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THE STRUCTURE OF MULTIPLICATION AND ADDITION IN SIMPLE C*-ALGEBRAS

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A C*-algebra is called simple if it contains no non-trivial closed two sided ideals. Many examples of simple C*-algebras have appeared in the literature (we mention without claiming any completeness [1], [2], [4], [7], [9], [10], [12], [13], [19], [22], [23]), but so far not many general properties of these algebras are known. The only really positive result seems to be Sakai's theorem on derivations of simple C*-algebras [20], [21], [22].

If some sort of classification of C*-algebras is at all possible such a classification should certainly begin with simple C*-algebras. The present paper is intended as a first (small) step in this direction. To be precise, we consider only simple C*-algebras with unit. If A is such an algebra then A is algebraically simple, i.e. A contains no non-trivial two sided ideal at all (for the proof note that any proper ideal I in A has empty intersection with the open set of invertible elements in A, and that the closure of I is again an ideal). Let $0 \neq b \in A$. Then

$$J = \left\{ \sum_{i=1}^{n} x_i b y_i \;\middle|\; x_1,\ldots,x_n,y_1,\ldots,y_n \in A \right\}$$

is a non-zero two sided ideal of A so that $J=A$. Thus, for any $a \in A$ and $0 \neq b \in A$ there are elements $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ in A such that $a = \sum_{i=1}^{n} x_i b y_i$. We denote by (a/b) the smallest integer n for which, given $b \neq 0$, such a representation of a exists. This number reflects in some sense the interplay between multiplication and addition in A.

Our approach is inspired by the classification of factors by Murray and von Neumann [17]. We divide first the class of all simple C*-algebras with unit into two classes, namely finite and infinite algebras. In this paper we are concerned mainly with the finite case. It turns out that, in finite algebras, (a/b) and a second number $(a/b)^{\sim}$ (defined in section 3) play a rôle analogous to that of the ratio $[p/q]$ introduced by Murray and von Neumann for two projections p and q in a finite factor. One can even construct in certain (called factorial) algebras a dimension function in much the same way as in [17]. This

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dimension function is defined for every element x in the algebra (not only for projections) and measures the size of the "support projection" of x. It determines, given a, b in the algebra, the numbers (a/b) and $(a/b)^{\sim}$.

The class of factorial algebras contains, besides II_1-factors, a large class of simple AF-algebras and, in particular, all UHF-algebras (for the definition of UHF- and AF-algebras see [13] and [2]). The general results we obtain for factorial algebras seem to be new and non-trivial even for UHF-algebras. But, of course, there are finite simple C*-algebras which are not factorial (5.3).

The reader is assumed to be familiar with the theory of operator algebras. We use the notation of [5], [6].

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1. Definitions.

In this section we give the basic definitions, and prove some related propositions which will be used frequently in the paper. Throughout this section A is a C*-algebra which, for convenience, is assumed to be represented on Hilbert space.

1.1. We restate the definition of (a/b) given in the introduction.

DEFINITION. Let A be simple with unit $\mathbf{1}$ and $a, b \in A$, $b \neq 0$. We define

$$(a/b) = \min\left\{n \in \mathbf{N} \;\middle|\; \exists\, x_1, \ldots, x_n, y_1, \ldots, y_n \in A \text{ such that } a = \sum_{i=1}^{n} x_i b y_i\right\}.$$

Moreover, we define

$$a \precsim b, \quad \text{if } (a/b) = 1$$

$$a \approx b, \quad \text{if } a \precsim b \text{ and } b \precsim a$$

The following formal properties of (a/b) are readily verified.

(I) $(a+b/c) \leq (a/c) + (b/c) \qquad (c \neq 0)$

(II) $(a/c) \leq (a/b)(b/c) \qquad (b, c \neq 0)$

(III) The relation $\precsim$ is transitive, and the relation $\approx$ is an equivalence relation.

(IV) Let M_n be the algebra of $n \times n$ complex matrices with unit $\mathbf{1}_n$ and let $\{e_{ij} \mid i, j = 1, \ldots, n\}$ be a self-adjoint system of matrix units in M_n. Let $a, b \in A$. Then $(a/b) \leq n$ holds in A if and only if $e_{11} \otimes a \precsim \mathbf{1}_n \otimes b$ in $M_n \otimes A$.

PROOF OF (IV). Let $a, b \in A$ and $a=\sum_{i=1}^n x_i b y_i$ with $x_i, y_i \in A$. Then

$$e_{11}\otimes a = \left(\sum_{i=1}^n e_{1i}\otimes x_i\right)(1_n\otimes b)\left(\sum_{i=1}^n e_{i1}\otimes y_i\right).$$

Conversely, if $e_{11}\otimes a = x(1_n\otimes b)y$ with $x, y \in M_n\otimes A$, then without loss of generality we may assume $x=(e_{11}\otimes 1)x$ and $y=y(e_{11}\otimes 1)$. Thus, x and y have the representation

$$x = \sum_{i=1}^n e_{1i}\otimes x_i \quad \text{and} \quad y = \sum_{i=1}^n e_{i1}\otimes y_i \quad \text{with } x_i, y_i \in A,$$

whence $a=\sum_{i=1}^n x_i b y_i$.

1.2. Next, we generalize the concept of the reduced algebra $A_p = pAp$ where p is a projection in A to arbitrary elements of A. Given $\varepsilon>0$, let f_ε be the continuous function on $\mathbf{R}$ defined by

$$f_\varepsilon \equiv 0 \text{ on } [-\infty, \varepsilon/2],$$

$$f_\varepsilon \text{ is linear on } [\varepsilon/2, \varepsilon],$$

$$f_\varepsilon \equiv 1 \text{ on } [\varepsilon, \infty].$$

When a is in A, we denote throughout the paper by $|a|$ the absolute value $|a| = (a^*a)^{\frac{1}{2}}$ of a.

DEFINITION. Given $a \in A$, set

$$A_a = A_{|a|} = \bigcup_{\varepsilon>0} f_\varepsilon(|a|)Af_\varepsilon(|a|)$$

It is easy to see that the closure of A_a coincides with the closure of $|a|A|a|$, and is a simple C*-subalgebra of A, if A is simple.

One of the main features of the reduced algebra is the property that $x \in A_a$ implies $x \precsim a$. In fact, if $x=f_\varepsilon(|a|)yf_\varepsilon(|a|)$ then $x \precsim f_\varepsilon(|a|) \precsim a^*a \precsim a$.

1.3. PROPOSITION. *Let $x \in A$ with polar decomposition $x=u|x|$. Then $uf(|x|)$ is in A for any continuous function f on* $\mathbf{R}$ *that vanishes in* 0.

