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AUTOMATIC CONTINUITY IN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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Introduction.

The basic question in the branch of mathematics that has become known as "automatic continuity" may be phrased in its fullest generality as follows: if T is a linear map from the Banach space A to the Banach space B under what (algebraic) conditions on T, A and/or B may it be inferred that T is a bounded linear map?

In this paper the space A will be $C^n([0,1]$, the Banach algebra of n times continuously differentiable, complex valued functions defined on the unit interval $[0,1]$ (with $n \geqq 0$, fixed), and for the purposes of these introductory remarks, T may be thought of as an algebra homomorphism into the Banach algebra B.

The theme of the present investigation may perhaps best be explained in the following way: if $A = C([0,1])$, the algebra of continuous complex valued functions on $[0,1]$ and $T: A \to B$ is a homomorphism, then it has recently been established that T may be discontinuous [4], [5], but it has been known for some time that T is necessarily continuous (with respect to the sup-norm on A) on some dense subalgebra of A [1, Theorem 4.1]. This fact is a straightforward application of a basic result (Theorem 1.1 below) which establishes a bilinear continuity for T. The class of Banach algebras to which this theorem applies (the Silov algebras, cf. Definition 0.5 below) includes the algebras $C^n([0,1)]$, but the analogue of the $C([0,1])$-result just mentioned does not hold. Indeed, as pointed out in [2, Theorem 6.3], there exists an algebra isomorphism of $C^1([0,1])$ into a Banach algebra B which is discontinuous on every dense subalgebra of $C^1([0,1])$. However, it turns out that if we consider the dense subalgebras $C^k([0,1])$ of $C^n([0,1])$ $(k > n)$ in their own topologies, then continuity results may be obtained: for instance, in Corollary 1.18 it is shown that any derivation (Definition 0.6) on $C^n([0,1])$ is continuous when restricted to $C^{2n}([0,1])$. And in Theorem 2.6 it is shown that if $T: C^n([0,1]) \to B$ is a

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homomorphism and B has finite dimensional (Jacobson-)radical, then T is continuous on $C^{2n+1}([0,1])$.

We list now the basic concepts and facts needed in the sequel. For proofs we may refer to Allan Sinclair's book [11], in particular section 1.1.

DEFINITION 0.1. If $T: A \to B$ is a linear map and A, B are Banach spaces, then the *separating space* of T, $\mathfrak{S}(T)$ is defined by

$$\mathfrak{S}(T) = \{y \in B \mid \exists \{x_n\} \subset A,\ x_n \to 0 \text{ for which } Tx_n \to y\}.$$

This space measures the discontinuity of T because $\mathfrak{S}T) = \{0\}$ if and only if T is continuous, by the closed graph theorem.

LEMMA 0.2. *Let A, B, C, D be Banach spaces, $T: A \to B$ be continuous, $S: B \to C$ be linear and $R: C \to D$ be continuous. Then*

i) *$\mathfrak{S}(S)$ is a closed linear subspace of C.*
ii) *$\mathfrak{S}(ST) \subseteqq \mathfrak{S}(S)$.*
iii) *$(R\mathfrak{S}(S))^{-} = \mathfrak{S}(RS)$* ($^{-}$ norm closure).
iv) *RS is continuous if and only if $R\mathfrak{S}(S) = \{0\}$.*

DEFINITION 0.3. If A is a Banach algebra, B a Banach space and $T: A \to B$ a linear map, then *the continuity ideal* of T, $\mathscr{I}(T)$, is defined by

$$\{x \in A \mid y \mapsto T(xy) \text{ is continuous}\}.$$

REMARK 0.4. If $T: A \to B$ is a homomorphism then Lemma 0.2 iv) shows that

$$\mathscr{I}(T) = \{x \in A \mid T(x)\mathfrak{S}(T) = \{0\}\}.$$

As mentioned before the maps under consideration will have as their domain of definition the Silov algebras $C^n([0,1])$, $n \geqq 0$.

DEFINITION 0.5. A commutative unital Banach algebra A viewed as an algebra of functions on its maximal ideal space Φ_A is a *Silov algebra* when the following holds: if F_1 and F_2 are any two closed and disjoint subsets of Φ_A then there is $a \in A$ such that $a(F_1) = \{0\}$ and $a(F_2) = \{1\}$.

The algebras $C^n([0,1])$ $(n \geqq 0)$ will always be equipped with their natural Banach algebra norm

$$\|f\|_n = \max_{t \in [0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}.$$

DEFINITION 0.6. If A is a Banach algebra and M a (two-sided) Banach A-module, i.e. there is a continuous homomorphism and a continuous anti-homomorphism into the bounded linear operators on M, usually denoted by $a \cdot m$ and $m \cdot a$, respectively, then a *derivation* $D: A \to M$ is a linear map satisfying

$$D(fg) = f \cdot Dg + Df \cdot g$$

for all $f, g \in A$.

1. Implications of bilinear continuity.

If D is a derivation with continuity ideal $\mathcal{J}(D)$ then D is known to be continuous as a bilinear map on $\mathcal{J}(D)$. This fact is noted in [2, Theorem 1.1] and the proof depends on $\mathcal{J}(D)$ being closed. In the case of a homomorphism, the continuity ideal is not necessarily closed and weaker results on continuous bilinearity should be expected. In fact, the best general result appears to be Sinclair's improvement of [1, Theorem 3.7], of which we quote the case we shall need:

THEOREM 1.1 [9, Theorem 2.2]. *Let A be a Silov algebra, B a Banach algebra and T: $A \to B$ a homomorphism with separating space $\mathfrak{S}(T)$ and continuity ideal*

$$\mathcal{J}(T) = \{f \in A \mid T(f)\mathfrak{S}(T) = \{0\}\} .$$

Then there exists a finite set $F \subset \Phi_A$, the singularity set of T, such that if $J(F)$ is the minimal ideal of functions vanishing in neighborhoods of F, then $J(F) \subseteq \mathcal{J}(T)$. Moreover there exists a constant M such that

$$(*) \qquad \|T(fg)\| \leqq M\|f\| \, \|g\|$$

for all $f, g \in J(F)$.

Actually, the development that follows uses only the conclusion of this theorem and not that T is a homomorphism; consequently, it is also valid for any separable map [7, Theorem 2.2] and we shall simply assume, until further notice, that

(1.2) T is a linear map for which the continuity ideal $\mathcal{J}(T)$ (Definition 0.3) has finite hull, and (*) holds.

PROPOSITION 1.3. $\|T(fg)\| \leqq M\|f\| \, \|g\|$ *for any* $f \in \mathcal{J}(T) \cap \overline{J(F)}$, $g \in \overline{J(F)}$.

PROOF. Suppose first that $f \in \overline{J(F)}$, $g \in J(F)$ and $\{f_n\} \subset J(F)$ with $f_n \to f$. By Theorem 1.1

$$\|T(f_n g)\| \leqq M\|f_n\| \, \|g\|$$

and passing to the limit we obtain $\|T(fg)\| \leqq M\|f\| \, \|g\|$.

If $f \in \mathscr{I}(T) \cap \overline{J(F)}$, $\{g_n\} \subset J(F)$ with $g_n \to g \in \overline{J(F)}$, then, by the first paragraph

$$\|T(fg_n)\| \leqq M\|f\| \, \|g_n\| ,$$

and since $f \in \mathscr{I}(T)$, $T(fg_n) \to T(fg)$ so that

$$\|T(fg)\| \leqq M\|f\| \, \|g\| .$$

With one additional assumption on A we are in a position to prove our main result on bilinear continuity.

