one X_{i_0} contains more than one point. We then pick $x_0, y_0 \in X_{i_0}$, $x_0 \neq y_0$, and let A_0 consist of all $f\colon \sum_{i\in I} X_i \to \mathbf{R}$ such that $f \mid X_i \in A_i$ for all $i \in I$ and such that there exists a finite subset J of I with $i_0 \in J$ and

$$f\Big|\bigcup_{i\in I\setminus J} X_i = \tfrac{1}{2}(f(x_0)+f(y_0))\,.$$

The proof that the norm-closure A of A_0 does indeed satisfy (2.2), (2.3), and (2.4) and that $(\sum_{i\in I} X_i, A)$ has the strong Dini property is a straightforward (but lengthy) verification. Furthermore, it is trivial to check that if each (X_i, A_i) has the Riesz interpolation property, so has $(\sum_{i\in I} X_i, A)$.

We say that a topological space X is *locally separable* if each point in X has an open separable neighbourhood. Then, mimicking the proof that each paracompact, locally compact Hausdorff space is the disjoint union of locally compact, σ-compact, Hausdorff spaces (cf. [2, pp. 241–242]), we see that each completely metrizable, locally separable space is the disjoint union of Polish spaces. In view of the above remark the following result is then clear:

3.5. PROPOSITION. *Every completely metrizable, locally separable space is homeomorphic to the extreme boundary of a simplex.*

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