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CROSSED PRODUCTS, DIRECT INTEGRALS AND CONNES' CLASSIFICATION OF TYPE III FACTORS

COLIN SUTHERLAND

Recently, in [1], A. Connes has given a classification of factors of type III (acting on separable Hilbert spaces) into those of type III_λ $(0 \leqq \lambda \leqq 1)$. The primary object of this note is to show that if a separable Hilbert space $\mathscr{H}$, and a Borel set $B \subset [0,1]$ are given, then the set of factors on $\mathscr{H}$ of type III_λ for some $\lambda \in B$ is Borel with respect to the Effros Borel structure.

The literature of the subject contains references to similar problems, and, in particular the papers of O. Nielsen [6] and O. Marechal [5] should be mentioned. For $x \in (0,\frac{1}{2})$, let $\mathscr{R}_x$ be the Powers factor, so that $\mathscr{R}_x$ is of type III_λ where

$$\lambda = \frac{1-2x}{1+2x} \qquad (x \in [0,\tfrac{1}{2}))$$

(see [2]). Nielsen has shown that if $\mathscr{F}$ denotes the set of all type III factors on a given separable Hilbert space $\mathscr{H}$ then $\{\mathscr{M} \in \mathscr{F} : \mathscr{M} \otimes \mathscr{R}_x \cong \mathscr{M}\}$ is analytic in $\mathscr{F}$; the question of whether or not this set is Borel or not is closely related with our work here. Marechal has described a topology on $\mathscr{F}$ which generates the Effros Borel structure, and with respect to which the map $x \in (0,\frac{1}{2}) \to \mathscr{R}_x$ is of first Baire class. We shall show that the set

$$\{(\mathscr{M},\lambda) : \ \mathscr{M} \in \mathscr{F}, \text{ and } \mathscr{M} \text{ is of type } \mathrm{III}_\lambda\}$$

is Borel in $\mathscr{F} \times [0,1]$; the map $x \to \mathscr{R}_x$ is an explicit cross-section of this set.

The principal tools for the investigation are Takesaki's ([9]) characterization of type III_λ factors in terms of crossed products and an easy modification of the authors results on measurable fields of modular automorphisms [8]. We first develop some results of a general nature concerning "Borel fields" of automorphism groups, and crossed products; although these results are essentially trivial they are vital for later arguments, and will also be used in a forthcoming paper analyzing the type of the left regular representation of certain groups.

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1. Crossed products and direct integrals.

Let G be a locally compact group, and $\mathscr{M}$ a von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. We denote by $\operatorname{Aut}(\mathscr{M})$ the group of all (*)-automorphisms of $\mathscr{M}$. A *continuous action* of G on $\mathscr{M}$ means a homomorphism $\alpha: g \in G \to \alpha_g \in \operatorname{Aut}(\mathscr{M})$ which is continuous in the sense that for each $x \in \mathscr{M}$ the map $g \to \alpha_g(x)$ is σ-strong *-continuous. We recall that in this situation we may construct the crossed product $\mathscr{R}(\mathscr{M};\alpha)$, of $\mathscr{M}$ by α; it is the von Neumann algebra on $L^2(\mathscr{H}; G)$ generated by the operators

$$(\pi(\alpha)(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g) \qquad x \in \mathscr{M}$$

and

$$(\lambda(h)\xi)(g) = \xi(h^{-1}g) \qquad h \in G$$

where $\xi \in L^2(\mathscr{H}; G)$.

We shall restrict attention to von Neumann algebras $\mathscr{M}$ on separable Hilbert spaces, and to separable locally compact groups; we shall omit the qualification. For the theory of Borel fields of von Neumann algebras we refer to [3] and [7]. Throughout Γ will denote a standard Borel space; by a measure on Γ we will mean a Borel measure.

DEFINITION 1.1. Let G be a locally compact group, and $\{\gamma \to \mathscr{M}(\gamma) : \gamma \in \Gamma\}$ a Borel field of von Neumann algebras. For each γ, let α^γ be a continuous action of G on $\mathscr{M}(\gamma)$. We shall say $\gamma \to \alpha^\gamma$ is a Borel field of continuous actions if, for each $g \in G$, and each Borel operator field $\gamma \to x(\gamma) \in \mathscr{M}(\gamma)$, the operator field $\gamma \to \alpha_g^\gamma(x(\gamma))$ is Borel.

