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INTEGRABLE GROUP ACTIONS ON VON NEUMANN ALGEBRAS

W. L. PASCHKE[1]

1. Introduction.

In what follows, M will denote a von Neumann algebra with separable predual, G a separable locally compact abelian group. We shall be concerned with *integrable* actions α (recently introduced and studied by A. Connes and M. Takesaki in [4]) of G as *-automorphisms on M, and in particular with certain relationships among the fixed-point algebra M_α, the crossed product $W^*(M,G,\alpha)$ of M by α [13], and the spectral invariant $\Gamma(\alpha)$ [3] in this setting. Section 2 below extends the analysis undertaken in [10] for compact-group actions to integrable actions; the main result in section 2 is, roughly, that if α is an integrable action of G on M satisfying a certain spectral condition, then $W^*(M,G,\alpha)$ is isomorphic to the algebra of bounded module maps on a self-dual inner product module over M_α (so in particular M_α and $W^*(M,G,\alpha)$ have isomorphic centers). In section 3, we use this result to show that if α is any continuous action of G on M such that $M_\alpha' \cap M \subseteq M_\alpha$, then $\Gamma(\alpha) \subseteq \Gamma(\alpha \otimes \beta)$ for any continuous action β of G on a von Neumann algebra. Finally, we consider in section 4 the canonical implementation $\{U_t : t \in G\}$ of α by unitaries on a Hilbert space on which M acts standardly, and show that if α is integrable, then the von Neumann algebra generated by M and the U_t's is a homomorphic image of both $W^*(M,G,\alpha)$ and the abovementioned module map algebra.

We now recall some definitions and establish our notation. A *continuous action* of G on M is a homomorphism $t \to \alpha_t$ from G into the group of *-automorphisms of M such that for each $x \in M$, the map $t \to \alpha_t(x)$ is continuous with respect to the ultraweak topology on M (or equivalently, as a one-line computation shows, with respect to the ultrastrong* topology on M). We let

$$M_\alpha = \{x \in M : \alpha_t(x) = x \ \forall t \in G\} ,$$

a von Neumann subalgebra of M. To define the crossed product $W^*(M,G,\alpha)$ for such an action α, we represent M faithfully and normally on a separable Hilbert space H, and consider the Hilbert space $L^2(G,H)$ of measurable, norm-

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square integrable functions from G into H. For $s \in G$, $x \in M$, we define operators L_s, $\Lambda(x)$ on $L^2(G,H)$ by

$$(L_s\Phi)(t) = \Phi(t-s) \qquad (\Lambda(x)\Phi)(t) = \alpha_{-t}(x)(\Phi(t))$$

($\Phi \in L^2(G,H)$) and let $W^*(M,G,\alpha)$ be the von Neumann algebra on $L^2(G, H)$ generated by these operators. This definition of $W^*(M,G,\alpha)$ does not in fact depend on the particular faithful normal representation of M we use (3.4 of [13]).

We next define the spectral invariants $\mathrm{sp}(\alpha)$ and $\Gamma(\alpha)$ as in [1] and [2], respectively. For $f \in L^1(G)$, $x \in M$, there is an element $\theta_\alpha(f)(x) \in M$ such that

$$\varphi(\theta_\alpha(f)(x)) = \int_G f(t)\,\varphi(\alpha_t(x))\,dt \qquad \forall\,\varphi \in M_* .$$

The map θ_α is a homomorphism from the convolution algebra $L^1(G)$ into the algebra of ultraweakly continuous linear operators on M (with $\|\theta_\alpha(f)\| \leqq \|f\|$), so $\theta_\alpha^{-1}(0)$ is a closed ideal of $L^1(G)$. Let $\mathrm{sp}(\alpha)$ be the zero-set in $\hat{G}$ (the dual group of G) of this ideal, i.e.

$$\mathrm{sp}(\alpha) = \{\chi \in \hat{G} : \hat{f}(\chi) = 0 \ \forall f \in \theta_\alpha^{-1}(0)\} .$$

If e is a non-zero projection in M_α, let $\alpha^{(e)}$ be the action of G on eMe obtained by restriction of α. We set

$$\Gamma(\alpha) = \bigcap \{\mathrm{sp}(\alpha^{(e)}) : e \text{ a non-zero projection in } M_\alpha\} .$$

Following [13], we define the dual action of $\hat{G}$ on $W^*(M,G,\alpha)$ by letting W_χ, for $\chi \in \hat{G}$, be the unitary operator on $L^2(G,H)$ given by $(W_\chi\Phi)(t) = \chi(t)\Phi(t)$ ($\Phi \in L^2(G,H)$); the dual action $\hat{\alpha}$ is defined by $\hat{\alpha}_\chi(T) = W_\chi^* T W_\chi$ ($T \in W^*(M,G,\alpha)$). (One checks easily that each $\hat{\alpha}_\chi$ maps the generators of $W^*(M,G,\alpha)$ to scalar multiples of themselves).

Because M has separable predual, we may assume that M acts on a separable Hilbert space H with a cyclic and separating vector. In this situation (actually in the more general context of arbitrary von Neumann algebras acting standardly on Hilbert space), U. Haagerup [7] and T. Digernes [5], [6] construct a "canonical implementation" of the action α by a strongly continuous group of unitaries $\{U_t : t \in G\}$ on H. (See section 4 below). We quote 3.13 of [6], which identifies generators for the commutant of $W^*(M,G,\alpha)$ under these circumstances.