THEOREM 1.4. *Suppose A is a Silov algebra in which each closed primary ideal (i.e. contained in a unique maximal ideal) has finite codimension, suppose B is a Banach space and T: $A \to B$ a linear map for which* (1.2) *holds. Then there exists a constant N such that*

$$\|T(fg)\| \leqq N\|f\| \, \|g\|, \quad f \in \mathscr{I}(T), \ g \in \overline{\mathscr{I}(T)} .$$

PROOF. Evidently the argument at the beginning of the proof of Proposition 1.3 will establish this Theorem, once the above inequality has been proved for $f,g \in \mathscr{I}(T)$. We do this in two steps. Suppose first that $f \in \mathscr{I}(T)$ and $g \in \overline{J(F)}$. Since $\overline{J(F)} \subseteqq \overline{\mathscr{I}(T)}$, the assumption on closed primary ideals of A implies that $\overline{J(F)}$ is a closed subspace of finite codimension in $\overline{\mathscr{I}(T)}$. Hence we can select vectors $\{h_1, \ldots, h_n\} \subset \mathscr{I}(T)$ so that if $H = \operatorname{span}\{h_1, \ldots, h_n\}$ then

$$\overline{\mathscr{I}(T)} = \overline{J(F)} \oplus H .$$

Let P be the projection of $\overline{\mathscr{I}(T)}$ onto $\overline{J(F)}$ along H. Now $f \in \mathscr{I}(T)$ so $f = a + b$ with $a \in \overline{J(F)}$, $b \in H$. As $H \subseteqq \mathscr{I}(T)$, $a \in \overline{J(F)} \cap \overline{\mathscr{I}(T)}$. We can write

$$b = \sum_{i=1}^{n} \alpha_i(f) h_i ,$$

where α_i are continuous linear functionals on $\overline{\mathscr{I}(T)}$. With $g \in \overline{J(F)}$ we then obtain the following estimates

$$\begin{aligned} \|T(fg)\| &\leqq \|T(ag) + \|T(\textstyle\sum \alpha_i(f) h_i g)\| \\ &\leqq \|T(ag)\| + \sum_{i=1}^{n} |\alpha_i(f)| \, \|T(h_i g)\| \\ &\leqq M\|a\| \, \|g\| + \|f\| \sum_{i=1}^{n} \|\alpha_i\| \, \|T(h_i g)\| \end{aligned}$$

(by Proposition 1.3.)

$$\leqq M\|P\| \, \|f\| \, \|g\| + \|f\| \, \|g\| \left(\sum_{i=1}^{n} \|\alpha_i\| \, A_i \right)$$

where $A_i = \sup_{\|g\|=1} \|T(h_i g)\| < \infty$, as $h_i \in \mathscr{I}(T)$.

Letting

$$K = M\|P\| + \sum_{i=1}^{n} \|\alpha_i\| \, A_i \, ,$$

we have then established that

$$\|T(fg)\| \leqq K\|f\| \, \|g\|, \quad f \in \mathscr{I}(T), \; g \in \overline{J(F)} \, .$$

Now suppose $f, g \in \mathscr{I}(T)$ and write $g = c + d$ with $c \in \overline{J(F)}$ and $d \in H$, hence $c \in \mathscr{I}(T) \cap \overline{J(F)}$. Then

$$\begin{aligned} \|T(fg)\| &\leqq \|T(fc)\| + \|T(fd)\| \\ &\leqq K\|f\| \, \|c\| + \sum_{i=1}^{n} \|\alpha_i\| \, A_i \, \|g\| \, \|f\| \\ &\leqq \left(K\|P\| + \sum_{i=1}^{n} \|\alpha_i\| \, A_i \right) \|f\| \, \|g\| \, . \\ &= N\|f\| \, \|g\| \, . \end{aligned}$$

This proves the theorem.

The algebras $C^n([0,1])$ are Silov algebras and it is easy to see that they satisfy the hypotheses of Theorem 1.4. Moreover, we may assume without loss of generality that the singularity set F for the mapping T is a singleton, say $F = \{0\}$: Suppose $F = \{\lambda_1, \ldots, \lambda_p\}$ and let $\{e_1, \ldots, e_p\}$ be functions in $C^n([0,1])$ that satisfy: $e_i = 1$ in a neighborhood of λ_i and $= 0$ in a neighborhood of λ_j, $j \neq i$, $i = 1, \ldots, p$. Letting $e_0 = 1 - \sum_{i=1}^{p} e_i$ we have

$$T(f) = T\left(\sum_{i=0}^{p} e_i f \right) = \sum_{i=0}^{p} T(e_i f) = \sum_{i=0}^{p} T_i(f) \, .$$

It is well known that T_0 is continuous and T_i has singularity set $F_i = \{\lambda_i\}$, $i = 1, \ldots, p$. Moreover, it is easy to check that (1.2) holds with respect to T and F if and only if (1.2) holds for T_i and F_i, $i = 1, \ldots, p$.

Letting

$$M_{n,k} = \{ f \in C^n([0,1]) \mid f^{(j)}(0) = 0, \; j = 0, \ldots, k \}$$

we shall make heavy use of the following structure theorem:

LEMMA 1.5 ([2, Theorem 2.1], [3, Theorem 3.1]).

i) $M_{n,k}^2 = z^{k+1} M_{n,k}, \quad k=0,\ldots,n-1.$
ii) $M_{n,n}^2 = z^n M_{n,n},$

where $z\colon t \to t,\ t \in [0,1]$.

The natural maps $S_{n,k}\colon M_{n,k} \to M_{n,k}^2$ defined by

$$S_{n,k}(f) = \begin{cases} z^{k+1} f & \text{if } 0 \leqq k < n \\ z^k f & \text{if } k = n \end{cases}$$

for any $f \in M_{n,k}$, give rise to a way of norming $M_{n,k}^2$. We define the graph norms $|\cdot|_{n,k}$ by

$$|f|_{n,k} = \|f\| + \|S_{n,k}^{-1} f\|, \quad f \in M_{n,k}^2 .$$

Letting $B_{n,k} = (M_{n,k}^2, |\cdot|_{n,k})$, $k=0,\ldots,n$ we make the following very useful observation.

PROPOSITION 1.6. *Each* $B_{n,k}$ *is a Banach algebra.*

PROOF is routine.

REMARK. At least in the case $B_{n,n}$ there is another way of describing and norming the functions in question, in fact

$$B_{n,n} = \left\{ f \in C^n([0,1]) \;\middle|\; \frac{f^{(j)}(t)}{t^j} \to 0 \text{ as } t \to 0,\ j=0,1,\ldots,n \right\}.$$

To see this, suppose first that $f \in M_{n,n}^2$. By Lemma 1.5 there is $g \in M_{n,n}$ such that $f = z^n g$ and an application of Leibniz' rule will show that $f^{(j)}(t)/t^j \to 0$ as $t \to 0$, $j=0,\ldots,n$. Conversely, if $f^{(j)}(t)/t^j \to 0$, $t \to 0$, $j=0,\ldots,n$ then $f \in M_{n,n}$ and hence

$$\exists g \in A_n = \{ g \in C^n([0,1]) \mid t^j g^{(j)}(t) \to 0 \text{ as } t \to 0,\ j=0,\ldots,n \}$$

such that $f = z^n g$ [2, proof of Theorem 2.1]. An application of Leibniz' rule to the function $f(t)/t^n$, $t \in]0,1]$, followed by repeated uses of l'Hopital's rule will show that $g \in M_{n,n}$, that is, $f \in M_{n,n}^2$.

The expressions used to characterize $B_{n,n}$ may be used to define a norm on $B_{n,n}$:

$$|||f||| = \sum_{j=0}^{n} \sup_{t\in[0,1]} \left|\frac{f^{(j)}(t)}{t^j}\right|, \quad f \in M^2_{n,n}.$$

It is not hard to see that $|||\cdot|||$ is an algebra norm, equivalent with $|\cdot|_{n,n}$.

We are now in a position to establish the fact that will enable us to translate the bilinear continuity of Theorem 1.4 into statements about linear continuity. First a definition:

DEFINITION 1.7. For given n and any $k=0,\ldots,n$ we let τ denote the canonical map

$$\tau\colon M_{n,k} \otimes M_{n,k} \to M^2_{n,k}$$

from the algebraic tensor product defined as the linear extension of the map: $f\otimes g \to fg$, $f, g \in M_{n,k}$.