It follows readily (from e.g. [4]) that if $\gamma \to \alpha^\gamma$ is a Borel field of continuous actions of G on $\{\gamma \to \mathscr{M}(\gamma): \gamma \in \Gamma\}$, and if $\gamma \to x(\gamma) \in \mathscr{M}(\gamma)$ is a Borel operator field, then the map

$$(g,\gamma) \in G \times \Gamma \to \alpha_g^\gamma(x(\gamma))$$

is Borel as a function of two variables.

The proofs of the following propositions are trivial and left to the reader.

PROPOSITION 1.2. *Let $\gamma \to \alpha^\gamma$ be a Borel field of continuous actions of G on $\{\gamma \to \mathscr{M}(\gamma) : \gamma \in \Gamma\}$. Then the field of von Neumann algebras $\gamma \to \mathscr{R}(\mathscr{M}(\gamma);\alpha^\gamma)$ is Borel.*

DEFINITION 1.3. Let $\gamma \to \alpha^\gamma$ be a Borel field of continuous actions of G on $\{\gamma \to \mathscr{M}(\gamma) : \gamma \in \Gamma\}$, and let μ be a measure on Γ. Put $\mathscr{M} = \int_\Gamma^\oplus \mathscr{M}(\gamma)\,d\mu(\gamma)$. For $x \in \mathscr{M}$ and $g \in G$, define

$$\alpha_g(x) = \int_\Gamma^\oplus \alpha_g^\gamma(x(\gamma))\,d\mu(\gamma)\,,$$

where $x=\int_\Gamma^\oplus x(\gamma)\,d\mu(\gamma)$. We write $\alpha_g=\int_\Gamma^\oplus \alpha_g^\gamma\,d\mu(\gamma)$.

PROPOSITION 1.4. *Let $\gamma \to \alpha^\gamma$, $\{\gamma \to \mathscr{M}(\gamma) : \gamma \in \Gamma\}$, $\mathscr{M}$ and $\alpha_g=\int_\Gamma^\oplus \alpha_g^\gamma\,d\mu(\gamma)$ be as in* 1.3. *Then*

i) *α is a continuous action of G on $\mathscr{M}$; we write $\alpha=\int_\Gamma^\oplus \alpha^\gamma\,d\mu(\gamma)$.*
ii) *$\mathscr{R}(\int_\Gamma^\oplus \mathscr{M}(\gamma)\,d\mu(\gamma), \int_\Gamma^\oplus \alpha^\gamma\,d\mu(\gamma))$ is unitarily equivalent with $\int_\Gamma^\oplus \mathscr{R}(\mathscr{M}(\gamma);\alpha^\gamma)\,d\mu(\gamma)$.*

Suppose now that the group G in question is *abelian*; and α is a continuous action of G on the von Neumann algebra $\mathscr{M}$. Following [9], we define a continuous action θ of the dual $\hat{G}$ of G on $\mathscr{R}(\mathscr{M};\alpha)$ by

$$\theta_p(y) = v(p)yv(p)^*$$

where

$$(v(p)\xi)(g) = \overline{\langle p,g\rangle}\xi(g)\,,$$

$p \in \hat{G}$, $g \in G$, $\xi \in L^2(\mathscr{H};G)$. It is readily verified that

$$(*)\qquad \begin{cases} \theta_p(\pi(\alpha)(x)) = \pi(\alpha)(x), & x \in \mathscr{M}\,,\\ \theta_p(\lambda(g)) = \overline{\langle p,g\rangle}\lambda(g), & g \in G\,. \end{cases}$$

Using the identities (*) it is trivial to verify,

PROPOSITION 1.5. *Let $\gamma \to \alpha^\gamma$ be a Borel field of continuous actions of the locally compact separable abelian group G on $\{\gamma \to \mathscr{M}(\gamma) : \gamma \in \Gamma\}$. For each $\gamma \in \Gamma$, let θ^γ denote the continuous action of $\hat{G}$ on $\mathscr{R}(\mathscr{M}(\gamma);\alpha^\gamma)$ dual to α^γ. Then $\gamma \to \theta^\gamma$ is Borel. Furthermore, if $\theta=\int_\Gamma^\oplus \theta^\gamma\,d\mu(\gamma)$, then θ is dual to $\alpha=\int_\Gamma^\oplus \alpha^\gamma\,d\mu(\gamma)$ on $\mathscr{R}(\mathscr{M};\alpha)$.*