1.1. THEOREM. *$W^*(M,G,\alpha)'$ is generated by the operators $\Lambda'(w)$ ($w \in M'$) and R_s ($s \in G$) defined on $L^2(G,H)$ by*

$$(\Lambda'(w)\Phi)(t) = w(\Phi(t)) \qquad (R_s\Phi)(t) = U_s(\Phi(s+t)) \qquad (\Phi \in L^2(G,H)) .$$

2. Integrable actions.

Let α be a continuous action of G on M. For $x \in M$ and K a compact subset of G, the integral (taken in the ultraweak sense)

$$\langle x, x \rangle_K = \int_K \alpha_t(x^* x)\, dt$$

defines an element of M. Let X_α denote the set of those $x \in M$ such that

$$\int_G \alpha_t(x^* x)\, dt$$

exists, i.e. such that the increasing net $\{\langle x, x \rangle_K : K^{\text{compact}} \subseteqq G\}$ is norm-bounded. Clearly, X_α is a left ideal of M and a right M_α-module. Following [4], we say that α is *integrable* if X_α is ultraweakly dense in M. It is immediate that if G is compact, then any continuous action of G is integrable; our goal in this section is to generalize much of the analysis in [10] on compact groups of automorphisms to the setting of integrable actions of locally compact abelian groups.

One elementary but useful example of an integrable action arises from the left regular representation of G on $L^2(G)$. For $t \in G$, let V_t be the unitary operator on $L^2(G)$ defined by $(V_t\xi)(s) = \xi(s-t)$. The action λ of G on $B(L^2(G))$ defined by $\lambda_t(S) = V_t S V_t^*$ is easily seen to be integrable. (Indeed, one checks that X_λ contains an ultraweakly dense subalgebra of the algebra $L^\infty(G)$ of multiplication operators on $L^2(G)$.) If α is any continuous action of G on M, and β is an integrable action of G on another von Neumann algebra N, then the tensor product action $\alpha \otimes \beta$ of G on $M \otimes N$ defined by $(\alpha\otimes\beta)_t = \alpha_t \otimes \beta_t$ is integrable because $X_{\alpha\otimes\beta}$ contains all operators of the form $x \otimes y$ ($x \in M$, $y \in X_\beta$). By passing from α to $\alpha \otimes \lambda$, then, we have a convenient way of converting continuous actions into integrable ones.

If α is an integrable action of G on M, we define $\langle \cdot, \cdot \rangle : X_\alpha \times X_\alpha \to M_\alpha$ by

$$\langle x, y \rangle = \int_G \alpha_t(y^* x)\, dt \qquad (x, y \in X_\alpha).$$

(One checks that the integral in question exists as the ultraweak limit of corresponding integrals over compact subsets of G.) This makes X_α into an inner product module over M_α (see [9] or, for a summary discussion, [10]), i.e. $\langle x, x \rangle \geqq 0$, $\langle x, x \rangle = 0$ implies $x = 0$, $\langle x, y \rangle^* = \langle y, x \rangle$, and $\langle xa, y \rangle = \langle x, y \rangle a$ ($x, y \in X_\alpha$, $a \in M_\alpha$). Let $\bar{X}_\alpha$ be the inner product module "completion" of X_α to a self-dual module over M_α in the sense of section 3 of [9], $A(\bar{X}_\alpha)$ the (von Neumann) algebra of bounded module maps of $\bar{X}_\alpha$ into itself. Our immediate aim is to obtain an isomorphism of $A(\bar{X}_\alpha)$ with a reduced subalgebra of $W^*(M, G, \alpha)$ by a central projection as was done in [10] in the compact case.

For $x \in X_\alpha$, $\xi \in H$, define $x\odot\xi\colon G \to H$ by $(x\odot\xi)(s)=\alpha_{-s}(x)\xi$. We have

$$\int_G \|(x\odot\xi)(s)\|^2\,ds = \int_G (\alpha_{-s}(x^*x)\xi,\xi)\,ds < \infty ,$$

so $x\odot\xi \in L^2(G,H)$. Let $X_\alpha\odot H$ denote the closed linear span in $L^2(G,H)$ of $\{x\odot\xi : x \in X_\alpha,\ \xi \in H\}$, and let P_α be the projection of $L^2(G,H)$ on $X_\alpha\odot H$. As in [10], but for less transparent reasons, we have

2.1. Proposition. $P_\alpha \in$ center $(W^*(M, G, \alpha))$.

Prooof. We assume that M acts on H with a cyclic and separating vector, so 1.1 above is applicable. For $x \in X_\alpha$, $w \in M'$, $\xi \in H$, and $t \in G$, one verifies immediately that

$$\Lambda'(w)(x\odot\xi) = x\odot w\xi, \qquad R_t(x\odot\xi) = x\odot U_t\xi ,$$

so $X_\alpha\odot H$ is invariant under $W^*(M, G,\alpha)'$. But also

$$\Lambda(y)(x\odot\xi) = yx\odot\xi \qquad (y \in M)$$

$$L_t(x\odot\xi) = \alpha_t(x)\odot\xi ,$$

so $X_\alpha\odot H$ is invariant under $W^*(M, G, \alpha)$ as well.

We let $\langle X_\alpha, X_\alpha\rangle$ denote the linear span of $\{\langle x, y\rangle : x, y \in X_\alpha\}$, so $\langle X_\alpha, X_\alpha\rangle$ is a two-sided ideal of M_α. Further, $X_\alpha^* X_\alpha$ will denote the subspace of M spanned by $\{y^*x : x, y \in X_\alpha\}$, and for $\chi \in \hat{G}$, we set

$$M(\alpha, \chi) = \{y \in M : \alpha_t(y)=\chi(t)y \ \forall t \in G\} .$$

Define $E_\chi\colon X_\alpha^* X_\alpha \to M(\alpha, \chi)$ by

$$E_\chi(y) = \int_G \overline{\chi(t)}\alpha_t(y)\,dt .$$

Finally, for a subset S of M, $L(S)$ will denote the left annihilator of S in M.