We equip $M_{n,k}\otimes M_{n,k}$ with the greatest cross norm γ, defined by

$$\gamma(t) = \inf \sum \|f_i\| \, \|g_i\|$$

with the inf over all representations of the tensor

$$t = \sum_{i=1}^{p} f_i \otimes g_i \in M_{n,k} \otimes M_{n,k}.$$

PROPOSITION 1.8. *When k is $n-1$ or n, when the algebraic tensor product $M_{n,k}\otimes M_{n,k}$ is equipped with the greatest crossnorm and $M^2_{n,k}$ with the graph norm, then*

$$\tau\colon M_{n,k} \otimes M_{n,k} \to B_{n,k}$$

is a continuous and open map.

PROOF. τ is obviously surjective and since $C^n([0,1])$ is separable the Borel graph theorem [12, Appendix] will establish the openness, once we have shown τ to be continuous. To do this it suffices to find $C>0$ such that

$$|fg|_{n,k} \leq C\|f\| \, \|g\| \qquad \text{for any } f, g \in M_{n,k};$$

in fact, then if $t=\sum_{i=1}^{p} f_i\otimes g_i$, we obtain

$$|\tau(t)|_{n,k} \leq \sum_{i=1}^{p} |\tau(f_i\otimes g_i)|_{n,k} = \sum_{i=1}^{p} |f_i g_i|_{n,k} \leq C \sum_{i=1}^{p} \|f_i\| \, \|g_i\|.$$

The argument is based on the following application of the mean value theorem: if $0\leq i, j$ and $i+j\leq n$, then

$$\left|\frac{f^{(i)}(t)}{t^j}\right| \leq \|f^{(i+j)}\|_\infty \qquad \text{whenever } f \in M_{n,i+j-1}.$$

PROOF. Let i be given and let $j=1$. Then

$$\left|\frac{f^{(i)}(t)}{t}\right| = |f^{(i+1)}(\tau)| \leqq \|f^{(i+1)}\|_{\infty,[0,t]},$$

where $\tau \in]0,t[$ and where the notation $\|\cdot\|_{\infty,[0,s]}$ for $s \in [0,1]$ indicates the sup-norm on the interval $[0,s]$. Suppose the claim is correct for the given i, for j and for any interval $[0,\zeta] \subseteq [0,1]$. Then

$$\begin{aligned}\left|\frac{f^{(i)}(t)}{t^{j+1}}\right| &= \left|\frac{1}{t}\cdot\frac{f^{(i)}(t)}{t^j}\right| \leqq \left|\frac{1}{t}\right| \|f^{(i+j)}\|_{\infty,[0,t]} \\ &= \frac{1}{t}|f^{(i+j)}(\tau)| = \frac{\tau}{t}\left|\frac{f^{(i+j)}(\tau)}{\tau}\right| \\ &\leqq |f^{(i+j+1)}(\tau_1)| \leqq \|f^{(i+j+1)}\|_\infty\end{aligned}$$

where $0<\tau_1<\tau<t$.

Now, let $f,g \in M_{n,k}$ and consider

$$\begin{aligned}\left\|\frac{fg}{z^n}\right\| &= \sup_{t\in]0,1]} \sum_{k=0}^{n} \frac{1}{k!}\left|\left(\frac{fg}{z^n}\right)^{(k)}(t)\right| \\ &\leqq \sup_{t\in]0,1]} \sum_{j=0}^{n} \frac{1}{j!} \sum_{p=0}^{j} \left|\binom{j}{p}\frac{d^p}{dt^p}\left(\frac{1}{t^n}\right)\right| |(fg)^{(j-p)}(t)| \\ &\leqq \sup_{t\in]0,1]} \sum_{j=0}^{n}\sum_{p=0}^{j}\sum_{l=0}^{j-p} C_{njpl}\left|\frac{f^{(l)}(t)g^{(j-p-l)}(t)}{t^{n+p}}\right| \\ &= \sup_{t\in]0,1]} \sum_j\sum_p\sum_l C_{njpl}\left|\frac{f^{(l)}(t)}{t^{n-l}}\right|\left|\frac{g^{(j-p-l)}(t)}{t^{l+p}}\right| \\ &\leqq \sup_{t\in]0,1]} \sum_{jpl} C_{njpl}\|f^{(n)}\|_\infty \|g^{(j)}\|_\infty \\ &\leqq C\|f\|\,\|g\|.\end{aligned}$$

With this technical result behind us we are ready to state and prove the main continuity results of this section.

THEOREM 1.9. *Suppose T: $C^n([0,1]) \to B$ is a linear map with singularity set $F=\{0\}$ for which* (1.2) *is valid. If $\mathscr{I}(T) \subseteq M_{n,0}$ has closure equal to $M_{n,n}$, then T is continuous on $M_{n,n}\mathscr{I}(T) \subseteq B_{n,n}$,* [here

$$M_{n,n}\mathscr{I}(T) = \operatorname{span}\{ab \mid a \in M_{n,n},\ b \in \mathscr{I}(T)\}$$

and the inclusion signifies that $M_{n,n}\mathscr{I}(T)$ is equipped with the graph normal].

PROOF. By Proposition 1.8, $\tau\colon M_{n,n}\otimes M_{n,n} \to B_{n,n}$ is open and continuous. Moreover, $M_{n,n}\otimes\mathscr{J}(T)$ is a subalgebra of $M_{n,n}\otimes M_{n,n}$ and $M_{n,n}\otimes\mathscr{J}(T)\cap\ker\tau$ is dense in $\ker\tau$ (with respect to the greatest cross norm on $M_{n,n}\otimes M_{n,n}$). To see this, recall the algebra A_n of [2], alluded to in the Remark following Proposition 1.6. A_n has a bounded approximate identity $\{e_m\}$ (see [2]) and $\{e_m\}$ is an (unbounded) approximate idendity of $M_{n,n}$: the mapping $g \mapsto z^n g\colon A_n \to M_{n,n}$ is bicontinuous, and since $e_m a \to a$ for each $a \in A_n$, it follows that $e_m z^n a \to z^n a$, hence $e_m f \to f$ for every $f \in M_{n,n}$. Since each e_m may be chosen to vanish in a neighborhood of 0, it follows that $e_m f \in J(F) \subset \mathscr{J}(T)$ for every $f \in M_{n,n}$.

Now, if $\sum_{i=1}^{p} f_i \otimes g_i \in \ker\tau$, then

$$\sum_{i=1}^{p} f_i \otimes e_m g_i \in M_{n,n} \otimes \mathscr{J}(T) \cap \ker\tau$$

for each m, because

$$\tau\left(\sum_{i=1}^{p} f_i \otimes e_m g_i\right) = \sum_{i=1}^{p} f_i e_m g_i = e_m \sum_{i=1}^{p} f_i g_i = 0 .$$

Since there are finitely many terms in the sum the density claim follows.

Let $f \in M_{n,n}\mathscr{J}(T)$. Since τ is open $\exists$ constant $L>0$ such that we can find

$$\sum_{i=1}^{p} f_i \otimes g_i \in M_{n,n} \otimes M_{n,n}$$

with

$$\sum_{i=1}^{p} f_i g_i = f \quad \text{and} \quad \sum_{i=1}^{p} \|f_i\| \, \|g_i\| < L|f|_{n,n} .$$

Since $(M_{n,n}\otimes\mathscr{J}(T))\cap\ker\tau$ is dense in $\ker\tau$ we may assume $\sum f_i \otimes g_i \in M_{n,n}\otimes\mathscr{J}(T)$. Since $f_i \in M_{n,n} = \mathscr{J}(T)^{-}$, Theorem 1.4 applies:

$$\|T(f_i g_i)\| \leqq N\|f_i\| \, \|g_i\|, \qquad i=1,\ldots,p .$$

Consequently,

$$\|T(f)\| = \left\|T\left(\sum_{i=1}^{p} f_i g_i\right)\right\| \leqq \sum_{i=1}^{p} \|T(f_i g_i)\|$$

$$\leqq \sum_{i=1}^{p} N\|f_i\| \, \|g_i\| = N \sum_{i=1}^{p} \|f_i\| \, \|g_i\|$$

$$< NL|f|_{n,n} ,$$

which proves the theorem.