2. Connes' classification of type III factors.

Let $\mathscr{H}$ be a fixed separable (infinite dimensional) Hilbert space, and $\mathscr{F}$ denote the set of all type III factors on $\mathscr{H}$. It is known (see [7]) that $\mathscr{F}$, equipped with the relative Effros Borel structure is a standard Borel space. We shall consider $\mathscr{F}$ as the base space for a certain Borel field of type III factors, namely $\mathscr{M} \in \mathscr{F} \to \mathscr{M}$.

Let ω be a faithful normal state on $\mathscr{L}(\mathscr{H})$, the set of all bounded operators on $\mathscr{H}$. The restriction $\omega|_\mathscr{M}$ of ω to $\mathscr{M} \in \mathscr{F}$ is again faithful and normal, and thus uniquely determines a 1-parameter automorphism group of $\mathscr{M}$ i.e. the modular automorphism group of $\mathscr{M}$ associated with $\omega|_{\mathscr{M}}$. We denote this group by $\{\sigma_t^{\mathscr{M}} : t \in \mathbf{R}\}$. Also, we denote the representation of $\mathscr{M}$ deduced from $\omega|_{\mathscr{M}}$ by $\pi_{\mathscr{M}}$, and the modular automorphism group of $\pi_{\mathscr{M}}(\mathscr{M})$ by $\tilde{\sigma}_t^{\mathscr{M}}$.

LEMMA 2.1. i) *The field* $\mathscr{M} \in \mathscr{F} \to \pi_{\mathscr{M}}(\mathscr{M})$ *of von Neumann algebras is Borel.* ii) *The field* $\mathscr{M} \in \mathscr{F} \to \tilde{\sigma}^{\mathscr{M}}$ *of continuous actions of* $\mathbf{R}$ *is Borel.*

PROOF. We adapt the proof given in [8] for measurable fields of weights to the Borel case.

Let $\mathscr{M} \to x_j(\mathscr{M})$ be countably many Borel choice functions for the field $\mathscr{M} \in \mathscr{F} \to \mathscr{M}$. Let $\eta_{\mathscr{M}}$ be the canonical injection of $\mathscr{M}$ into the full left Hilbert algebra $\mathfrak{A}(\mathscr{M})$ determined by $\omega|_{\mathscr{M}}$ on $\mathscr{M}$, and put $\xi_j(\mathscr{M}) = \eta_{\mathscr{M}}(x_j(\mathscr{M}))$. Evidently, the vector fields $\mathscr{M} \to \xi_j(\mathscr{M})$ have the properties

i) $\xi_j(\mathscr{M})$ are dense in $\mathfrak{A}(\mathscr{M})$ with respect to the $\#$-norm
ii) the maps $\mathscr{M} \to (\xi_j(\mathscr{M}) \mid \xi_k(\mathscr{M})) = \omega(x_k(\mathscr{M})^* x_j(\mathscr{M}))$ are Borel
iii) the vector fields $\mathscr{M} \to \xi_j(\mathscr{M})\xi_k(\mathscr{M})$ and $\mathscr{M} \to \xi_j(\mathscr{M})^\#$ are Borel.

Thus if $\mathscr{H}(\mathscr{M})$ and $\mathscr{D}^\#(\mathscr{M})$ respectively are the Hilbert space completion and domain of the sharp operation (with graph norm) of $\mathfrak{A}(\mathscr{M})$, both $\mathscr{M} \to \mathscr{H}(\mathscr{M})$ and $\mathscr{M} \to \mathscr{D}^\#(\mathscr{M})$ are Borel fields of Hilbert spaces, and the field of canonical injections $i(\mathscr{M})$: $\mathscr{D}^\#(\mathscr{M}) \to \mathscr{H}(\mathscr{M})$ carries Borel vector fields to Borel vector fields. Put $K(\mathscr{M}) = i(\mathscr{M})i(\mathscr{M})^*$; then $K(\mathscr{M})$: $\mathscr{H}(\mathscr{M}) \to \mathscr{H}(\mathscr{M})$ and carries Borel fields to Borel fields. Since the modular operator $\Delta(\mathscr{M})$ of $\mathfrak{A}(\mathscr{M})$ is defined by