2.2. Proposition. *If α is an integrable action of G on M, then $\langle X_\alpha, X_\alpha\rangle$ is ultraweakly dense in M_α and*

$$L(M(\alpha, \chi)) = L(E_\chi(X_\alpha^* X_\alpha)) \quad \forall \chi \in \hat{G} .$$

Proof. Let q be a central projection of M_α annihilating $\langle X_\alpha, X_\alpha\rangle$. For $x \in X_\alpha$, we have

$$0 = q\left(\int_G \alpha_t(x^*x)\,dt\right)q = \int_G \alpha_t(qx^*xq)\,dt\,,$$

forcing $xq=0$. Since X_α is ultraweakly dense in M, we must have $q=0$, so $\langle X_\alpha, X_\alpha\rangle$ is ultraweakly dense in M_α.

Take $\chi \in \hat{G}$ and suppose $a \in L(E_\chi(X_\alpha^* X_\alpha))$. For $x \in M(\alpha,\chi)$ and $y_1 \in X_\alpha$, we have

$$\alpha_t((y_1x)^*(y_1x)) = x^*\alpha_t(y_1^*y_1)x \qquad \forall t \in G\,,$$

so $y_1x \in X_\alpha$. If $y_2 \in X_\alpha$, then $a\langle y_2, y_1\rangle x = aE_\chi(y_2^*y_1x)=0$. Since $\langle X_\alpha, X_\alpha\rangle$ is ultraweakly dense in M_α, this implies that $ax=0$, hence $a \in L(M(\alpha,\chi))$.

Arguing now exactly as in section 3 of [10], we construct a faithful normal *-representation of $A(\bar{X}_\alpha)$ on $X_\alpha \odot H$ such that $\varrho(T)(x\odot\xi)=Tx\odot\xi$ for every $T\in A(\bar{X}_\alpha)$ such that $TX_\alpha \subseteq X_\alpha$.

2.3. Proposition. *If α is an integrable action of G on M, then* $\varrho(A(\overline{X_\alpha})) = W^*(M,G,\alpha)|_{X_\alpha\odot H}$.

Proof. To see that $W^*(M,G,\alpha)|_{X_\alpha\odot H} \subseteq \varrho(A(\overline{X_\alpha}))$, proceed exactly as in the first part of the proof of 3.1 of [10], replacing M by X_α throughout. For the reverse inclusion, we may assume that M has a cyclic and separating vector, so that 1.1 above is applicable. The operators $T\in A(\overline{X_\alpha})$ such that $T(X_\alpha)\subseteq X_\alpha$ are ultraweakly dense in $A(\overline{X_\alpha})$ by 2.5 of [10]. For such a T (and for $w \in M'$, $x \in X_\alpha$, $s \in G$, and $\xi \in H$) we have

$$\begin{aligned}\varrho(T)\Lambda'(w)(x\odot\xi) &= \varrho(T)(x\odot w\xi)\\ &= Tx\odot w\xi = \Lambda'(w)\varrho(T)(x\odot\xi)\end{aligned}$$

and

$$\varrho(T)R_s(x\odot\xi) = Tx\odot U_s\xi = R_s\varrho(T)(x\odot\xi)\,,$$

so $\varrho(T)$ commutes with $W^*(M,G,\alpha)'|_{X_\alpha\odot H}$, which gives the desired inclusion.

If $y \in$ center M_α, it is immediate that $Ty(\tau)=\tau\cdot y$ ($\tau \in \overline{X_\alpha}$) defines a bounded module map in the center of $A(\overline{X_\alpha})$. Using the ultraweak density of $\langle X_\alpha, X_\alpha\rangle$ in M_α, a routine argument shows that the center of $A(\overline{X_\alpha})$ consists precisely of maps of this form. Notice that for such y we have

$$\varrho(Ty)(x\odot\xi) = xy\odot\xi = x\odot y\xi \qquad (x \in X_\alpha,\ \xi \in H)\,.$$

By 2.3, then, we have

2.4. REMARK. Let α be an integrable action of G on M. If $T \in$ center $(W^*(M,G,\alpha))$, there is a $y \in$ center M_α such that

$$T(x \odot \xi) = xy \odot \xi = x \odot y\xi \quad (x \in X_\alpha,\ \xi \in H).$$

It is of interest to know when $X_\alpha \odot H = L^2(G,H)$, so that ϱ is a *-isomorphism of $A(\overline{X_\alpha})$ with $W^*(M,G,\alpha)$. The necessary and sufficient condition given for this in the compact case given in 4.6 of [10] generalizes to integrable actions.

2.5. PROPOSITION. *Let α be an integrable action of G on M. Then $X_\alpha \odot H = L^2(G,H)$ if and only if $L(M(\alpha,\chi)) = 0\ \forall \chi \in \hat{G}$.*

PROOF. We need only make certain modifications in the proof of 4.6 of [10]. Suppose first that $X_\alpha \odot H = L^2(G,H)$, and take $\chi \in \hat{G}$. Let $\eta \in (M(\alpha,\chi)H)^\perp$. Given $y_1, y_2 \in X_\alpha$, define $\Phi \in L^2(G,H)$ by

$$\Phi(t) = \chi(t)\alpha_{-t}(y_1)\eta$$

(that is $\Phi = W_\chi(y_1 \odot \eta)$). For any $\xi \in H$, we have

$$(y_2 \odot \xi, \Phi) = (E_\chi(y_1^* y_2)\xi, \eta) = 0.$$

This shows that $\Phi = 0$, and hence $\eta = 0$ because of the ultraweak density of X_α in M. Since $(M(\alpha,\chi)H)^\perp = 0$, we must have $L(M(\alpha,\chi)) = 0$.