PROOF. If P is a complex polynomial without constant coefficient, then $uP(|x|)$ is in A. Let $\{P_n\}$ be a sequence of polynomials such that $P_n(|x|)$ converges to $f(|x|)$ in norm. Then $uP_n(|x|)$ converges to $uf(|x|)$ in norm and the assertion follows.

1.4. Given a in A with polar decomposition $a=u|a|$ and $\varepsilon>0$ we know from 1.3 that $z_\varepsilon = uf_\varepsilon(|a|)$ is in A.

As $f_{\varepsilon/2}(|a|)f_\varepsilon(|a|)=f_\varepsilon(|a|)$, the mapping Φ_ε defined by

$$\Phi_\varepsilon(x) = z_{\varepsilon/2}xz_{\varepsilon/2}^*$$

is an isomorphism of $f_\varepsilon(|a|)Af_\varepsilon(|a|)$ onto $f_\varepsilon(|a^*|)Af_\varepsilon(|a^*|)$. Its inverse is

$$\Phi_\varepsilon^{-1}(y) = z_{\varepsilon/2}^*yz_{\varepsilon/2} .$$

The isomorphisms Φ_ε $(\varepsilon>0)$ extend to an isomorphism Φ from A_a onto A_{a^*}. Obviously $x\approx\Phi(x)$ for all $x \in A_a$.

1.5. We say that two positive elements h, h' of A are equivalent in the sense of Murray–von Neumann and write $h\sim h'$ if there is $x \in A$ such that $h=x^*x$ and $h'=xx^*$. If Φ: $A_a \to A_{a^*}$ is the isomorphism defined in 1.4 and $h \in A_a$ is positive, then obviously $h\sim\Phi(h)$.

1.6. We say that $x \in A$ is orthogonal to $y \in A$ and write $x\perp y$, if $xy=yx =x^*y=yx^*=0$.

PROPOSITION. *Let $a, b \in A$, $b\neq 0$, $(a/b)=n$, and let $b_1,\dots,b_n$ be pairwise orthogonal elements of A such that $b_i\gtrsim b$ $(i=1,\dots,n)$. Then $a\lesssim\sum_{i=1}^n |b_i|^{\frac{1}{2}}$.*

PROOF. Let $b_i=u_i|b_i|$ be the polar decomposition of b_i. By 1.3 we have $u_i|b_i|^{\frac{1}{4}} \in A$. Since the b_i are orthogonal, also $|b_i||b_j|=0$ for $i\neq j$.

If $a=\sum_{i=1}^n x_iby_i$ and $b=s_ib_it_i$ $(i=1,\dots,n)$, then

$$a = \sum_{i=1}^n x_is_ib_it_iy_i = \left(\sum_{i=1}^n x_is_iu_i|b_i|^{\frac{1}{4}}\right)\left(\sum_{i=1}^n |b_i|^{\frac{1}{2}}\right)\left(\sum_{i=1}^n |b_i|^{\frac{1}{4}}t_iy_i\right)$$

and this is exactly what we wanted to show.

1.7. LEMMA. *Let e, b be positive elements of A and a, x, y be arbitrary elements of A such that $b=xay$ and $ea=ae=a$. Then there is $z \in A$ such that $zz^*=b$ and $e(z^*z)=z^*z$. If $\|b\|\leq 1$ then $z^*z\leq e$ ($\leq$ is the usual order on the self-adjoint part of A).*

PROOF. Set $z_0=x(ayy^*a^*)^{\frac{1}{2}}$. Then

$$z_0z_0^* = xayy^*a^*x^* = bb^* = b^2 .$$

On the other hand

$$z_0^*z_0 = (ayy^*a^*)^{\frac{1}{2}}x^*x(ayy^*a^*)^{\frac{1}{2}}$$

whence $e(z_0^*z_0)=z_0^*z_0$. If $z_0=u|z_0|$ is the polar decomposition of z_0, then $z=u|z_0|^{\frac{1}{2}}$ is in A by 1.3. It follows $zz^*=|z_0^*|=b$ and $ez^*z=e|z_0|=|z_0|=z^*z$. The second assertion is obvious.

The following special case is important. Let $p, q \in A$ be projections and $p \precsim q$. If we apply the Lemma to $a = e = q$ and $b = p$, we see that there is a projection $q' \leqq q$ in A such that $q' \sim p$.

1.8. PROPOSITION. *Let A be simple and a, b non-zero elements of A. There is $0 \neq y \in A$ such that $yy^* \in A_a$ and $y^*y \in A_b$, and there are non-zero elements $z \in A_a$, $z' \in A_b$ such that $z \approx z'$.*

PROOF. Given $d \neq 0$ in A, the set

$$J = \{c \mid dxc = 0 \text{ for all } x \in A\}$$

is a closed ideal in A. From $J \neq A$ (for instance $|d| \notin J$) we conclude $J = \{0\}$. Thus, given $a, b \in A$ and $\varepsilon > 0$ with $\varepsilon < \|a\|$, $\varepsilon < \|b\|$, there is $x \in A$ such that $y = f_\varepsilon(|a|)xf_\varepsilon(|b|) \neq 0$. It follows $yy^* \in A_a$ and $y^*y \in A_b$. With Φ: $A_{yy^*} \to A_{y^*y}$ defined as in 1.4 and $z \in A_{yy^*}$ we get $z \approx \Phi(z)$. Therefore z and $z' = \Phi(z)$ have the desired properties.

1.9. PROPOSITION. *Let A be simple and $p \in A$ a projection. Let $a \neq 0$ be an element of A such that $(p/a) = n$. Then there is $\delta > 0$ such that $(p/b) \leqq n$ whenever $\|a - b\| < \delta$, and such that $(p/f_\varepsilon(|a|)) \leqq n$ whenever $\varepsilon < \delta$.*

PROOF. Let $p = \sum_{i=1}^n x_i a y_i$. Without loss of generality we assume $x_i = px_i$ and $y_i = y_i p$.

There is a $\delta > 0$ such that $\sum_{i=1}^n x_i b y_i$ is invertible in A_p whenever $\|a - b\| < \delta$. In this case obviously $(p/b) \leqq n$. If $u|a|$ is the polar decomposition of a and $\varepsilon < \delta$, then

$$\begin{aligned} \|a - af_\varepsilon(|a|)\| &= \|u(|a| - |a|f_\varepsilon(|a|))\| \\ &= \||a| - |a|f_\varepsilon(|a|)\| < \delta . \end{aligned}$$

If z_ε is defined as in 1.4, then $af_\varepsilon(|a|) = z_{\varepsilon/2}|a|f_\varepsilon(|a|)$. This shows that

$$f_\varepsilon(|a|) \gtrsim |a|f_\varepsilon(|a|) \gtrsim af_\varepsilon(|a|)$$

whence

$$(p/f_\varepsilon(|a|)) \leqq (p/af_\varepsilon(|a|)) \leqq n .$$

1.10. The following is interesting in connection with other comparison theories in operator algebras (cf. in particular [16]).