$$\Delta(\mathscr{M}) = K(\mathscr{M})^{-1}(I - K(\mathscr{M})),$$

for any Borel vector field $\mathscr{M} \to \xi(\mathscr{M}) \in \mathfrak{A}(\mathscr{M})$ we also have $\mathscr{M} \to \Delta(\mathscr{M})^{\frac{1}{2}}$ $\xi(\mathscr{M})$ a Borel vector field. In order to prove (i) it is now sufficient to note that the operator fields

$$\mathscr{M} \to \pi_{\mathscr{M}}(x_j(\mathscr{M})) = \pi_l(\xi_j(\mathscr{M}))$$

are Borel choice functions for $\mathscr{M} \to \pi_{\mathscr{M}}(\mathscr{M})$ (where π_l is the "left regular" representation of $\mathfrak{A}(\mathscr{M})$ on $\mathscr{H}(\mathscr{M})$).

To prove (ii), we note that since $\mathscr{M} \to \Delta(\mathscr{M})^{\frac{1}{2}}\, \xi_j(\mathscr{M})$ is a Borel operator field, so is $\mathscr{M} \to \Delta(\mathscr{M})^{it}\, \xi_j(\mathscr{M})$ for any fixed $t \in \mathbf{R}$. But

$$\tilde{\sigma}_t^{\mathscr{M}}(\pi_{\mathscr{M}}(x_j(\mathscr{M})) = \pi_l(\Delta(\mathscr{M})^{it}\xi_j(\mathscr{M}))),$$

so that $\mathscr{M} \to \tilde{\sigma}_t^{\mathscr{M}}(\pi_{\mathscr{M}}(x_j(\mathscr{M})))$ is Borel for each fixed t and j. But the $\pi_{\mathscr{M}}(x_j(\mathscr{M}))$ are dense in $\pi_{\mathscr{M}}(\mathscr{M})$ with respect to the σ-strong $*$-operator topology, so that if $\mathscr{M} \to y(\mathscr{M})$ is any Borel field with $y(\mathscr{M}) \in \pi_{\mathscr{M}}(\mathscr{M})$, the field $\mathscr{M} \to \sigma_t^{\mathscr{M}}(y(\mathscr{M}))$ is also Borel.

THEOREM 2.2. *Suppose $B \subset [0,1]$ is a Borel set, and let $\mathscr{F}_B$ denote the set of factors on a given separable Hilbert space of type* III_λ *for some $\lambda \in B$. Then $\mathscr{F}_B$ is a Borel set.*

PROOF. By Lemma 2.1, the field of continuous actions $\mathscr{M} \to \tilde{\sigma}^{\mathscr{M}}$ is Borel, and thus by Proposition 1.2, the field of von Neumann algebras $\mathscr{M} \in \mathscr{F} \to \mathscr{R}(\pi_{\mathscr{M}}(\mathscr{M}); \tilde{\sigma}^{\mathscr{M}})$ is Borel, as is the field of dual actions $\mathscr{M} \to \theta^{\mathscr{M}}$ (Proposition 1.5). Let $Z(\mathscr{M}; \sigma^{\mathscr{M}})$ denote the centre of $\mathscr{R}(\pi_{\mathscr{M}}(\mathscr{M}); \tilde{\sigma}^{\mathscr{M}})$; by [3], $\mathscr{M} \in \mathscr{F} \to Z(\mathscr{M}; \sigma^{\mathscr{M}})$ is Borel. Thus if $\tilde{\theta}^{\mathscr{M}}$ denotes the restriction of $\theta^{\mathscr{M}}$ to $Z(\mathscr{M}; \sigma^{\mathscr{M}})$, $\mathscr{M} \to \tilde{\theta}^{\mathscr{M}}$ is a Borel field of continuous actions of $\mathbf{R}$.