Conversely, suppose that $L(M(\alpha,\chi)) = 0\ \forall \chi \in \hat{G}$. Let $\Phi \in (X_\alpha \odot H)^\perp$ and take $\xi \in H$. We will show that

$$((\Lambda(y)\Phi)(t), \xi) = 0 \text{ a.e.} \quad \forall y \in X_\alpha^*,$$

which forces $\Phi = 0$. Given $y \in X_\alpha^*$, define f on G by $f(t) = ((\Lambda(y)\Phi)(t), \xi)$. We have $f \in L^1(G)$ because $y^* \odot \xi \in L^2(G,H)$ and

$$\begin{aligned}\int_G |f(t)|\,dt &= \int_G |(\alpha_{-t}(y)(\Phi(t)), \xi)|\,dt \\ &= \int_G |(\Phi(t), (y^* \odot \xi)(t))|\,dt \\ &\leqq \|\Phi\|\,\|y^* \odot \xi\|.\end{aligned}$$

Given $\chi \in \hat{G}$ and $\delta > 0$, we can find $\xi_1, \ldots, \xi_\eta \in H$ and $y_1, \ldots, y_n \in M(\alpha,\chi)$ such that

$$\left\| \xi - \sum_{i=1}^{n} y_i \xi_i \right\| < \delta$$

(since $[M(\alpha,\chi)H]=H$). Set $\eta=\xi-\sum_{i=1}^{n} y_i\xi_i$. We have

$$|\hat{f}(\chi)| \leqq \left|\int_G \overline{\chi(t)}(\alpha_{-t}(y)(\Phi(t)),\eta)\,dt\right| + \\ +\left|\sum_{i=1}^{n}\int_G \overline{\chi(t)}(\alpha_{-t}(y)(\Phi(t)),y_i\xi_i)\,dt\right| .$$

For $i=1,2,\ldots,n$, though, the ith term in the second expression equals $(\Phi, y^*y_i\odot\xi_i)$ (because $y_i\in M(\alpha,\chi)$) and hence vanishes by our choice of Φ, while

$$\begin{aligned}\int_G |(\alpha_{-t}(y)(\Phi(t)),\eta)|\,dt &\leqq \int_G \|\Phi(t)\|\;\|\alpha_{-t}(y^*)\eta\|\,dt \\ &\leqq \|\Phi\|\left(\int_G (\alpha_{-t}(yy^*)\eta,\eta)\,dt\right)^{\frac{1}{2}} \\ &\leqq \|\Phi\|\;\|\langle y^*,y^*\rangle\|^{\frac{1}{2}}\delta .\end{aligned}$$

As δ was arbitrary, we have $\hat{f}(\chi)=0$ $\forall\,\chi\in\hat{G}$ and hence $f=0$ a.e. as required.

3. Invariant Γ for integrable actions.

Theorem 3.3.2 of [4] states that if α is a continuous action of G on M, then $\Gamma(\alpha)$ is the kernel of the restriction of the dual action $\hat{\alpha}$ to the center of $W^*(M,G,\alpha)$. This result, together with the results of the preceding section of the present paper, can be used to give a proof of our next proposition. We prefer, however, to give a self-contained proof whose main ideas are similar to those of the proof of the theorem just cited.

3.1. THEOREM. *If α is an integrable action of G on M, then $\Gamma(\alpha)$ consists precisely of those $\chi\in\hat{G}$ such that*

(i) $L(M(\alpha,\chi))=0$;

(ii) $M(\alpha,\chi)\subseteqq(\text{center } M_\alpha)'$.

We will require the following lemma, which is a modification of 9.5 of [13].

3.2. LEMMA. *Let β be a continuous action of G on a commutative von Neumann algebra A. If p is a projection in A such that $\beta_{s_0}(p)\neq p$ for some $s_0\in G$, then there is a non-zero projection $q\leqq p$ and a neighborhood U of s_0 in G such that $\beta_s(q)q=0$ $\forall\, s\in U$.*

PROOF OF LEMMA. We may assume that $\vee \{\beta_t(p) : t \in G\} = 1$; otherwise set $p_0 = \vee \{\beta_t(p) : t \in G\}$ and replace A by Ap_0, β by $\beta^{(p_0)}$. Appealing to 9.5 of [13] (which is stated for the special case $G = \mathbf{R}$, but whose proof, *mutatis mutandis*, is valid for arbitrary locally compact abelian G), we find a non-zero projection $q_1 \in A$ and a neighborhood U of s_0 such that $\beta_t(q_1)q_1 = 0 \ \forall t \in U$. By our initial assumption on p, we must have $\beta_{s_1}(q_1)p \neq 0$ for some $s_1 \in G$. Set $q = \beta_{s_1}(q_1)p$. Then for $t \in U$, we have

$$\beta_t(q)q = \beta_t(p)\beta_{t+s_1}(q_1)p = \beta_t(p)\beta_{s_1}(\beta_t(q_1)q_1)p = 0 .$$

PROOF OF 3.1. We first show that if $\chi_0 \in \hat{G}$ is such that $L(M(\alpha,\chi_0)) \neq 0$, then $\chi_0 \notin \Gamma(\alpha)$. Let p be a non-zero projection such that $pM(\alpha,\chi_0) = 0$, and let P be the projection of $L^2(G,H)$ on the closed linear span in $L^2(G,H)$ of $\{x \odot p\xi : x \in X_\alpha,\ \xi \in H\}$. Because $p \neq 0$ and X_α is ultraweakly dense in M, we have $P \neq 0$, and also of course $P \leqq P_\alpha$. For any $y_1, y_2 \in X_\alpha$, $\xi_1, \xi_2 \in H$,

$$(W_{\chi_0}(y_1 \odot \xi_1), y_2 \odot p\xi_2) = (pE_{\chi_0}(y_2^* y_1)\xi_1, \xi_2) = 0$$

so $PW_{\chi_0}P_\alpha = 0$. Hence P_α cannot commute with W_{χ_0}, that is, $\hat{\alpha}_{\chi_0}(P_\alpha) \neq P_\alpha$.