PROPOSITION. *Let A be simple, $p \in A$ a projection and $h \in A$ positive. If $(p/h) = n$, then there are $x_1, \ldots, x_n \in A$ such that*

$$p = \sum_{i=1}^{n} x_i h x_i^* .$$

PROOF. We consider first the case $n=1$. By 1.9 we have then $(p/f_\varepsilon(h))=1$, if $\varepsilon>0$ is small enough. Lemma 1.7 applied to $a=f_\varepsilon(h)$, $e=f_{\varepsilon/2}(h)$ and $b=p$ shows the existence of $z \in A$ such that $zz^*=p$ and $f_{\varepsilon/2}(h)z^*z=z^*z$. Obviously, there is a (unique) positive k in the C*-algebra genbrated by h such that $f_{\varepsilon/2}(h)=kh$. We get $p=zk^{\frac{1}{2}}hk^{\frac{1}{2}}z^*$.

Let now n be an arbitrary natural number and $(p/h)=n$. By 1.1.IV this implies $e_{11}\otimes p \lesssim \mathbf{1}_n \otimes h$ in $M_n \otimes A$. By the reasoning above there is $x \in M_n \otimes A$ such that $e_{11}\otimes p = x(\mathbf{1}_n\otimes h)x^*$. We may assume that x has the form $x=\sum_{i=1}^n e_{1i}\otimes x_i$ $(x_i \in A)$. It follows

$$e_{11}\otimes p = \left(\sum_{i=1}^{n} e_{1i}\otimes x_i\right)(\mathbf{1}_n\otimes h)\left(\sum_{i=1}^{n} e_{i1}\otimes x_i^*\right)$$

and $p=\sum_{i=1}^n x_i h x_i^*$.

Let now A be simple with unit $\mathbf{1}$ and $h \in A$ positive. There are $x_1, \ldots, x_n$ in A such that $\mathbf{1}=\sum_{i=1}^n x_i h x_i^*$. If $a \in A$ is positive, then $a=\sum_{i=1}^n a^{\frac{1}{2}} x_i h x_i^* a^{\frac{1}{2}}$.

Thus, the proposition shows in particular that any positive $a \in A$ can be written as $a=\sum_{i=1}^n y_i h y_i^*$ with $y_1, \ldots, y_n \in A$.

2. Finite and infinite algebras.

Throughout this section A denotes a simple C*-algebra with unit.

2.1. DEFINITION. We call A infinite if there is a non-invertible element a of A such that $a \approx \mathbf{1}$. A is said to be finite if it is not infinite (that is if $(\mathbf{1}/a) \geqq 2$ for every non-invertible a in A).

2.2. We start with some considerations concerning infinite algebras. Examples of such algebras are the Calkin algebra and type III-factors in a separable Hilbert space. There exist also separable infinite algebras [8.2.1].

PROPOSITON. *If A is infinite, then A contains two orthogonal projections p and q such that $p \sim q \sim \mathbf{1}$ in A.*

PROOF. Let $a \in A$ be non-invertible and $a \approx \mathbf{1}$. Then either $|a|$ or $|a^*|$ is non-invertible. Let us assume without loss of generality that $|a|$ is not invertible. By

1.9 there is $\varepsilon > 0$ such that $f_\varepsilon(|a|) \approx \mathbf{1}$. Lemma 1.7 applied to $a = f_\varepsilon(|a|)$, $e = f_{\varepsilon/2}(|a|)$ and $b = \mathbf{1}$ shows that there is a projection $r \sim 1$ in A such that $f_{\varepsilon/2}(|a|)r = r$. Since $r \in A_a$ and $0 \in \mathrm{Sp}\,(|a|)$ we see that $r \neq \mathbf{1}$.

Let $u \in A$ be such that $u^*u = \mathbf{1}$, $uu^* = r$ and set $n = (\mathbf{1}/1 - r)$. The projections

$$\mathbf{1} - r, u(\mathbf{1} - r)u^*, \ldots, u^{n-1}(\mathbf{1} - r)u^{*n-1}$$

are pairwise orthogonal and equivalent ($\sim$ and $\approx$), so that with

$$s = (\mathbf{1} - r) + u(\mathbf{1} - r)u^* + \ldots + u^{n-1}(\mathbf{1} - r)u^{*n-1}$$

we get $(\mathbf{1}/s) = 1$ by 1.6. On the other hand $\mathbf{1} - s = u^{n-1}ru^{*n-1}$ whence $\mathbf{1} - s \sim r \sim 1$.

For q we take now $\mathbf{1} - s$ and for p we take the projection $p \leqq s$ with $p \sim \mathbf{1}$ whose existence is asserted at the end of 1.7.

In particular, the proposition shows that A is finite if and only if $x^*x = \mathbf{1}$ implies $xx^* = \mathbf{1}$ ($x \in A$). If there is a finite trace on A then A can not be infinite (any non-trivial trace on A is faithful [5,6.2.2]). This shows that a simple C*-algebra with unit which is finite in the sense of Dixmier [8] is also finite in the sense of the definition above.

We call a projection p in A finite or infinite if A_p is finite or infinite respectively. If p is infinite it follows in standard fashion from 2.2 that there are infinitely many orthogonal projections q_i in A_p all equivalent in the sense of Murray–von Neumann to p. We conclude from 1.6 that $(a/p) = 1$ for all $a \in A$. We conjecture that, in an infinite simple C*-algebra with unit, $(a/b) = 1$ holds for any $a, b \in A$, $(b \neq 0)$.

2.3. Contrary to this we have in finite algebras the following.

Proposition. *Let A be an infinite dimensional finite simple* C**-algebra with unit and let n be a natural number. There is $a \in A$ such that $(\mathbf{1}/a) \geqq n$.*

Proof. There is a positive element h of A that has infinite spectrum [15,2.5]. For this h there exist non-zero continuous functions $g_1, \ldots, g_n$ (with disjoint supports) on $\mathrm{Sp}\,(h)$ such that the elements $\bar{g}_i = g_i(h)$ are pairwise orthogonal. Repeated application of 1.8 shows that there are non-zero elements $f_i \in A_{\bar{g}_i}$ ($i = 1, \ldots, n$) such that $f_1 \gtrapprox f_2 \gtrapprox \cdots \gtrapprox f_n$. (If $0 < k < n$ and $f_1, \ldots, f_k$ are constructed, choose $z \in A_{f_k} \subset A_{\bar{g}_k}$ and $z' \in A_{\bar{g}_{k+1}}$ such that $z \approx z'$ and write $f_{k+1} = z'$.)