It is known that a homomorphism defined on $C(X)$, the algebra of continuous functions on the compact Hausdorff space X, is continuous on some dense subalgebra, and consequently splits into a continuous and a singular part [1, Theorem 4.3]. No analogue of this is possible for $C^n([0,1])$ as [2, Theorem 6.3] gives an example of an isomorphism of $C^1([0,1])$ which is discontinuous on every dense subalgebra of $C^1([0,1])$. However, by means of Theorem 1.9 we are able to establish a partial analogue by showing that any homomorphism of $C^n([0,1])$ will be continuous on a fairly "large" part of $C^{2n}([0,1])$, namely $J(F)\cap C^{2n}([0,1])$.

The argument is based on the following

LEMMA 1.10. *$M_{2n,2n}\subseteq M_{n,n}^2$ and the injection $M_{2n,2n}\to B_{n,n}$ is continuous.*

PROOF. If $f\in M_{2n,2n}$ then standard applications of Leibniz' and l'Hopital's rules will show that $f/z^n\in M_{n,n}$, that is, $f\in M_{n,n}^2$ by Lemma 1.5. To show that $\exists\, C>0$ such that $|f|_{n,n}\leqq C\|f\|_{2n}$ where $\|\cdot\|_e$ denotes the norm in $C^e([0,1])$ we once again use the observation that was mentioned in the proof of Proposition 1.8: if $0\leqq i,j$ and $i+j\leqq p$ then

$$\left|\frac{f^{(i)}(t)}{t^j}\right| \leqq \|f^{(i+j)}\|_\infty$$

for every $t\in\,]0,1]$ and every $f\in M_{p,i+j-1}$. So let $f\in M_{2n,2n}$:

$$|f|_{n,n} = \|f\|_n+\left\|\frac{f}{z^n}\right\|_n \leqq \|f\|_{2n}+\left\|\frac{f}{z^n}\right\|_n,$$

and

$$\begin{aligned}\left\|\frac{f}{z^n}\right\|_n &= \sup_{t\in]0,1]}\sum_{k=0}^{n}\frac{1}{k!}\left|\left(\frac{f}{z^n}\right)^{(k)}(t)\right| \\ &\leqq \sup_{t\in]0,1]}\sum_{k=0}^{n}\sum_{j=0}^{k}c_{k,j}\left|\frac{f^{(k-j)}(t)}{t^{n+j}}\right| \\ &\leqq \sum_{k=0}^{n}\sum_{j=0}^{k}c_{k,j}\|f^{(k+n)}\|_\infty \\ &\leqq C\|f\|_{2n}\end{aligned}$$

which finishes the proof.

COROLLARY 1.11. *Let v: $C^n([0,1])\to B$ be a homorphism into the Banach algebra B with singularity set F. Then v is continuous on $C^{2n}([0,1])\cap J(F)$.*

PROOF. By virtue of the remarks following Theorem 1.4 we may assume that v has one singularity point, 0. It is immaterial that v is no longer a homomorphism. Combining Lemma 1.10 and Theorem 1.9 we obtain, for any $f \in C^{2n}([0,1]) \cap J(\{0\})$:

$$\|v(f)\| < NL|f|_{2n,2n} \leqq (1+C)NL\|f\|_{2n}.$$

Consequently, the original homomorphism v is continuous on $C^{2n}([0,1] \cap J(F)$.

REMARK. We may extend the restriction of v (to $J(F) \cap C^{2n}([0,1])$) by continuity to

$$M_{2n,2n}(F) = \{ f \in C^{2n}([0,1]) \mid f^{(j)}(t)=0 \text{ for } j=0,\dots,2n, \text{ all } t \in F\}$$

and thus obtain a splitting

$$v = \mu + \lambda$$

on $M_{2n,2n}(F)$. Clearly μ is a continuous homomorphism and by arguments like the relevant parts of the proof of [1, Theorem 4.3] it may be shown that λ is a homomorphism of $M_{2n,2n}(F)$ into the radical of $v(M_{2n,2n}(F))^{-}$. In fact, straightforward applications of the continuity of μ and the definition of the separating space will show that

$$\mathfrak{S}(v|_{C^{2n}}) = \mathfrak{S}(\lambda) = \lambda(M_{2n,2n}(F))^{-}.$$

Returning now to the map T of Theorem 1.9 we address ourselves to the possibility that the continuity ideal $\mathscr{I}(T)$ be slightly larger. If the norm closure $\mathscr{I}(T)^{-} \supseteqq M_{n,n-1}$ then it turns out that T becomes continuous on $C^{2n}([0,1])$. As we shall see this situation arises when T is a derivation.

THEOREM 1.12. *If $\mathscr{I}(T)^{-} \supseteqq M_{n,n-1}$ then T is continuous on $B_{n,n-1}$.*

We establish this after the following algebraic fact.

LEMMA 1.13. *$M_{n,n-1}^2 = z^n M_{n,n-1} = pM_{n,n-1}$, where $p \in M_{n,n-1} \backslash M_{n,n}$ is any function that vanishes only at* 0.

PROOF. The conditions on p ensure that [3, Lemma 3.2] applies to show that $z^{2n}/p \in M_{n,n-1}$. If $f = z^n g \in M_{n,n-1}^2$, then

$$f = p \cdot \frac{z^{2n}}{p} \cdot \frac{g}{z^n} = p \frac{\frac{z^{2n}}{p} g}{z^n} \in pM_{n,n-1}.$$

The other inclusion is obvious.

PROOF OF THEOREM 1.12. Since $z^n \in M_{n,n-1}$ we can find a function $p \in \mathscr{I}(T) \backslash M_{n,n}$ which vanishes only at 0. Let

$$J = \text{span}\,(p, J(0))\,.$$

Then J is a Souslin space, because $J(0)$ is a countable union of separable Banach spaces, e.g. functions in $C^n([0,1])$ vanishing on $[0,1/m]$, $m=2,3,\ldots$ and because $J(0)$ is of codimension 1 in J. Consider the algebraic tensor product of $M_{n,n-1} \otimes J$ of the normal linear spaces $M_{n,n-1}$ and J and equip this tensor product with the greatest cross norm. Since

$$M_{n,n-1} \otimes J \subseteq M_{n,n-1} \otimes M_{n,n-1}$$

we may consider the map τ of Proposition 1.8:

$$\tau\colon M_{n,n-1} \otimes J \to B_{n,n-1}\,.$$

We claim that τ is continuous, surjective and open. The continuity of τ follows as in Proposition 1.8. The surjectivity is immediate from the definition of J and Lemma 1.13, and since $M_{n,n-1} \otimes J$ as the algebraic tensor product of two Souslin spaces is itself a Souslin space [8], the Borel graph theorem yields the openness of τ. (A readily accessible reference for the basics of Souslin spaces (or analytic spaces) is in the appendix of F. Treves' book [12]. This appendix is devoted to a proof of the Borel graph theorem). The rest of the argument is identical with the last part of the proof of Theorem 1.9:

Since τ is open there exists a constant L such that for every $f \in B_{n,n-1}$ we can find

$$\sum_{i=1}^{p} f_i \otimes g_i \in M_{n,n-1} \otimes J$$

such that $f = \sum f_i g_i$ and

$$\|\sum f_i\|\,\|g_i\| < L|f|_{n,n-1}\,.$$

By Theorem 1.4

$$\|T(f_i g_i)\| \leqq C\|f_i\|\,\|g_i\|$$

and hence

$$\|T(f)\| \leqq \sum_{i=1}^{q} \|T(f_i g_i)\| \leqq \sum_{i=1}^{q} C\|f_i\|\,\|g_i\| < CL|f|_{n,n-1}\,.$$