By 3.2 (applied to the restriction of $\hat{\alpha}$ to the center of $W^*(M,G,\alpha)$), there is a non-zero projection $E \in$ center $(W^*(M,G,\alpha))$, $E \leqq P$, and a neighborhood U of χ_0 in $\hat{G}$ such that $\hat{\alpha}_\chi(E)E = 0 \ \forall \chi \in U$. Since $E(X_\alpha \odot H) \neq 0$, there is, by 2.4., a non-zero projection $e \in$ center (M_α) such that

$$E(y \odot \xi) = y \odot e\xi \qquad (y \in X_\alpha,\ \xi \in H) .$$

Take $\chi \in U$. Then $EW_\chi E = 0$, so for any $y_1, y_2 \in X_\alpha$ and $\xi_1, \xi_2 \in H$, we have

$$\begin{aligned} 0 &= (EW_\chi E(y_1 \odot \xi_1), y_2 \odot \xi_2) \\ &= (W_\chi(y_1 \odot e\xi_1), y_2 \odot e\xi_2) \\ &= \int_G \overline{\chi(t)}(e\alpha_t(y_2^* y_1)e\xi_1, \xi_2)\,dt . \end{aligned}$$

This shows that $eE_\chi(X_\alpha^* X_\alpha)e = 0 \ \forall \chi \in U$. There is a continuous function $f \in L^1(G)$ such that $\hat{f}(\chi_0) \neq 0$, $\operatorname{supp}(\hat{f}) \subseteqq U$, $\hat{f} \in L^1(\hat{G})$, and

$$f(t) = \int_{\hat{G}} \hat{f}(\chi)\chi(t)\,d\chi \qquad (t \in G) ,$$

$d\chi$ being Haar measure on $\hat{G}$, appropriately normalized. Take $y_1, y_2 \in X_\alpha$ and let φ be a normal positive functional on M. One checks that the function $t \to \varphi(e\alpha_t(y_2^* y_1)e)$ belongs to $L^1(G)$. Fubini's theorem gives

$$\begin{aligned}\varphi(e\theta_\alpha(f)(y_2^*y_1)e) &= \int_G f(t)\varphi(e\alpha_t(y_2^*y_1)e)\,dt \\ &= \int_G\int_{\hat G} \hat f(\chi)\chi(t)\varphi(e\alpha_t(y_2^*y_1)e)\,d\chi\,dt \\ &= \int_U \hat f(\chi)\varphi(eE_{\bar\chi}(y_2^*y_1)e)\,d\chi = 0\,,\end{aligned}$$

since $E_{\bar\chi}(X_\alpha^*X_\alpha)=E_\chi(X_\alpha^*X_\alpha)^*$. But $X_\alpha^*X_\alpha$ is ultraweakly dense in M, and $\theta_\alpha(f)$ is ultraweakly continuous, so $e\theta_\alpha(f)(M)e=0$, that is, $\theta_{\alpha^{(e)}}(f)=0$. Since $\hat f(\chi_0)\neq 0$, we have $\chi_0 \notin \mathrm{sp}(\alpha^{(e)})$ and hence $\chi_0 \notin \Gamma(\alpha)$. We have shown that $\chi \in \Gamma(\alpha)$ implies $L(M(\alpha,\chi))=0$.

We next show that if $\chi \in \Gamma(\alpha)$, then $M(\alpha,\chi)\subseteqq(\text{center } M_\alpha)'$. Let e be a projection in center M_α. We claim that $L(eM(\alpha,\chi)e)=M(1-e)$. (If p is the projection in M such that $L(eM(\alpha,\chi)e)=Mp$, then certainly $p\leqq 1-e$. We must show that $pe=0$. Since the action α leaves $eM(\alpha,\chi)e$ globally invariant, we see that $p \in M_\alpha$. Let $q=pe$, so q is a projection in M_α and $qM(\alpha,\chi)q=0$. If $q\neq 0$, then we would have $(qMq)_{\alpha^{(q)}}=qM_\alpha q$, so $\chi \in \Gamma(\alpha^{(q)})$, and

$$(qMq)(\alpha^{(q)},\chi) = qM(\alpha,\chi)q = 0\,,$$

contradicting what was shown in the previous paragraph.) Likewise, $L((1-e)M(\alpha,\chi)(1-e))=Me$. Take $x,y \in M(\alpha,\chi)$. Then

$$(ey^*-y^*e)(exe) = ey^*exe-y^*exe = 0\,,$$

since $y^*ex \in M_\alpha$. Also

$$(ey^*-y^*e)((1-e)x(1-e)) = ey^*(1-e)x(1-e) = 0\,,$$

since $y^*(1-e)x \in M_\alpha$. This shows that $ey^*-y^*e \in M(1-e)\cap Me$, so y^*, and hence y, commutes with e. We conclude that $\chi \in \Gamma(\alpha)$ implies $M(\alpha,\chi)\subseteqq(\text{center } M_\alpha)'$.

Now suppose that $\chi \in \hat G$ satisfies $L(M(\alpha,\chi))=0$ and $M(\alpha,\chi)\subseteqq(\text{center } M_\alpha)'$. Let e be a non-zero projection in center M_α. If $f\in L^1(G)$ is such that $\theta_\alpha(f)(eMe)=0$, then in particular

$$0 = \theta_\alpha(f)(M(\alpha,\chi)^*e) = \hat f(\chi)M(\alpha,\chi)^*e\,.$$

Since $L(M(\alpha,\chi))=0$, we must have $\hat f(\chi)=0$. This shows that $\chi \in \bigcap\{\mathrm{sp}(\alpha^{(e)}) : e$ a non-zero projection in center $M_\alpha\}$. But by 2.2.2(b) of [2] (which is stated for factors M, but whose proof is in fact valid for arbitrary M), this last set is $\Gamma(\alpha)$.

3.3. Corollary. *Let α be an integrable action of G on M. Then $\Gamma(\alpha)=\hat G$ if and only if $L(M(\alpha,\chi))=0$ $\forall\,\chi \in \hat G$ and* center $M_\alpha\subseteqq$center M.

PROOF. If $L(M(\alpha,\chi))=0\ \forall \chi \in \hat{G}$ and center $M_\alpha \subseteq$ center M, then $\Gamma(\alpha)=\hat{G}$ by 3.1 above.