If we assume that $(\mathbf{1}/f_n) \leqq n - 1$ we get $\mathbf{1} \lessapprox \sum_{i=1}^{n-1} |f_i|^{\frac{1}{2}}$ by 1.6. This is a contradiction to the finiteness of A since the expression on the right hand side is certainly not invertible. In fact, $f_n \perp \sum_{i=1}^{n-1} |f_i|^{\frac{1}{2}}$.

More generally, the proposition, combined with 1.1 II, shows that for each $b \in A$ there is $a \in A$ such that $(b/a) \geqq n$.

2.4. Proposition. *Let A be finite. Then every projection p in A is finite.*

Proof. If $a \in A_p$ is not invertible in A_p and $p = xay$ with $x, y \in A_p$, then $z = \mathbf{1} - p + a$ is not invertible in A and $\mathbf{1} = (\mathbf{1} - p + x)z(\mathbf{1} - p + y)$ is a contradiction.

2.5. Proposition. *Let A be finite and $p, q \in A$ projections. Then $p \approx q$ if and only if $p \sim q$.*

Proof. Let $p \approx q$. According to 1.7 there is a projection $q' \leqq q$ with $q' \sim p$ and, consequently, $q' \approx p$. If we assume that $q' \neq q$, then $q' \approx p \approx q$ is a contradiction to the finiteness of q (2.4).

We do not believe that this proposition holds in general for infinite algebras.

3. Factorial algebras.

We introduce now a class of simple C*-algebras for which detailed information about (a/b) and the comparison theory with respect to $\lesssim$ can be obtained through a dimension function. It will be shown in section 5 that this class contains, besides II_1-factors, all UHF-algebras and also the algebras studied in 6.4–6.6 of [12]. In this section A will denote a simple C*-algebra with unit.

3.1. Definition. Let $a, b \in A$, $b \neq 0$. We define with the convention $\sup \varnothing = 0$

$$(a/b)^{\sim} = \sup \{ n \mid \exists\, b_1, \ldots, b_n \in A_a \text{ such that } b_i \gtrsim b \ (i = 1, \ldots, n) \text{ and } b_i \perp b_j \text{ for } i \neq j \} .$$

Thus, $(a/b)^{\sim}$ is either a natural number or ∞. It follows from 2.2 through a standard argument that $(1/a)^{\sim} = \infty$ for any $a \in A$, if A is infinite (for completeness we may set $(a/0)^{\sim} = \infty$ for all a). If A is finite, however, $(a/b)^{\sim}$ is finite for every $a \in A$ and $0 \neq b \in A$. In fact, the following holds.

3.2. Proposition. *If A is finite, then*

$$(a/b)^{\sim} \leqq (a/b) \quad \textit{for } a \in A,\ 0 \neq b \in A .$$

PROOF. Let $(a/b)=n$ and assume that there are pairwise orthogonal elements $b_1,\ldots,b_{n+1}$ in A_a such that $b_i \gtrsim b$ $(i=1,\ldots,n+1)$. There is an $\varepsilon>0$ such that $f_\varepsilon(|a|)b_i=b_i$ $(i=1,\ldots,n+1)$. Set

$$c = \sum_{i=1}^{n} |b_i|^{\frac{1}{2}} .$$

Then by 1.6 $c \gtrsim a \gtrsim f_\varepsilon(|a|)$.

There are $x,y \in A$ such that $f_\varepsilon(|a|)=xcy$. With $d=\mathbf{1}-f_\varepsilon(|a|)$ this yields

$$\mathbf{1} = (xc^{\frac{1}{3}}+d^{\frac{1}{3}})(c^{\frac{1}{3}}+d^{\frac{1}{3}})(c^{\frac{1}{3}}y+d^{\frac{1}{3}}) .$$

This is a contradiction since $c^{\frac{1}{3}}+d^{\frac{1}{3}}$ is orthogonal to b_{n+1} and thus not invertible.

3.3. DEFINITION. Let A be finite. A sequence of non-zero elements $\{h_n\}$ in A is said to be a fundamental sequence if $(1/h_n) \to \infty$ and if there is a natural number K such that for all $a \in A$ and $n \in \mathbf{N}$

$$(a/h_n)^{\sim} \leqq (a/h_n) \leqq (a/h_n)^{\sim}+K$$

A is called factorial if there is a fundamental sequence in A.

It will be seen later in 3.9 that only the cases $K=1$ and $K=2$ occur. In II_1-factors and UHF-algebras K may be chosen equal to 1, in the example 6.5 of [12] K has to be chosen equal to 2.

3.4. The following Lemma together with 1.1 II is essential for the construction of a dimension function.

LEMMA. *Let A be factorial and let h be an element of a fundamental sequence in A. Then*

$$(a/h)^{\sim} \geqq (a/b)^{\sim}((b/h)-K) \quad \textit{for all } a \in A,\ 0 \neq b \in A .$$

PROOF. If $K \geqq (b/h)$, the inequality is trivially true. Thus assume $(b/h)-K>0$ and let $(a/b)^{\sim}=r$ and $(b/h)=s$. Let $b_1,\ldots,b_r \in A_a$ be pairwise orthogonal and $b_i \gtrsim b$ $(i=1,\ldots,r)$.

We have $(b_i/h)^{\sim} \geqq (b_i/h)-K \geqq s-K$ $(i=1,\ldots,r)$. For each $i=1,\ldots,r$ there are orthogonal elements $c_1^i,\ldots,c_{s-K}^i \in A_{b_i} \subset A_a$, such that $c_j^i \gtrsim h$ $(i=1,\ldots,r,$ $j=1,\ldots,s-K)$.

Since $c_j^i \perp c_k^l$ whenever $i \neq l$ or $j \neq k$, the assertion follows.

3.5. THEOREM. *Let A be factorial and $\{h_n\}$ a fundamental sequence in A. For each $a \in A$ the limit*

$$\lambda(a) = \lim_{n\to\infty} (a/h_n)(1/h_n)^{-1}$$

exists and is independent of the fundamental sequence. λ is called the dimension function on A.