REMARK. There is another way to prove Theorem 1.12 without tensor products, using uniqueness of norm and Lemma 1.13: Choose $p \in \mathscr{I}(T)$ such that $M_{n,n-1}^2 = pM_{n,n-1}$ and norm $M_{n,n-1}^2$ with the norm:

$$|||f|||_p = \|f\| + \left\|\frac{f}{p}\right\|$$

to get a Banach algebra norm equivalent to $|\cdot|_{n,n-1}$ by uniqueness of norms for semi-simple algebras. Now, if $f \in M^2_{n,n-1}$, $f=pg$ with $g \in M_{n,n-1}$, so

$$\|T(f)\| = \|T(pg)\| \leqq K_p\|g\| ,$$

since $p \in \mathscr{J}(T)$; this yields

$$\|T(f)\| \leqq K_p\|g\| \leqq K_p|||f|||_p \leqq K_pL|f|_{n,n-1} .$$

COROLLARY 1.14. *If* $\mathscr{J}(T)^{-} \supseteq M_{n,n-1}$ *then* T *is continuous on* $C^{2n}([0,1])$.

PROOF. It is clear from the definition of the graph norms that $B_{n,n} \subset B_{n,n-1}$, isometrically. Consequently, Lemma 1.10 shows that T is continuous on $M_{2n,2n}$ and since $M_{2n,2n}$ is a closed subspace of $C^{2n}([0,1])$ of finite codimension, the Corollary follows.

We end this section by a short study of a certain class of operators to which the results of this section may be applied.

Let A be a commutative Banach algebra and let M be a left Banach-A-module, i.e. a Banach space together with a continuous homomorphism ϱ: $A \to \mathscr{B}(M)$.

DEFINITION 1.15. A linear map S: $A \to M$ is of class G if there exists a bilinear map L: $A \times A \to M$ with the property that $L(a,\cdot)$ is continuous for each $a \in A$ such that

$$S(ab) = \varrho(a)S(b) + L(a,b)$$

for every $a,b \in A$. The continuity ideal $\mathscr{J}(S)$ of $S \in G$ is

$$\begin{aligned}\mathscr{J}(S) &= \{a \in A \mid b \mapsto S(ab) \text{ is continuous}\} \\ &= \{a \in A \mid b \mapsto \varrho(a)S(b) \text{ is continuous}\} .\end{aligned}$$

EXAMPLES. Suppose M is a two-sided module with respect to left module multiplication $\varrho_1(\cdot)$ and right module multiplication $\varrho_2(\cdot)$ and suppose D: $A \to M$ is a derivation, i.e., D satisfies

$$D(ab) = \varrho_1(a)D(b) + D(a)\varrho_2(b)$$

for every $a,b \in A$. Then D is of class G, because $b \mapsto D(a)\varrho_2(b)$ is continuous for each $a \in A$.

LEMMA 1.16. *Assume A to be a Silov algebra. Let $S \in G$. Then $\mathscr{I}(S)$ is a closed ideal with finite hull F, called the singularity set of S. Moreover, there exists a constant M such that*

$$\|S(ab)\| \leqq M\|a\| \, \|b\|$$

for every $a, b \in \mathscr{I}(S)$.

PROOF. If $\{a_n\} \subset \mathscr{I}(S)$ with $a_n \to a$ then $\varrho(a_n)S(b) \to \varrho(a)S(b)$ for every $b \in A$ and since

$$\|\varrho(a_n)Sb\| \leqq C\|a_n\| \, \|Sb\| ,$$

the uniform boundedness principle yields the continuity of $b \mapsto \varrho(a)Sb$, and the boundedness of S as a bilinear map on $\mathscr{I}(S) \times \mathscr{I}(S)$. The finiteness of $F = \text{hull}\,(\mathscr{I}(S))$ follows from [7, Theorem 2.2].

We now specialize to $A = C^n([0,1])$. *If S*: $C^n([0,1]) \to M$ is a map of class G with singularity set $F = \{t_1, \ldots, t_p\}$ we may choose, as usual, functions $e_1, \ldots, e_p \in C^n([0,1])$ such that e_j is identically one in a neighborhood of t_j and vanishes on a neighborhood of $F \backslash \{t_j\}$. Then $S_j = S \circ e_j$ is a map of class G with one singularity point, t_j, for which we may borrow an argument from [2] to prove the following

LEMMA 1.17. *Let S: $C^n([0,1]) \to M$ be a map of class G with one singularity point, which we may take to be* 0. *Then $M_{n,n-1} \subseteqq \mathscr{I}(S)$.*

PROOF. As in [2, Theorem 3.2], since $\mathscr{I}(S)$ is known to be closed and $M_{n,n}$ has finite codimension, it suffices to show that $z^n \in \mathscr{I}(S|_{M_{n,n}})$, i.e. that $\varrho(z^n)S$ is continuous on $M_{n,n}$. By the factorization argument in [2, proof of Theorem 3.2], if $\{f_k\} \subset M_{n,n}$ and $f_k \to 0$ then there is a sequence $\{a_k\} \subset M_{n,n}$ and a $b \in M_{n,n} \subset \mathscr{I}(S)$ so that $z^n f_k = a_k b$ and $a_k \to 0$. Then

$$\varrho(z^n)S(f_k) = \varrho(a_k)S(b) + \varrho(b)S(a_k) - \varrho(f_k)S(z^n) \to 0$$

as $k \to \infty$.

COROLLARY 1.18. *Every map S of class G defined on $C^n([0,1])$ is continuous on $C^{2n}([0,1])$.*

PROOF. As we remarked before Lemma 1.17 and before Lemma 1.5 we may assume that S has one singularity point, 0. By the previous Lemma $M_{n,n-1} \subseteqq \mathscr{I}(S)$ and then Corollary 1.14 applies.

2. C^{2n+1}-continuity of C^n-homomorphisms.

As Corollary 1.18 indicates there is some reason to expect that if a mapping of the types considered here is defined on C^n then it may well be continuous when restricted to C^k for suitable $k>n$. In the case of homomorhisms, supporting evidence is to be found in Dales' and McClure's work on higher point derivations [3]: Let $d_0, d_1, \ldots, d_k$ be a non-degenerate higher point derivation on $C^n([0,1])$ associated with the point 0. Construct a mapping of $C^n([0,1])$. into $B(\mathbf{C}^{k+1})$ (the algebra of $(k+1)\times(k+1)$ complex matrices, i.e. assuming a basis chosen on $\mathbf{C}^{k+1}$, the algebra of linear operators on $\mathbf{C}^{k+1}$) as follows:

$$f \in C^n([0,1]) \to (\alpha_{ij}(f))_{i,j=0}^{k},$$

where $\alpha_{ij}\equiv 0$ if $j>i$ and $\alpha_{ij}(f)=d_{i-j}(f)$ if $j\leqq i$. The rules for matrix multiplication and the definition of a higher point derivation immediately show $f\to(\alpha_{ij}(f))$ to be a homomorphism. Now, by [3, Example 2.5] we have $k\leqq 2n$, provided that $d_1\neq 0$. Moreover, as the proof of [3, Theorem 3.3] shows, each functional d_j is a linear combination of (extensions of) the canonical higher derivations $f\to f^{(i)}(0)$, $i=1,\ldots,k$. With $k\leqq 2n$ it follows that all d_j and thus $f\to(\alpha_{ij}(f))$ are continuous on $C^{2n}([0,1])$.