Conversely, suppose that $\Gamma(\alpha)=\hat{G}$. Then by 3.1 $L(M(\alpha,\chi))=0\ \forall \chi \in \hat{G}$ and the elements of center M_α commute with the subspace of M spanned by the $M(\alpha,\chi)$'s. The latter, however, is ultraweakly dense in M. (Suppose φ is a normal positive linear functional on M annihilating all the $M(\alpha,\chi)$'s. For $y_1, y_2 \in X_\alpha$, define $f \in L^1(G)$ by $f(t)=\varphi(\alpha_t(y_2^* y_1))$. We have

$$\hat{f}(\chi) = \varphi(E_\chi(y_2^* y_1)) = 0 \qquad \forall \chi \in \hat{G},$$

so, since f is continuous, $\varphi(X_\alpha^* X_\alpha)=0$, forcing $\varphi=0$.) This shows that center $M_\alpha \subseteq$ center M.

Our next result concerns the behavior of the invariant Γ with respect to tensor products of continuous actions of G on von Neumann algebras. Its proof will require the following reduction to the integrable case.

3.4. LEMMA. *Let α be a continuous action of G on M. Write $\tilde{M}=M\otimes B(L^2(G))$ and let $\tilde{\alpha}$ be the action $\alpha\otimes\lambda$ of G on $\tilde{M}$ (where λ is as in section 2 above). Then $\Gamma(\tilde{\alpha})=\Gamma(\alpha)$. Further, if $M'_\alpha \cap M \subseteq M_\alpha$, then $\tilde{M}'_{\tilde{\alpha}} \cap \tilde{M} \subseteq \tilde{M}_{\tilde{\alpha}}$.*

PROOF. It follows without difficulty from the definition of Γ that $\Gamma(\alpha) = \Gamma(\alpha\otimes 1)$, where 1 is the trivial action of G on $B(L^2(G))$. The actions $\alpha\otimes 1$ and $\alpha\otimes\lambda$ are equivalent ($\sim$) in the sense of 2.2.3 of [2] because $\{1\otimes V_t : t \in G\}$ (see section 2) is a strongly continuous group of unitaries in $\tilde{M}_{\alpha\otimes 1}$ satisfying

$$(\alpha\otimes\lambda)_t(T) = (1\otimes V_t)(\alpha\otimes 1)_t(T)(1\otimes V_t)^* \qquad \text{for } t \in G,\ T \in \tilde{M}.$$

It now follows from 2.2.4(c) of [2] (which is stated only for factors, but which in fact is valid for arbitrary von Neumann algebras) that $\Gamma(\alpha\otimes 1)=\Gamma(\alpha\otimes\lambda)$.

Suppose now that $M'_\alpha \cap M \subseteq M_\alpha$, and take $T \in \tilde{M}'_{\tilde{\alpha}} \cap \tilde{M}$. If φ is an ultraweakly continuous linear functional on $B(L^2(G))$, there is a unique "slice map" $R_\varphi \colon \tilde{M} \to M$ satisfying

$$R_\varphi(x\otimes a)=\varphi(a)x \qquad (x \in M,\ a \in B(L^2(G)))\,.$$

It follows that

$$R_\varphi(T(x\otimes 1)) = R_\varphi(T)x \quad \text{and} \quad R_\varphi((x\otimes 1)T) = xR_\varphi(T) \qquad \forall x \in M\,.$$

If $x \in M_\alpha$, then $x\otimes 1 \in \tilde{M}_{\tilde{\alpha}}$, and we have $R_\varphi(T)x = xR_\varphi(T)$. Hence $R_\varphi(T) \in M'_\alpha \cap M \subseteq M_\alpha$. By theorem 1 of [14], this means that $T \in M_\alpha \otimes B(L^2(G))$. It is well-known that $B(L^2(G))'_\lambda = B(L^2(G))_\lambda$, so by considering slice maps from $M_\alpha\otimes B(L^2(G))$ to $B(L^2(G))$, we see in like manner that $T \in M_\alpha \otimes B(L^2(G))_\lambda \subseteq \tilde{M}_{\tilde{\alpha}}$.

3.5. THEOREM. *Let α be a continuous action of G on M such that $M'_\alpha \cap M \subseteq M_\alpha$. Then $\Gamma(\alpha) \subseteq \Gamma(\alpha \otimes \beta)$ for any continuous action β of G on a von Neumann algebra N.*

PROOF. Using 3.5 to replace (M, α) by $(\tilde{M}, \tilde{\alpha})$ if necessary, we may assume that α is integrable and that for each $\chi \in \hat{G}$, $M(\alpha, \chi)$ contains a unitary u_χ. (Note that if $U_\chi \in B(L^2(G))$ is defined by $U_\chi \xi = \chi \xi$, then $\lambda_t(U_\chi) = \bar{\chi}(t) U_\chi$ for $\chi \in \hat{G}$, $t \in G$.)

Take $\chi \in \Gamma(\alpha)$. We will show that if $T \in$ center $(M \otimes N)_{\alpha \otimes \beta}$, then T commutes with each element of $(M \otimes N)(\alpha \otimes \beta, \chi)$; we can then conclude from 3.1 above that $\chi \in \Gamma(\alpha \otimes \beta)$. Since $M_\alpha \otimes 1 \subseteq (M \otimes N)_{\alpha \otimes \beta}$, T must commute with $M_\alpha \otimes 1$. Arguing with slice maps as in the proof of 3.4, we see that $T \in (M'_\alpha \cap M) \otimes N$. But $M'_\alpha \cap M \subseteq M_\alpha$ by assumption, so in fact $T \in$ (center M_α)$\otimes N$. Since $\chi \in \Gamma(\alpha)$, the unitary $u_\chi \in M(\alpha, \chi)$ commutes with center M_α by 3.1, so $T(u_\chi \otimes 1) = (u_\chi \otimes 1)T$. For arbitrary $S \in (M \otimes N)(\alpha \otimes \beta, \chi)$, we have $(u_\chi^* \otimes 1)S \in (M \otimes N)_{\alpha \otimes \beta}$, so

$$\begin{aligned} TS &= T(u_\chi \otimes 1)(u_\chi^* \otimes 1)S = (u_\chi \otimes 1)T[(u_\chi^* \otimes 1)S] \\ &= (u_\chi \otimes 1)[(u_\chi^* \otimes 1)S]T = ST, \end{aligned}$$

which is what we wanted.