PROOF. For $n, m \in \mathbf{N}$ write $t_n = (1/h_n)$, $s_n = (a/h_n)$, $r_{nm} = (h_n/h_m)$. Then $t_n \to \infty$ as $n \to \infty$, by assumption, and $s_n \geqq (1/a)^{-1} t_n \to \infty$ as $n \to \infty$, by 1.1 II. Also $r_{nm} \to \infty$ if n is fixed and $m \to \infty$. From $t_m \leqq r_{nm} t_n$ and

$$s_m \geqq (a/h_m)^{\sim} \geqq (s_n - K)(r_{nm} - K)$$

(this is 1.1 II and 3.4) we get the following inequality

$$(*) \qquad \frac{s_n - K}{t_n}\,\frac{r_{nm} - K}{r_{nm}} \leqq \frac{s_m}{t_m}$$

(we assume here that n is so large that $s_n - K$ is positive).

Let n be fixed. If we let m tend to infinity in (*), we get

$$(s_n - K)/t_n \leqq \lim_{m\to\infty} \inf s_m/t_m .$$

If $n \to \infty$, this leads to

$$\lim_{n\to\infty} \sup s_n/t_n \leqq \lim_{m\to\infty} \inf s_m/t_m .$$

This argument is due to Murray and von Neumann [17].

If $\{h_n'\}$ is a second fundamental sequence in A then the sequence $\{k_n\}$ defined by $k_{2n} = h_n$ and $k_{2n+1} = h_n'$ is of course again a fundamental sequence. As $(a/k_n)(1/k_n)^{-1}$ converges this shows that the value $\lambda(a)$ does not depend on the choice of the fundamental sequence.

3.6. Let A be factorial and λ the dimension function on A. The following properties are immediate consequence of the definition of λ.

(a) $a \precsim b \Rightarrow \lambda(a) \leqq \lambda(b)$
  $a \approx b \Rightarrow \lambda(a) = \lambda(b)$

(b) $(1/a)^{-1} \leqq \lambda(a) \leqq 1 \quad (a \neq 0)$
  $\lambda(a) = 0 \Leftrightarrow a = 0$

(c) $\lambda(a+b) \leqq \lambda(a) + \lambda(b)$
  $a \perp b \Rightarrow \lambda(a+b) = \lambda(a) + \lambda(b)$

(d) $\lambda(ab) \leqq \min\{\lambda(a), \lambda(b)\}$

(e) $\lambda(a) = \lambda(a^*a) = \lambda(aa^*) = \lambda(a^*)$

(f) $\lambda(a) = \lim_{\varepsilon\to 0} \lambda(f_\varepsilon(|a|))$.

PROOF. Let $\{h_n\}$ be a fundamental sequence in A.

(a) If $a \lesssim b$, then $(a/h_n) \leqq (b/h_n)$.

(b) follows from $(1/h_n) \leqq (1/a)(a/h_n)$.

(c) The first assertion follows from 1.1 I. If $a \perp b$, then

$$(a+b/h_n)^{\sim} \geqq (a/h_n)^{\sim} + (b/h_n)^{\sim} .$$

(d) We have $(ab/h_n) \leqq \min\{(a/h_n), (b/h_n)\}$.

(e) Using 1.4 we see that $(a^*a/h_n)^{\sim} = (aa^*/h_n)^{\sim}$, and $(a/h_n)^{\sim} = (a^*a//h_n)^{\sim}$ is obvious.

(f) This follows from the definiton of $(a/h_n)^{\sim}$.

3.7. In factorial algebras the comparison theory is to a large extent determined by the dimension function.

THEOREM. *Let A be factorial, $a, b \in A$, $b \neq 0$, λ the dimension function on A and $N \in \mathbf{N}$.*

(i) *If $(a/b) \leqq N$, then $\lambda(a) \leqq N\lambda(b)$*

(ii) *If $\lambda(a) < N\lambda(b)$, then $(a/b) \leqq N$.*

We remark, however, that if $h \in A$ is positive, then $\lambda(h) = \lambda(h^2)$ though $h \approx h^2$ does not hold in general. The algebra in [12,6.5] contains two projections p, q such that $\lambda(p) = \lambda(q)$ while p and q are not comparable.

PROOF OF THE THEOREM. Let $\{h_n\}$ be a fundamental sequence in A. (i) follows from the inequality $(a/h_n) \leqq (a/b)(b/h_n)$. To prove (ii) choose $n \in \mathbf{N}$ such that $N(b/h_n)^{\sim} > (a/h_n)$ and set $(b/h_n)^{\sim} = r$. There are orthogonal elements $c_1, \ldots, c_r$ in A_b such that $c_i \gtrsim h_n$ $(i = 1, \ldots, r)$. Since $Nr > (a/h_n)$, we conclude from 1.6 that in $M_N \otimes A$ we have

$$e_{11} \otimes a \lesssim \sum_{\substack{i=1,\ldots,N \\ j=1,\ldots,r}} e_{ii} \otimes |c_j|^{\frac{1}{2}}$$

(In fact $e_{ii} \otimes c_j \gtrsim e_{11} \otimes h_n$).

If we denote the sum on the right hand side by x, then $x \lesssim 1_N \otimes b$. According to 1.1 IV this means just $(a/b) \leqq N$.

3.8. THEOREM. *Let A be factorial, $a, b \in A$, $b \neq 0$, λ the dimension function on A and $N \in \mathbf{N}$.*

(i) *If $(a/b)^{\sim} \geqq N$, then $\lambda(a) \geqq N\lambda(b)$*

(ii) *If $\lambda(a) > N\lambda(b)$, then $(a/b)^{\sim} \geqq N$.*

PROOF. Let $\{h_n\}$ be a fundamental sequence in A. By 3.4

$$(a/h_n)^{\sim} \geqq (a/b)^{\sim}((b/h_n)-K),$$

whence (i).

For the proof of (ii) choose $n \in \mathbf{N}$ such that $(a/h_n)^{\sim} > N(b/h_n)$ and set $(b/h_n) = r$. There are orthogonal elements $c_1, \ldots, c_{rN}$ in A_a such that $c_i \gtrsim h_n$ $(i=1,\ldots,rN)$. Write

$$d_j = \sum_{i=(j-1)r+1}^{jr} |c_i|^{\frac{1}{2}} \qquad (j=1,\ldots,N).$$

Another application of 1.6 shows $d_j \gtrsim b$ $(j=1,\ldots,N)$. Since the d_j are elements of A_a and pairwise orthogonal, the proof is complete.

3.9. COROLLARY. *In a factorial algebra A we have for all a, b in A $(b \neq 0)$,*

$$\lambda(a)/\lambda(b) - 1 \leqq (a/b)^{\sim} \leqq (a/b) \leqq \lambda(a)/\lambda(b) + 1.$$

In particular $(a/b) \leqq (a/b)^{\sim} + 2$.