It turns out that a similar result holds for an arbitrary homomorphism ν of $C^n([0,1])$ into a finite dimensional Banach algebra: ν becomes continuous when restricted to $C^{2n+1}([0,1])$ (Corollary 2.4). It is not yet known whether $2n+1$ is best possible. (Added in proof May 16, 1977: In a paper in preparation we shall prove, among other results, theorem 2.6 with $2n+1$ replaced by $2n$.)

The proof of this is based on the following stability result which should be compared with [10, Lemma 2.3]. Note that the result is not a direct corollary of [10, Lemma 2.3] because (with the notation introduced below) $z_0^\alpha \notin C^n([0,1])$ for $\alpha\in\,]0,n[\setminus\{1,2,\ldots,n-1\}$.

THEOREM 2.1. *Let $n\geqq 0$, let ν be a homomorphism of $C^n([0,1])$ with singularity set F and let $t_0\in F$. Let $e_0\in C^n([0,1])$ be a function that is identically one in a neighborhood of t_0 and vanishes on a neighborhood of $F\setminus\{t_0\}$. Let $\nu_0=\nu\circ e_0$ and denote the separating space of ν_0 by $\mathfrak{S}$. Let*

$$f_0 \in M_{n,n,t_0} = \{f\in C^n([0,1]) \mid f^{(j)}(t_0) = 0,\ j=0,\ldots,n\}.$$

If $z_0(t)=t-t_0$, $t\in[0,1]$ and $\mathfrak{S}_\alpha=(\nu(|z_0|^\alpha f_0)\mathfrak{S})^-$ for $\alpha>0$ then $\mathfrak{S}_\alpha=\mathfrak{S}_\beta$ for all $\alpha,\beta>0$.

PROOF. For any $\alpha>0$, $|z_0|^\alpha$ is a multiplier of M_{n,n,t_0}, that is,

$$|z_0|^\alpha M_{n,n,t_0} \subseteqq M_{n,n,t_0}.$$

Since M_{n,n,t_0} is of finite codimension in $C^n([0,1])$, $\mathfrak{S}_\alpha$ is the separating space of $v_\alpha = v(|z_0|^\alpha f_0)v_0$, considered as a mapping on M_{n,n,t_0}. If $0<\alpha<\beta$ then $\mathfrak{S}_\beta \subseteqq \mathfrak{S}_\alpha$, because if

$$m = \lim_k v(|z_0|^\beta f_0)\, v_0(f_k)$$

where $f_k \to 0$, then $|z_0|^{\beta-\alpha} f_k \to 0$ and

$$\begin{aligned} v(|z_0|^{\beta-\alpha} f_k) &= v(|z_0|^\alpha f_0)\, v_0(|z_0|^{\beta-\alpha} f_k) \\ &= v(|z_0|^\beta f_0) v_0(f_k) \to m\,. \end{aligned}$$

For $0<\alpha$ let $\tilde{\mathfrak{S}}_\alpha$ be the separating space of the map

$$\tilde{v}_\alpha \colon f \mapsto v_0(|z_0|^\alpha f), \qquad f \in M_{n,n,t_0}\,.$$

This is well-defined since $|z_0|^\alpha f \in M_{n,n,t_0}$ if $f \in M_{n,n,t_0}$. Arguing as above we see that

$$\tilde{\mathfrak{S}}_\beta \subseteqq \tilde{\mathfrak{S}}_\alpha \subseteqq \mathfrak{S}, \qquad \text{if } 0<\alpha<\beta\,.$$

Now suppose that for certain α and β, $0<\alpha<\beta$ we have $\tilde{\mathfrak{S}}_\alpha = \tilde{\mathfrak{S}}_\beta$. We then assert that $\mathfrak{S}_\alpha = \mathfrak{S}_\xi$ for all $\xi \geqq \alpha$. First, if $\tilde{\mathfrak{S}}_\alpha = \tilde{\mathfrak{S}}_\beta$, then $\mathfrak{S}_{\alpha+\gamma} = \mathfrak{S}_{\beta+\gamma}$ for all $\gamma \geqq 0$, for

$$\mathfrak{S}_{\beta+\gamma} = (v(f_0|z_0|^\gamma)\tilde{\mathfrak{S}}_\beta)^-$$

by Lemma 0.2 iii) and hence

$$\begin{aligned} \mathfrak{S}_{\beta+\gamma} &= (v(f_0|z_0|^\gamma)\tilde{\mathfrak{S}}_\alpha)^- \\ &= \mathfrak{S}_{\alpha+\gamma}\,. \end{aligned}$$

Next, we claim that for each $\gamma \geqq 0$

$$\mathfrak{S}_{\alpha+\gamma} = \mathfrak{S}_{\beta+\gamma+m(\beta-\alpha)}, \qquad m = 0, 1, 2, \ldots\,.$$

We have proved the case $m=0$. Assume equality holds for $m=k$. Letting $\gamma' = (\beta-\alpha)+\gamma$ we then have

$$\begin{aligned} \mathfrak{S}_{\beta+\gamma+(k+1)(\beta-\alpha)} &= \mathfrak{S}_{\beta+\gamma'+k(\beta-\alpha)} \\ &= \mathfrak{S}_{\alpha+\gamma'} = \mathfrak{S}_{\alpha+(\beta-\alpha)+\gamma} = \mathfrak{S}_{\beta+\gamma} \\ &= \mathfrak{S}_{\alpha+\gamma}\,. \end{aligned}$$

If $\xi>\alpha$, pick $\delta = \beta + m(\beta-\alpha) > \xi$ and then $\mathfrak{S}_\alpha \supseteqq \mathfrak{S}_\xi \supseteqq \mathfrak{S}_\delta = \mathfrak{S}_\alpha$. Hence $\mathfrak{S}_\xi = \mathfrak{S}_\alpha$ for any $\xi \geqq \alpha$.

To complete the argument suppose there exist α, β, $0<\alpha<\beta$ such that $\mathfrak{S}_\alpha \neq \mathfrak{S}_\beta$. By the above $\tilde{\mathfrak{S}}_\alpha \neq \tilde{\mathfrak{S}}_\beta$. Moreover, if we pick α_m, $m = 1, 2, \ldots$ such that

$$\alpha_1 + \ldots + \alpha_m < \alpha_1 + \ldots + \alpha_{m+1} < \alpha < \beta ,$$

then

$$\tilde{\mathfrak{S}}_{\alpha_1+\ldots+\alpha_m} \supsetneqq \tilde{\mathfrak{S}}_{\alpha_1+\ldots+\alpha_{m+1}} \qquad \text{for } m=1,2,\ldots . \tag{2.2}$$

Otherwise, if for some M, $\tilde{\mathfrak{S}}_{\alpha_1+\ldots+\alpha_M} = \tilde{\mathfrak{S}}_{\alpha_1+\ldots+\alpha_{M+1}}$ then $\mathfrak{S}_\xi = \mathfrak{S}_{\alpha_1+\ldots+\alpha_M}$ for all $\xi \geqq \alpha_1 + \ldots + \alpha_M$. In particular $\mathfrak{S}_\alpha = \mathfrak{S}_\beta$ which would be a contradiction. However (2.2) can not hold for all m by the general stability theorem [6, Proposition 2.1]. This completes the proof of the theorem.

If A and B are Banach algebras with identities and $v: A \to B$ is a homomorphism then v defines a homomorphism $\tilde{v}$ of A into $\mathscr{B}(B)$, the bounded linear operators on B, via the left regular representation:

$$\tilde{v}(a)b = v(a)b, \qquad a \in A,\ b \in B .$$

Since B has a unit $\|\tilde{v}(a)\| = \|v(a)\|$, and consequently v is continuous if and only if $\tilde{v}$ is continuous.