By Takesaki's result [9] we know that for $t \in \mathbf{R}$ and $\mathscr{M} \in \mathscr{F}$, $e^t \in S(\mathscr{M})$ (the modular spectrum of $\mathscr{M}$) if and only if $\tilde{\theta}_t^{\mathscr{M}} = \text{identity}$. We shall use this to show that for any $a \in (0,1)$, the set $\mathscr{F}_{(a,1]}$ is Borel; since sets of the form $(a,1]$ generate the Borel structure in $[0,1]$, and the map $B \subset [0,1] \to \mathscr{F}_B$ is a lattice isomorphism, the proof will then be complete.

Let $\mathscr{M} \to z_j(\mathscr{M})$ be a sequence of Borel choice functions for $\mathscr{M} \in \mathscr{F} \to Z(\mathscr{M}; \sigma^{\mathscr{M}})$, and consider the maps $\Phi_j \colon \mathscr{F} \times \mathbf{R} \to Z(\mathscr{M}; \sigma^{\mathscr{M}})$ given by

$$\Phi_j(\mathscr{M}, t) = \tilde{\theta}_t^{\mathscr{M}}(z_j(\mathscr{M})) .$$

By 1.5, the Φ_j are Borel. Since $e^t \in S(\mathscr{M})$ if and only if $\tilde{\theta}_t^{\mathscr{M}} = \text{identity}$, i.e. if and only if $\Phi_j(\mathscr{M}, t) = z_j(\mathscr{M})$ for all j, the set

$$\mathscr{G} = \{(\mathscr{M}, t) \in \mathscr{F} \times \mathbf{R} : \ e^t \in S(\mathscr{M})\}$$

is Borel in $\mathscr{F} \times \mathbf{R}$.

For $a \in (0,1)$, *put* $\mathscr{G}_a = \mathscr{G} \cap (\mathscr{F} \times (\log a, 0))$, and put

$$\mathscr{G}_0 = (\mathscr{F} \times \{t \in \mathbf{R} : \ t < 0\}) - \mathscr{G} .$$

Finally let P be the projection of $\mathscr{F} \times \mathbf{R}$ on $\mathscr{F}$.

Firstly, it is clear that $\mathscr{F}_{\{0\}} = P\mathscr{G}_0$ and thus that $\mathscr{F}_{\{0\}}$ is analytic in $\mathscr{F}$. Secondly, we claim that $\mathscr{F}_{(a,1]} = P\mathscr{G}_a$ – for $\mathscr{M} \in \mathscr{F}_{(a,1]}$ if and only if $(\mathscr{M}, t) \in \mathscr{G}$ for some $t \in (\log a, 0)$.

Finally we claim that $\mathscr{F}_{(0,a]}$ is analytic in $\mathscr{F}$. Put

$$\mathscr{G}'_a = \mathscr{G} \cap (\mathscr{F} \times (-\infty, \log a])$$

— clearly $\mathscr{G}'_a$ is Borel. For each integer $n \geqq 2$, define a map Φ_n on $\mathscr{F} \times \{(-\infty, 0]\}$ by $\Phi_n(\mathscr{M}, t) = (\mathscr{M}, t/n)$. Clearly the maps Φ_n are Borel. But now, $\mathscr{M} \in \mathscr{F}_{(0,a]}$ if and only if there exists a $t \in \mathbf{R}$ with $(\mathscr{M}, t) \in \mathscr{G}'_a$ and $\Phi_n(\mathscr{M}, t) \notin \mathscr{G}_a$ for all $n \geqq 2$. Hence

$$\mathscr{F}_{(0,a]} = P\left(\mathscr{G}'_a \cap \left(\mathscr{G} - \bigcup_{n=2}^{\infty} \Phi_n^{-1}\mathscr{G}_a\right)\right).$$

Thus, each of the sets $\mathscr{F}_{\{0\}}$, $\mathscr{F}_{(0,a]}$ and $\mathscr{F}_{(a,1]}$ are analytic in $\mathscr{F}$. By the separation theorem for analytic sets [4], each is Borel.

COROLLARY 2.3. *For* $\mathscr{M} \in \mathscr{F}$, *let* $r(\mathscr{M})$ *be the unique generator of* $S(\mathscr{M})$ *with* $r(\mathscr{M}) \in [0,1]$. *Then* r *is a Borel map.*

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MATHEMATICS DEPARTMENT
UNIVERSITY OF OREGON
EUGENE, OREGON 92603
U.S.A.