PROOF. Combine 3.7 and 3.8.

3.10. COROLLARY. *In a factorial algebra any sequence $\{h_n\}$ satisfying $(\mathbf{1}/h_n) \to \infty$, is a fundamental sequence.*

PROOF. This follows from 3.9.

3.11. COROLLARY. *Let A be factorial and let g, h be positive elements of A such that $\lambda(h) < \lambda(g)$. Then there exists $x \in A$ such that $x^*x = h$ and $xx^* \in A_g$.*

PROOF. By 3.7 and 3.6 (f) we have $h \lesssim f_\varepsilon(g)$ if $\varepsilon > 0$ is small enough. Thus Lemma 1.7 applied to $a = f_\varepsilon(g)$, $b = h$ and $e = f_{\varepsilon/2}(g)$ gives the assertion.

Since in the case of UHF-algebras and more general factorial AF-algebras λ is known (see section 4), this corollary and other results of this section can immediately be applied to comparability questions in such algebras.

4. The uniqueness of the trace on factorial algebras.

Let F be a II_1-factor with normalized trace τ. Given $x \in F$, let p_x be its support projection. It is fairly easy to see that F is factorial and $\lambda(x) = \tau(p_x)$. In this section we prove a general version of this result for arbitrary factorial algebras having a finite trace (4.2). As a consequence, we will see that, if there is a finite trace on a factorial algebra, then this trace is unique. This implies that every representation of finite type of a factorial algebra is a factor representation (cf. [5,6.8.7(ii)].

In this section A denotes a factorial simple C*-algebra with unit. By a (normalized) finite trace on A we mean a state τ on A such that $\tau(xy)=\tau(yx)$ for all $x,y \in A$. We remark incidentally that there exists a simple C*-algebra with unit having uncountably many different finite trace [10,2.17].

4.1. LEMMA. *Let τ be a finite trace on A and let $g,h \in A$ be positive and non-zero.*

(a) *For every $\varepsilon>0$, there is $\delta>0$ such that*

$$\tau(f_\delta(h)) \geqq ((h/g)-2)\tau(f_\varepsilon(g))\,.$$

(b) *For all ε, $0<\varepsilon<\|g\|$ and for all $\delta>0$*

$$\tau(f_\delta(h)) \leqq (f_\delta(h)/f_\varepsilon(g))\tau(f_{\varepsilon/2}(g))\,.$$

PROOF. (a) Set $(h/g)=n$. Since $f_\varepsilon(g)\lesssim g$, we get $(h/f_\varepsilon(g))\geqq n$ and $(h/f_\varepsilon(g))^\sim\geqq n-2$ for any $\varepsilon>0$ (3.9). There is $\delta>0$ and there exist orthogonal elements $c_1,\ldots,c_{n-2} \in f_{2\delta}(h)Af_{2\delta}(h)$ such that $c_i\gtrsim f_\varepsilon(g)$ $(i=1,\ldots,n-2)$. An application of Lemma 1.7 to $c_i, f_\delta(h), f_\varepsilon(g)$ in the place of a,e,b shows that there are $x_i \in A$ such that $x_ix_i^*=f_\varepsilon(g)$ and $x_i^*x_i\leqq f_\delta(h)$. From the definition of x_i in the proof of 1.7 it follows that the elements $x_1^*x_1,\ldots,x_{n-2}^*x_{n-2}$ are pairwise orthogonal and so, that

$$x_1^*x_1+\ldots+x_{n-2}^*x_{n-2} \leqq f_\delta(h)\,.$$

Now

$$\tau(f_\delta(h)) \geqq \tau\left(\sum_{i=1}^{n-2} x_i^*x_i\right) = \sum_{i=1}^{n-2} \tau(x_i^*x_i) = \sum_{i=1}^{n-2} \tau(x_ix_i^*) = (n-2)\tau(f_\varepsilon(g))\,.$$

(b) Given ε and δ, let $n=(f_\delta(h)/f_\varepsilon(g))$. In $M_n\otimes A$ we have $e_{11}\otimes f_\delta(h)\lesssim \mathbf{1}_n\otimes f_\varepsilon(g)$. By 1.7 there is $x \in M_n\otimes A$ such that

$$xx^* = e_{11}\otimes f_\delta(h) \quad \text{and} \quad x^*x \leqq \mathbf{1}_n\otimes f_{\varepsilon/2}(g)\,.$$

If x has the representation $x=\sum_{i=1}^n e_{1i}\otimes x_i$ with $x_i \in A$, then $f_\delta(h)=\sum_{i=1}^n x_ix_i^*$. On the other hand $x_i^*x_i\leqq f_{\varepsilon/2}(g)$ $(i=1,\ldots,n)$ and

$$\tau(f_\delta(h)) = \tau\left(\sum_{i=1}^{n} x_ix_i^*\right) = \sum_{i=1}^{n} \tau(x_i^*x_i) \leqq n\tau(f_{\varepsilon/2}(g))\,.$$

4.2. THEOREM. *Let A be factorial and let τ be a finite trace on A. Then*

$$\lambda(a) = \lim_{\delta\to\infty} \tau(f_\delta(|a|)) \quad \textit{for all } a \in A\,.$$

PROOF. Since $\lambda(a)=\lambda(|a|)$, it suffices to prove the assertion for positive a. The following inequalities follow from Lemma 4.1 for positive elements a, b of A and $\varepsilon, \delta > 0$.

(1) $$\tau(f_\varepsilon(b)) \leqq ((\mathbf{1}/b)-2)^{-1}\,.$$

This is 4.1 (a) applied to $h=\mathbf{1}$ and $g=b$. From 4.1 (b) we get replacing h by $\mathbf{1}$ and g by b

$$\tau(f_{\varepsilon/2}(b)) \geqq (\mathbf{1}/f_\varepsilon(b))^{-1}\,.$$

By 1.9 we have $(\mathbf{1}/f_\varepsilon(b))=(\mathbf{1}/b)$ for small ε and hence

(2) $$\tau(f_\varepsilon(b)) \geqq (\mathbf{1}/b)^{-1} \quad \text{for small } \varepsilon\,.$$

Lemma 4.1 (b) and (1), taken together, yield

$$\tau(f_\delta(a)) \leqq (f_\delta(a)/f_\varepsilon(b))((\mathbf{1}/b)-2)^{-1}\,.$$

As $(f_\delta(a)/f_\varepsilon(b)) \leqq (a/f_\varepsilon(b))$ and $(\mathbf{1}/b)=(\mathbf{1}/f_\varepsilon(b))$ for small ε (1.9), this leads to