In the particular case where B is finite dimensional and $A = C^n([0,1])$ we may assume, first, that our homomorphism v maps into $\mathscr{B}(B)$ and secondly, through a direct sum splitting of B as in [2, proof of Theorem 5.1], we may then assume that the singularity set of v consists of one point. With this reduction we have the following

Corollary 2.3. *Let v be a homomorphism of $C^n([0,1])$ with kernel K. Assume* codim $(K) < \infty$ *and* hull $(K) = \{t_0\}$. *Let $\mathscr{I}(v)$ be the continuity ideal of v. Then for all $\alpha > 0$ and $f \in M_{n,n,t_0}$,*

$$|z_0|^\alpha f \in \mathscr{I}(v), \qquad \textit{and} \qquad |z_0|^{n+\alpha} f \in K .$$

In particular $K \supseteqq M^3_{n,n,t_0}$.

Moreover $z_0^k f \in \mathscr{I}(v)$ for $k = 1,2,\ldots$.

Proof. By the finite codimension of K, $z_0^{2m} = |z_0|^{2m} \in K$ for m sufficiently large, and consequently $|z_0|^{2m} f \in K$ for large m and every $f \in C^n([0,1])$. Taking $e_0 \equiv 1$ in Theorem 2.1 we then obtain

$$\mathfrak{S}_\alpha = (v(|z_0|^\alpha f)\mathfrak{S})^- = \{0\}$$

for any $\alpha > 0$ and any $f \in M_{n,n,t_0}$, that is, $|z_0|^\alpha f \in \mathscr{I}(v)$. Next since codim (K) $< \infty$, $J \equiv v^{-1}(\mathfrak{S})$ is closed, so $M_{n,n,t_0} \subseteqq J$. Also since

$$\mathscr{I}(v) = \{f \in C^n([0,1]) \mid v(f)\mathfrak{S} = \{0\}\} ,$$

it follows that $J\mathscr{I}(v) \subseteqq K$. Therefore

$$K \supseteqq J\mathscr{J}(v) \supseteqq M_{n,n,t_0}\mathscr{J}(v) \supseteqq |z_0|^\alpha M^2_{n,n,t_0}$$
$$= |z_0|^\alpha z_0^n M_{n,n,t_0} = |z_0|^{n+\alpha} M_{n,n,t_0},$$

the last equality following from the fact that $z_0 M_{n,n,t_0} = |z_0| M_{n,n,t_0}$. To see this last observation let $f_1(t) = f(t)$ for $t \geqq t_0$ and $f_1(t) = 0$ for $t \leqq t_0$, and let $f_2 = f - f_1$. Then $f_1, f_2 \in M_{n,n,t_0}$,

$$z_0 f = |z_0| f_1 - |z_0| f_2, \quad \text{and} \quad |z_0| f = z_0 f_1 - z_0 f_2 .$$

Therefore $z_0 M_{n,n,t_0} = |z_0| M_{n,n,t_0}$. Clearly

$$M^3_{n,n,t_0} = z_0^{2n} M_{n,n,t_0} \subset K$$

and $z_0^k f \in \mathscr{J}(v)$ for $k = 1, 2, \ldots$ as required.

It seems quite probable that $K \supset M^2_{n,n,t_0} = z_0^n M_{n,n,t_0}$, but we have been unable to prove this.

COROLLARY 2.4. *Let $v: C^n([0,1]) \to B$ be a homomorphism into the finite dimensional Banach algebra B. Then v is continuous on $C^{2n+1}([0,1])$.*

PROOF. By the remarks preceding Corollary 2.3 we may assume that v has a one-point singularity set, $\{t_0\}$. Let $f \in M_{2n+1,2n+1,t_0}$ and let $g = |z_0|^{-n-1} f$. It is elementary that $g \in M_{n,n,t_0}$, in fact by the general version of the mean value theorem mentioned in the proof of Proposition 1.8 we get that there exist constants $c_{j,k}$ and C (for $0 \leqq j \leqq k \leqq n$) such that

$$\begin{aligned} \|g\|_n &= \sup_{t \neq t_0} \sum_{k=0}^{n} \frac{1}{k!} \left| \left(\frac{f}{z_0^{n+1}} \right)^{(k)} (t) \right| \\ &\leqq \sup_{t \neq t_0} \sum_{k=0}^{n} \sum_{j=0}^{k} c_{j,k} \left| \frac{f^{(k+j)}(t)}{(t-t_0)^{n+1+j}} \right| \\ &\leqq \sup_{t \neq t_0} \sum_{k=0}^{n} \sum_{j=0}^{k} c_{j,k} \|f^{(k+n+1)}\|_\infty \\ &\leqq C\|f\|_{2n+1} . \end{aligned}$$

Thus, $f \in M_{2n+1,2n+1,t_0}$ implies that there exists a $g \in M_{n,n,t_0}$ such that

$$f = |z_0|^{n+1} g \quad \text{and} \quad \|g\|_n \leqq C\|f\|_{2n+1},$$

since $|z_0|^{n+\frac{1}{2}} \in M_{n,n,t_0}$, by Corollary 2.3, and thus there is an N such that

$$\|v(f)\| = \|v(|z_0|^{n+1} g)\| \leqq N\|g\|_n \leqq NC\|f\|_{2n+1} .$$

Because $M_{2n+1,2n+1,t_0}$ is closed and of finite codimension in $C^{2n+1}([0,1])$, the Corollary is proved.

We shall prove a stronger version of Corollary 2.4 by showing that if $\nu: C^n([0,1]) \to B$ is a homomorphism and B has a finite dimensional radical then ν is continuous on $C^{2n+1}([0,1])$. This requires the use of Theorem 2.5; we present a proof of this theorem which was suggested by the referee and which considerably shortens our original argument.

THEOREM 2.5. *Suppose $n \geqq 0$. Let ν be a homomorphism of $C^n([0,1])$ into B, where B is a Banach algebra. Suppose ν is continuous on $C^k([0,1])$ for some $k > n$. Then ν is continuous on $C^{2n+1}([0,1])$.*

PROOF. Suppose ν has singularity set $F = \{t_0, \ldots, t_p\}$ and suppose $e_j \in C^\infty([0,1])$ are chosen identically 1 in a neighborhood of t_j and vanishing on a neighborhood of $F \backslash \{t_j\}$ for $j = 0, \ldots, p$. Let $\nu_j = \nu \circ e_j$ and $\nu_{p+1} = \nu - \sum_{j=0}^{p} \nu_j$. Since ν_{p+1} is known to be continuous [1, Corollary 3.9] it will suffice to show that each ν_j $(j = 0, \ldots, p)$ is continuous on $C^{2n+1}([0,1])$. Moreover, by the stability result (Theorem 2.1), it suffices to show that $\mathscr{I}(\nu_0)$, say, contains a power of z_0, because then we may argue exactly as in the proof of Corollary 2.4 to obtain the desired continuity result.

So suppose $\nu_0 = \nu e_0$ has singularity point $t_0 \in [0,1]$ and is continuous on some $C^k([0,1])$ by virtue of ν being continuous on $C^k([0,1])$. The continuity of ν on $C^k([0,1])$ means that B is a Banach-C^k module under the module multiplication $\varrho(f)b = \nu(f)b$ for every $b \in B$ and $f \in C^k([0,1])$.

The Banach space $C^n([0,1])$ is a Banach C^k-module under multiplication of functions and ν_0 is a C^k-module homomorphism from $C^n([0,1])$ into B, because

$$\nu_0(f_1 f_2) = \nu(f_1)\nu_0(f_2)$$

for every $f_1 \in C^k([0,1])$ and $f_2 \in C^n([0,1])$. Let

$$\mathscr{I}_k(\nu_0) = \{f \in C^k([0,1]) \mid \nu(f)\mathfrak{S}(\nu_0) = \{0\}\}\,.$$

Clearly, $\mathscr{I}_k(\nu_0)$ is a closed ideal in $C^k([0,1])$, because ν is continuous on $C^k([0,1])$. Thus $\mathscr{I}_k(\nu_0) \supseteqq M_{k,k,t_0}$, that is, $z_0^{k+1} \in \mathscr{I}_k(\nu_0)$. Let, as usual,

$$\begin{aligned} \mathscr{I}(\nu_0) &= \{f \in C^n([0,1]) \mid \nu_0(f\cdot) \text{ is continuous}\} \\ &= \{f \in C^n([0,1]) \mid \nu(f)\nu_0 \text{ is continuous}\} \\ &= \{f \in C^n([0,1]) \mid \nu(f)\mathfrak{S}(\nu_0) = \{0\}\} \end{aligned}$$

and note that $\mathscr{I}_k(\nu_0) = \mathscr{I}(\nu_0) \cap C^k([0,1])$. Hence $z_0^{k+1} \in \mathscr{I}(\nu_0)$ and the proof is complete.