It should be noted that the inclusion $\Gamma(\alpha) \subseteq \Gamma(\alpha \otimes \beta)$ fails in general without some sort of additional requirement on α. For what is probably the simplest possible example, let $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ and let M be the algebra of complex 2×2 matrices. Define an action α of G on M by

$$\alpha_{(1,0)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$\alpha_{(0,1)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\alpha_{(1,1)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

(These automorphisms are implemented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively.) One can check (using 3.1, for instance) that $\Gamma(\alpha) = \hat{G}$ but $\Gamma(\alpha \otimes \alpha)$ is trivial.

We remark that if M is properly infinite, then by 2.5.1 of [4], there is a faithful normal semifinite weight φ on M whose associated modular action σ^φ

of R on M is integrable and satisfies the relative commutant hypothesis of 3.5 above. From this it follows that for factors M and N we have (as announced in [3]) $\overline{S(M)S(N)} \subseteq S(M \otimes N)$, where $S(\cdot)$ is Connes's invariant [2].

4. Canonical implementation.

We recall here briefly certain aspects of the modular theory of faithful, normal, semi-finite (f.n.s.f.) weights on M. (For a summary discussion, see section 2 of [11] or 1.1 of [2]; the standard complete reference is [12].) Let $\varphi: M^+ \to [0, \infty]$ be such a weight. Our notation for the various objects associated with φ is as in [2], i.e.

$$\mathfrak{N}_\varphi = \{x \in M : \varphi(x^*x) < \infty\},$$

Hilbert space H_φ, injection $\eta_\varphi: \mathfrak{N}_\varphi \to H_\varphi$, representation π_φ of M on H_φ, "sharp" operator S_φ, modular conjugation operator J_φ, modular operator Δ_φ, and modular action σ^φ. Further, we let P_φ denote the closure in H_φ of the set $\{\pi_\varphi(x)J_\varphi\eta_\varphi(x) : x \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi\}$.

In [7], it is shown that P_φ is a cone and the following uniqueness result is obtained: if ψ is any other f.n.s.f. weight on M and $\sigma: \pi_\varphi(M) \to \pi_\psi(M)$ is an isomorphism, then there is a unique unitary $V: H_\varphi \to H_\psi$ such that

$$V\pi_\varphi(x)V^* = \sigma(\pi_\varphi(x)) \qquad (x \in M)$$

and $VP_\varphi = P_\psi$. For an action α of G on M, we write U_t^φ $(t \in G)$ for the unique unitary on H_φ such that $U_t^\varphi P_\varphi = P_\varphi$ and

$$U_t^\varphi \pi_\varphi(x)(U_t^\varphi)^* = \pi_\varphi(\alpha_t(x)) \qquad (x \in M),$$

and call $\{U_t^\varphi : t \in G\}$ the canonical unitary implementation of α on H_φ. We let $W^*(M, U_G^\varphi)$ denote the von Neumann algebra on H_φ generated by $\pi_\varphi(M)$ and the U_t^φ's.

4.1. Lemma. *For any two f.n.s.f. weights φ and ψ on M, the algebras $W^*(M, U_G^\varphi)$ and $W^*(M, U_G^\psi)$ are (spatially) isomorphic.*

Proof. Let $V: H_\varphi \to H_\psi$ be unitary such that $VP_\varphi = P_\psi$ and $V\pi_\varphi(x)V^* = \pi_\psi(x)$ $(x \in M)$. We have $VU_t^\varphi V^* P_\psi = P_\psi$ and

$$(VU_t^\varphi V^*)\pi_\psi(x)(VU_t^\varphi V^*) = \pi_\psi(\alpha_t(x)),$$

so by uniqueness $VU_t^\varphi V^* = U_t^\psi$ $(t \in G)$ and we see that V implements a spatial isomorphism of $W^*(M, U_G^\varphi)$ with $W^*(M, U_G^\psi)$.

4.2. Theorem. *Let α be an integrable action of G on M, φ a f.n.s.f. weight on M. Then $W^*(M, U_G^\varphi)$ is a homomorphic image of $W^*(M, G, \alpha)$.*

PROOF. By 4.1, it will suffice to prove the theorem for a particular, advantageously chosen weight φ. Accordingly, let M act on a Hilbert space H with cyclic and separating vector $\xi_0 \in H$. For $h \in M^+$, let

$$\varphi(h) = \int_G (\alpha_t(h)\xi_0, \xi_0)\,dt .$$

Because $\varphi(x^*x) < \infty \ \forall x \in X_\alpha$, we see that φ is an α-invariant f.n.s.f. weight on M. By α-invariance, there are unitaries W_t $(t \in G)$ on H_φ such that $W_t\eta_\varphi(x) = \eta_\varphi(\alpha_t(x))$. Clearly, W_t implements α_t and $W_t S_\varphi W_t^* = S_\varphi$, so W_t commutes with J_φ and we have

$$W_t\pi_\varphi(x)J_\varphi\eta_\varphi(x) = \pi_\varphi(\alpha_t(x))J_\varphi\eta_\varphi(\alpha_t(x)) \qquad (x \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi) .$$