(3) $$\tau(f_\delta(a)) \leqq (a/f_\varepsilon(b))((\mathbf{1}/f_\varepsilon(b))-2)^{-1}\,.$$

Using 4.1 (a) and (2) we get for small $\delta>0$, chosen in dependence on b, when a is fixed

(4) $$\tau(f_\delta(a)) \geqq ((a/b)-2)(\mathbf{1}/b)^{-1}\,.$$

Let $\{h_n\}$ be a fundamental sequence of positive elements in A such that $\|h_n\|=1$ for each n. Then $\{f_{\varepsilon_n}(h_n)\}$ is a fundamental sequence whenever $0<\varepsilon_n<1$ (3.10). If we replace b in (3) and (4) by h_n and let n tend to infinity then the right hand side of (3) as well as the right hand side of (4) tend to $\lambda(a)$. Consequently

$$\lambda(a) \leqq \lim_{\delta\to 0} \tau(f_\delta(a)) \leqq \lambda(a)\,.$$

4.3. Let A be factorial, τ a finite trace on A and $h=h^* \in A$. Then τ induces a positive measure $d\tau$ on the spectrum $\mathrm{Sp}\,(h)$ of h through the relation

$$\int_{\mathrm{Sp}\,(h)} f\,d\tau = \tau(f(h))$$

for each continuous function f on $\mathrm{Sp}\,(h)$. Let g be a positive continuous function on $\mathrm{Sp}\,(h)$ and let

$$\Delta_g = \{s \in \mathrm{Sp}\,(h) \mid g(s)>0\}\,.$$

As ε tends to 0, the functions $f_\varepsilon \circ g$ converge pointwise to the characteristic function χ_{Δ_g} of Δ_g and

$$\int_{\mathrm{Sp}\,(h)} \chi_{\Delta_g}\, d\tau = \lim_{\varepsilon \to 0} \int_{\mathrm{Sp}\,(h)} f_\varepsilon \circ g\, d\tau = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(g(h))) = \lambda(g(h)) .$$

This situation can be interpreted in the opposite sense. In fact, λ induces then a measure $\bar{\lambda}$ on Sp (h) by $\bar{\lambda}(U) = \lambda(u(h))$ where U is any open subset of Sp (h) and u is a positive continuous function on Sp (h) chosen in such a way that $u(s) > 0 \Leftrightarrow s \in U$. From this point of view we get

$$\tau(g(h)) = \int_{\mathrm{Sp}\,(h)} g\, d\bar{\lambda}$$

for all continuous functions g on Sp (h). This is an exact analogy to the definition of the trace on a II_1-factor in terms of the dimension function by Murray–von Neumann in [17], [18]. In particular this shows that, if there is a finite trace on a factorial algebra, then this trace is unique.

5. Factorial AF-algebras.

A simple AF-algebra with unit is always finite. We are now going to show that a large class of simple AF-algebras is even factorial. From the dimension group associated with the algebra (cf. [12]) it can immediately be seen if a given simple AF-algebra is factorial.

5.1. LEMMA. *Let A be an* AF-*algebra. Let a, e, e' be positive elements of A such that $\|a\| = \|e\| = \|e'\| = 1$ and such that $ea = a$, $e'e = e$ (this implies in particular $a \leqq e \leqq e'$). Then, for every $\varepsilon > 0$, there exists a projection p in A satisfying $\|pa - a\| < \varepsilon$ and $e'p = p$.*

PROOF. Let A' be the dense locally finite dimensional involutive subalgebra of A. Let $n \geqq 7$ be a natural number. There is $x = x^* \in A'$ such that $\|x - a\| < \varepsilon/n$ (n will be chosen appropriately at the end of the proof). Let s be the spectral projection of x corresponding to the interval $[\varepsilon/3, 1 + \varepsilon/n]$. Then

$$\|sx - a\| \leqq \|sx - x\| + \|x - a\| < 2\varepsilon/3 .$$

There exists $\bar{x}$ in A' such that $x\bar{x} = \bar{x}x = s$ and $s\bar{x} - \bar{x}s = \bar{x}$. We have $\|\bar{x}\| \leqq 3\varepsilon^{-1}$.

It follows that

$$\begin{aligned} \|se - s\| &= \|\bar{x}xe - \bar{x}x\| \leqq \|\bar{x}\|(\|(x-a)e\| + \|ae - x\|) \\ &\leqq (3/\varepsilon)(2\varepsilon/n) = 6/n . \end{aligned}$$

Since also $ae^2=a$, the same calculation shows $\|se^2-s\|\leqq 6/n$. Let b be the relative inverse of se^2s in A_s. We see that $\|b\|\leqq 1/(1-6/n)$.

Write $z=b^{\frac{1}{2}}se$. Then $zz^*=s$ and $p=z^*z=esbse$ is a projection. It is clear that $e'p=p$. Furthermore

$$\|z^*-s\| = \|esb^{\frac{1}{2}}-s\| \leqq \|(es-s)b^{\frac{1}{2}}\|+\|sb^{\frac{1}{2}}-s\|$$
$$\leqq 6n^{-1}(1-6/n)^{-\frac{1}{2}}+((1-6/n)^{-\frac{1}{2}}-1) = \alpha_n .$$

Obviously $\alpha_n \to 0$ as n tends to infinity. Now

$$\|p-s\| \leqq \|p-z^*s\|+\|z^*s-s\| = \|z^*(z-s)\|+\|(z^*-s)s\| \leqq 2\alpha_n .$$

Finally

$$\|pa-a\| \leqq \|pa-sa\|+\|sa-sx\|+\|sx-a\| < 2\alpha_n+\varepsilon/n+2\varepsilon/3$$

and this expression is smaller than ε, if n is sufficiently large.

5.2. THEOREM. *Let A be a simple* AF-*algebra with unit. Let $\{q_n\}$ be a sequence of projections in A such that $(1/q_n)\to\infty$. If there is a constant $K>0$ such that for all projections p in A and for all $n\in\mathbf{N}$*

$$(p/q_n) \leqq (p/q_n)^{\sim}+K$$

then $\{q_n\}$ is a fundamental sequence in A.