Finally, we prove a generalization of Corollary 2.4:

THEOREM 2.6. *Let* v: $C^n([0,1]) \to B$ *be a homomorphism and assume about* B *that* $B = v(C^n([0,1]))^-$. *Suppose* $R = \text{radical}(B)$ *is finite dimensional. Then* v *is continuous on* $C^{2n+1}([0,1])$.

PROOF. By the remarks following Theorem 2.1 we may replace v by the homomorphism $\tilde{v}$: $C^n([0,1]) \to \mathscr{B}(B)$, defined by $\tilde{v}(a)b = v(a)b$ for every $b \in B$ and every $a \in C^n([0,1])$. Since $\|v(a)\| = \|\tilde{v}(a)\|$ we are done when the continuity of $\tilde{v}$ on $C^{2n+1}([0,1])$ has been established. We note first that since R is an ideal, R is invariant under $\tilde{v}$:

$$b \in R,\ f \in C^n([0,1]) \ \Rightarrow\ \tilde{v}(f)b = v(f)b \in R\,.$$

Since R is finite dimensional we may choose a projection $P \in \mathscr{B}(B)$ onto R and define a map

$$\tilde{v}_P\colon C^n([0,1]) \to \mathscr{B}(B)$$

by $\tilde{v}_P(f) = P\tilde{v}(f)P$, $f \in C^n([0,1])$. The invariance of R shows that $\tilde{v}_P$ is a homomorphism:

$$\begin{aligned}\tilde{v}_P(f_1f_2) &= P\tilde{v}(f_1)\tilde{v}(f_2)P = P\tilde{v}(f_1)P\tilde{v}(f_2)P \\ &= P\tilde{v}(f_1)P \circ P\tilde{v}(f_2)P = \tilde{v}_P(f_1)\tilde{v}_P(f_2), \quad f_1, f_2 \in C^n([0,1])\,.\end{aligned}$$

$\tilde{v}_P(f)|_R = v_P(f)$ is also a homomorphism: If $b \in R$ then

$$\begin{aligned}v_P(f_1 f_2)b &= \tilde{v}_P(f_1 f_2)b = \tilde{v}_P(f_1)\tilde{v}_P(f_2)b \\ &= v_P(f_1)v_P(f_2)b\end{aligned}$$

by the invariance of R. Since R is finite dimensional v_P is continuous on C^{2n+1} (Corollary 2.4) i.e. there exists an M such that

$$\|v_P(f)\| \leqq M\|f\|_{2n+1} \quad \text{for every } f \in C^{2n+1}([0,1])\,.$$

But

$$\begin{aligned}\|\tilde{v}_P(f)b\| &= \|\tilde{v}_P(f)Pb\| = \|v_P(f)Pb\| \\ &\leqq \|v_P(f)\|\ \|Pb\| \\ &\leqq M\|f\|_{2n+1}\|P\|\ \|b\|\end{aligned}$$

so

$$\begin{aligned}\|\tilde{v}_P(f)\| &= \sup_{\|b\| \leqq 1} \|\tilde{v}_P(f)b\| \\ &\leqq M\|P\|\ \|f\|_{2n+1}\end{aligned}$$

which shows that $\tilde{v}_P$ is continuous on $C^{2n+1}([0,1])$.

Let $Q = I - P$ and consider $\tilde{v}_Q$: $C^n([0,1]) \to \mathscr{B}(B)$ defined by $\tilde{v}_Q(f) = Q\tilde{v}(f)Q$. This also defines a homomorphism:

$$\begin{aligned}\tilde{v}_Q(f_1 f_2) &= Q\tilde{v}(f_1 f_2)Q \\ &= Q\tilde{v}(f_1)\tilde{v}(f_2)Q = Q\tilde{v}(f_1)(Q+P)\tilde{v}(f_2)Q \\ &= Q\tilde{v}(f_1)Q\tilde{v}(f_2)Q = Q\tilde{v}(f_1)Q \circ Q\tilde{v}(f_2)Q \\ &= \tilde{v}_Q(f_1)\tilde{v}_Q(f_2), \qquad \text{for } f_1, f_2 \in C^n([0,1]) .\end{aligned}$$

Consider the map Qv: $C^n([0,1]) \to B$ and note that since $\mathfrak{S}(v) \subseteqq R$, Qv is continuous by Lemma 0.2. iv), [The fact that $\mathfrak{S}(v) \subseteqq R$ may be seen this way: Let φ be a multiplicative linear functional on B and note that φv is a multiplicative linear functional on $C^n([0,1])$. By the continuity of φ and φv and Lemma 0.2 iv) we get $\mathfrak{S} \subseteqq \ker \varphi$. This being true for any φ, we get $\mathfrak{S} \subseteqq R$]. Since Qv is continuous there exist a C such that

$$\|Qv(f)\| \leqq C\|f\|_n$$

and hence

$$\begin{aligned}\|\tilde{v}_Q(f)b\| &= \|Q\tilde{v}(f)Qb\| = \|Qv(f)Qb\| \\ &\leqq \|Qv(f)\| \, \|Qb\| \leqq C\|f\|_n\|Q\| \, \|b\| ,\end{aligned}$$

that is $\|\tilde{v}_Q(f)\| \leqq C\|Q\| \, \|f\|_n$, which proves the continuity of $\tilde{v}_Q$. Consider next the following matrix representation of $\tilde{v}$:

$$\tilde{v}(f)b = \begin{Bmatrix} \tilde{v}_Q(f) & 0 \\ P\tilde{v}(f)Q & \tilde{v}_P(f) \end{Bmatrix} \begin{Bmatrix} Qb \\ Pb \end{Bmatrix}$$

which is a schematic writing of the identity

$$\tilde{v}(f)b = (P+Q)\tilde{v}(f)(P+Q)b .$$

We have already discussed the continuity properties of the diagonal entries $\tilde{v}_Q$ and $\tilde{v}_P$. So consider the function $P\tilde{v}(\circ)Q$: $C^n([0,1]) \to \mathscr{B}(QB, R)$. Denote this map by $\tilde{v}_{PQ}$ and note that

$$\begin{aligned}\tilde{v}_{PQ}(f_1 f_2) &= P\tilde{v}(f_1)\tilde{v}(f_2)Q \\ &= P\tilde{v}(f_1)(P+Q)\tilde{v}(f_2)Q \\ &= P\tilde{v}(f_1)P\tilde{v}(f_2)Q + P\tilde{v}(f_1)Q\tilde{v}(f_2)Q \\ &= \tilde{v}_P(f_1)\tilde{v}_{PQ}(f_2) + \tilde{v}_{PQ}(f_1)\tilde{v}_Q(f_2) .\end{aligned}$$

$\mathscr{B}(QB, R)$ is a two-sided C^{2n+1}-module with left module multiplication given by $\tilde{v}$ and right module multiplication given by $\tilde{v}_Q$. With respect to these actions, $\tilde{v}_{PQ}$ is a derivation, as the above identity shows. By Corollary 1.18 we then obtain the continuity of $\tilde{v}_{PQ}$ on $C^{2(2n+1)}([0,1])$. Consequently, $\tilde{v}$ and thus v are continuous on $C^{4n+2}([0,1])$. But then Theorem 2.5 tells us that v is continuous on $C^{2n+1}([0,1])$, and this completes the proof.

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