It follows that $W_tP_\varphi = P_\varphi$, so $W_t = U_t^\varphi$. For $x \in \mathfrak{N}_\varphi$, define $\tilde{x} \in L^2(G,H)$ by $\tilde{x}(t) = \alpha_{-t}(x)\xi_0$. We have $\|\tilde{x}\| = \|\eta_\varphi(x)\|$, so the map $\eta_\varphi(x) \to \tilde{x}$ extends to an isometry V of H_φ onto a subspace $\tilde{H}_\varphi$ of $L^2(G,H)$. It is immediate that

$$\Lambda(y)\tilde{x} = (yx)^{\sim} = V(\pi_\varphi(y)\eta_\varphi(x)) ,$$

$$L_t\tilde{x} = (\alpha_t(x))^{\sim} = V(U_t^\varphi\eta_\varphi(x))$$

$(y \in M,\ x \in \mathfrak{N}_\varphi,\ t \in G)$, so $\tilde{H}_\varphi$ is $W^*(M,G,\alpha)$-invariant and $W^*(M,G,\alpha)|_{\tilde{H}_\varphi}$ is isomorphic to $W^*(M,U_G^\varphi)$.

We show next that 4.2 remains valid when the algebra $W^*(M,G,\alpha)$ is replaced by the algebra $A(\overline{X_\alpha})$ of section 2. For this, we need to know that $\eta_\varphi(X_\alpha)$ is dense in H_φ (a fact which does not follow merely from the ultraweak density of X_α in M).

4.3. LEMMA. *In the situation of* 4.2 *and its proof, we have* $\overline{\eta_\varphi(X_\alpha)} = H_\varphi$.

PROOF. Let Q be the projection of H_φ on $\overline{\eta_\varphi(X_\alpha)}$. Since X_α is a left ideal of M, $\overline{\eta_\varphi(X_\alpha)}$ is $\pi_\varphi(M)$-invariant, hence $Q \in \pi_\varphi(M)'$. By Tomita's theorem, there is a projection $q \in M$ such that $Q = J_\varphi\pi_\varphi(q)J_\varphi$. For $t \in G$, we have $\alpha_t(X_\alpha) = X_\alpha$, whence it follows that $\overline{\eta_\varphi(X_\alpha)}$ is an invariant subspace for U_t^φ, that is, Q commutes with each U_t^φ. Since J_φ commutes with each U_t^φ, we see that $\pi_\varphi(q)$ commutes with each U_t^φ, hence $q \in M_\alpha$. By 5.6 of [11], the α-invariance of φ implies that σ_r^φ commutes with α_t $(r \in \mathbf{R},\ t \in G)$. This shows that $\sigma_r^\varphi(X_\alpha) = X_\alpha$ $(r \in \mathbf{R})$. For $y \in \mathfrak{N}_\varphi$, we have $\Delta_\varphi^{ir}\eta_\varphi(y) = \eta_\varphi(\sigma_r^\varphi(y))$, so $\overline{\eta_\varphi(X_\alpha)}$ is an invariant subspace for each Δ_φ^{ir}, so Q commutes with each Δ_φ^{ir}, so $\pi_\varphi(q)$ commutes with each Δ_φ^{ir}, hence with $\Delta_\varphi^{\frac{1}{2}}$. For $y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, we have

$$Q\eta_\varphi(y) = J_\varphi\pi_\varphi(q)J_\varphi S_\varphi\eta_\varphi(y^*) = J_\varphi\pi_\varphi(q)\Delta_\varphi^{\frac{1}{2}}\eta_\varphi(y^*) = S_\varphi\eta_\varphi(qy^*) = \eta_\varphi(yq) .$$

This means that $\eta_\varphi((X_\alpha \cap X_\alpha^*)(1-q))=(0)$, that is, $(X_\alpha \cap X_\alpha^*)(1-q)=(0)$, which forces $q=1$, proving the lemma.

4.4. THEOREM. *Let α be an integrable action of G on M, φ a f.n.s.f. weight on M. Then $W^*(M, U_G^\varphi)$ is a homomorphic image of $A(\bar{X}_\alpha)$.*

PROOF. We use the notation of the proof of 4.2. Notice that for $x \in X_\alpha$, we have $V\eta_\varphi(x)=x\odot\xi_0$, so by 4.3, $VH_\varphi \subseteq X_\alpha \odot H$. Now just invoke 2.3.

We remark that 4.2. above can be improved considerably in the compact case. Y. Haga [8] has shown that if α is a continuous action of a compact abelian group G on M, with M acting in any fashion as a von Neumann algebra of operators on H, then for any implementation $\{U_t : t \in G\}$ of α by a strong-operator continuous group of unitaries on H, the algebra $W^*(M, U_G)$ is a homomorphic image of $W^*(M, G, \alpha)$. One cannot improve on 4.4 in the compact case in this manner, however; $W^*(M, U_G)$ need not be a homomorphic image of $A(\bar{M})$ even when G is compact. For instance, let M be the von Neumann algebra $B(H)\otimes I$ on $H\otimes H$, $\{V_t : t \in G\}$ a strong-operator continuous representation of G by non-scalar unitaries on H, and α the trivial action of G on M, which is implemented by the unitaries $U_t = I\otimes V_t$. One checks that $A(\bar{M})=B(H)$, whereas $W^*(M, U_G)$ has non-trivial center, so the latter is not a homomorphic image of the former.

It should also be pointed out that the conclusion of 4.2 fails in the non-compact case without special assumptions on the action of M on H and/or the implementation of α. For example, let G be a non-compact, locally compact abelian group, let $H=L^2(G)$ and $M=B(H)$, acting on H in the usual (but non-"standard") way. As in section 2, let λ be the action of G which comes from the regular representation, implemented by the translation operators V_t $(t \in G)$. One shows without difficulty that $W^*(M, G, \lambda) \approx M\otimes L^\infty(\hat{G})$, but of course $W^*(M, V_G)=M$. Since $\hat{G}$ is non-discrete, $W^*(M, V_G)$ cannot be a homomorphic image of $W^*(M, G, \lambda)$.

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UNIVERSITY OF KANSAS
LAWRENCE
KANSAS 66045
U.S.A.