PROOF. It is enough to show that we have $(h/q_n)\leqq(h/q_n)^{\sim}+K$ for all positive elements h of A.
In fact, we have for arbitrary $a\in A$ the relation $|a|^{\frac{1}{2}}\gtrsim a$ (1.6) so that, if the desired inequality holds for $|a|^{\frac{1}{2}}$, we get

$$(a/q_n) \leqq (|a|^{\frac{1}{2}}/q_n) \leqq (|a|^{\frac{1}{2}}/q_n)^{\sim}+K = (a/q_n)^{\sim}+K .$$

Thus, let $h\in A$ be positive and let q be an element of $\{q_n\}$. Write $\varepsilon_n=4^{-n}$. Lemma 5.1 applied to $a=f_{\varepsilon_1}(h)$, $e=f_{2\varepsilon_2}(h)$ $e'=f_{\varepsilon_2}(h)$ shows that there is a projection $p_1\in A$ satisfying

$$\|p_1f_{\varepsilon_1}(h)-f_{\varepsilon_1}(h)\| < \varepsilon_1 \quad \text{and} \quad f_{\varepsilon_2}(h)p_1 = p_1 .$$

Suppose that p_{n-1} has been constructed. Again by Lemma 5.1 applied to

$$a = f_{\varepsilon_n}(h)-p_{n-1},\ e = f_{2\varepsilon_{n+1}}(h)-p_{n-1} \quad \text{and} \quad e' = f_{\varepsilon_{n+1}}(h)-p_{n-1} ,$$

there is a projection r in $A_{(1-p_{n-1})}$ such that

$$\|r(f_{\varepsilon_n}(h)-p_{n-1})-(f_{\varepsilon_n}(h)-p_{n-1})\| < \varepsilon_n$$

and

$$(f_{\varepsilon_{n+1}}(h)-p_{n-1})r = r .$$

With $p_n = p_{n-1} + r$ this gives

$$\|p_n f_{\varepsilon_n}(h) - f_{\varepsilon_n}(h)\| < \varepsilon_n \quad \text{and} \quad f_{\varepsilon_{n+1}}(h) p_n = p_n .$$

Let $N = \sup_{n \in \mathbf{N}} (p_n/q)$. By Lemma 1.7 and 1.1 IV there exist projections s_n (one for each $n \in \mathbf{N}$) in $M_N \otimes A$ such that

$$s_n \sim e_{11} \otimes p_n \quad \text{and} \quad s_n \leqq \mathbf{1}_N \otimes q .$$

Since $M_N \otimes A$ is an AF-algebra, so is $B = (\mathbf{1}_N \otimes q)(M_N \otimes A)(\mathbf{1}_N \otimes q)$ (cf. [7, 1.5(iii)]). There are partial isometries v_n in B such that $v_n^* v_n = s_n$ and $v_n v_n^* \leqq s_{n+1}$. Every v_n has a unitary extension u_n in B with $u_n s_n = v_n$ and $u_n^* u_n = u_n u_n^* = \mathbf{1}_N \otimes q$ (this follows from 1.6 and 1.8 in [13] and the corresponding fact for finite-dimensional C*-algebras). Hence replacing s_{n+1} by $u_1^* \ldots u_n^* s_{n+1} u_n \ldots u_1$ we may even assume that $s_1 \leqq s_2 \leqq s_3 \leqq \ldots$.

There are elements x_n of $M_N \otimes A$ satisfying

$$x_n^* x_n = e_{11} \otimes p_n, \quad x_n x_n^* = s_n \quad \text{and} \quad s_n x_{n+1} = x_n .$$

By construction of the projections p_n we have $p_n h^{\frac{1}{2}} \to h^{\frac{1}{2}}$. In fact

$$\begin{aligned} \|p_n h^{\frac{1}{2}} - h^{\frac{1}{2}}\| &\leqq \|p_n h^{\frac{1}{2}} - p_n f_{\varepsilon_n}(h) h^{\frac{1}{2}}\| + \|p_n f_{\varepsilon_n}(h) h^{\frac{1}{2}} - f_{\varepsilon_n}(h) h^{\frac{1}{2}}\| \\ &\quad + \|f_{\varepsilon_n}(h) h^{\frac{1}{2}} - h^{\frac{1}{2}}\| \\ &\leqq \varepsilon_n^{\frac{1}{2}} + \varepsilon_n \|h^{\frac{1}{2}}\| + \varepsilon_n^{\frac{1}{2}} . \end{aligned}$$

Consequently the limit $y = \lim_{n \to \infty} x_n (e_{11} \otimes h^{\frac{1}{2}})$ exists in $M_N \otimes A$ and $y^* y = e_{11} \otimes h$ while $(\mathbf{1}_N \otimes q) y y^* = y y^*$.

Hence $e_{11} \otimes h = y^* (\mathbf{1}_N \otimes q) y$ and by 1.1 IV this shows $(h/q) \leqq N$.

To complete the proof we remark that

$$N = \sup_{n \in \mathbf{N}} (p_n/q) \leqq \sup_{n \in \mathbf{N}} (p_n/q)^{\sim} + K$$

by the hypothesis of the theorem, and that

$$\sup_{n \in \mathbf{N}} (p_n/q)^{\sim} \leqq (h/q)^{\sim}$$

since $p_n \in A_h$.

5.3. We adopt the notation of Elliott in [12]. Let A be an infinite dimensional simple AF-algebra with unit, E its set of projections and d the dimension on A, that is the map of E into $\sim$-equivalence classes. $d(E)$ is embedded into an ordered abelian group — the dimension group of A. The dimension group of $M_n \otimes A$ is the same as that of A.

For two projections $p, q \in E$ we see using 1.1 IV and 1.7

$$(p/q) \leqq n \quad \Leftrightarrow \quad d(p) \leqq nd(q)$$

$$(p/q)^{\sim} \geqq n \quad \Leftrightarrow \quad d(p) \geqq nd(q)$$

Thus, by Theorem 5.2, A is factorial if (and only if) for any two projections $p, q \in A$, there is $n \in \mathbf{N}$ such that

$$nd(q) \leqq d(p) \leqq (n+2)d(q) .$$

It is readily seen that this is true if the positive cone of the dimension group is a whole half-space as in [12,6.4–6.6].

5.4. Let A be an AF-algebra and let $h \in A$ be positive. In the course of the proof of Theorem 5 it was shown that there is a sequence of projections $p_1 \leqq p_2 \leqq p_3 \leqq \ldots$ in A_h such that $p_n h p_n$ converges to h as n tends to infinity. This may be viewed as a partial answer to the question of Dixmier, if in a matroid algebra there exist "sufficiently" many spectral projections for a given self-adjoint element [7,8.2]. We remark that, by a recent result of Bratteli [3], existence of "exact" spectral projections in AF-algebras can not be expected in general.

5.5. G. Elliott pointed out to me that, for simple AF-algebras (with unit), uniqueness of the trace implies the factorial property. This follows from a theorem of Goodearl and Handelman [14, 4.3] applied to the dimension group associated with the algebra, together with Theorem 5.2.

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