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COMPLEX PREDUALS OF L_1 AND SUBSPACES OF $l_\infty^n(\mathbf{C})$

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Introduction.

In the present paper we investigate the structure of complex preduals of L_1 and the problems concerning norm preserving extensions of compact operators. Most of the results are known in the real case, but the complex case does not follow from these in a straightforward manner; in fact, in many respects the complex case is much more complicated and requires often different proofs. Many of the proofs here give other methods to show the corresponding real result.

We now wish to indicate in greater detail the arrangement and the results of this paper.

In section 1 we start by investigating the structure of those finite dimensional spaces, which embed isometrically into l_∞^n for some n. This leads then up to the main result of the section, which states that if X is a complex predual of L_1 and $E_1 \subseteqq X$, $E_2 \subseteqq X$ are finite dimensional spaces, so that E_1 embeds isometrically into l_∞^n for some n, then for every $\varepsilon > 0$ there is an $F \subseteqq X$ with $E_1 \subseteqq F$, F isometric to l_∞^m for suitable m and $d(x,F) < \varepsilon$ for $x \in E_2$, $\|x\| \leqq 1$. The corresponding real result was proved by Lazar and Lindenstrauss [16]. The proof given here provides an alternative way of proving the real result. The main brick in the proof is a complex version of the Lazar selection theorem, recently proved in [26]. The result is then used to give a new and very short proof of the Hirsberg–Lazar characterization of preduals of L_1, whose unit ball contains an extreme point.

Section 2 is devoted to the study of norm preserving extensions of compact operators. We first prove that if E is a finite dimensional space, whose unit ball is the absolutely convex hull of finitely many points, then every point in B_E can be represented as a combination of extreme points so that the coordinate functions are continuous. The real case is due to Kalman [12]. While his proof is geometric, the proof here uses an argument on extension of operators, based on the main theorem of section 1, but in the real case we do not need this theorem.

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The previous results of the paper are then used to characterize those preduals X of L_1 with the property that every compact operator with image in X can be extended preserving the norm. The corresponding real result was proved by Lazar and our proof follows his ideas. We end the section by proving that every predual of l_1 is isomorphic to an $\mathscr{L}_{\infty,1}$-space, a result due to W. B. Johnson and the first named author.

0. Notations and preliminaries.

In this paper all Banach spaces are assumed to be complex spaces unless otherwise stated, and throughout the paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [22], let us just here recall that if X and Y are Banach spaces, then the Banach distance $d(X,Y)$ is defined by

$$d(X,Y) = \inf\{\|S\|\,\|S^{-1}\| \mid S \text{ is isomorphism of } X \text{ onto } Y\}$$

and if X and Y are not isomorphic we put $d(X,Y)=\infty$.

For every natural n we let $\{e_j \mid 1 \leqq j \leqq n\}$ denote the unit vector basis of l_1^n and $\{e_j^* \mid 1 \leqq j \leqq n\}$ its biorthogonal system, i.e. the unit vector basis of l_∞^n. Further we let T be the unit circle in C, and if x and y are vectors in a Banach space, then we say that x and y are T-*equivalent*, if there is a $t \in \mathsf{T}$ so that $x=ty$.

If X is a Banach space we let B_X denote the unit ball of X, and for a convex set $K \subseteqq X$ we let $\partial_e K$ denote the extreme points. A compact absolutely convex set $K \subseteqq X$ is called a *complex polytope*, if there exists a finite set $A \subseteqq \partial_e K$ so that $\partial_e K = \mathsf{T} \cdot A$.

Let E and F be locally convex spaces and denote by $c(F)$ the set of all non-empty convex subsets of F. If $K \subseteqq E$ is convex and $\varphi\colon K \to c(F)$, then φ is called *convex*, provided:

$$\lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) \subseteqq \varphi(\lambda x_1 + (1-\lambda)x_2)$$

for all $x_1, x_2 \in K$ and $\lambda \in [0,1]$.

The map φ is said to be *lower semicontinuous* if $\{x \in K \mid \varphi(x) \cap U \neq \emptyset\}$ is open for every open subset U of F. Finally when K is absolutely convex, we say that φ is T-*symmetric*, if $\varphi(tx) = t\varphi(x)$ for all $t \in T$ and $x \in K$. By a *selection* for φ we mean a map $f\colon K \to F$ such that $f(x) \in \varphi(x)$ for all $x \in K$. Else our general reference in convexity is Alfsen's book [1].

1. Structure theorems for preduals of L_1.

Before we prove the main theorems of this section mentioned in the introduction, we need the following easy proposition on complex polytopes:

1.1. PROPOSITION. *Let E be a finite dimensional Banach space, then B_E is a complex polytope, if and only if there is an $n \in \mathbf{N}$ and an operator $l_1^n \to E$ taking the unit ball of l_1^n onto the unit ball of E.*

PROOF. Assume first B_E is a complex polytope, and let $x_1, x_2, \ldots, x_n$ be extreme points of B_E, mutually non T-equivalent, and so that $\partial_e B_E = \mathbf{T} \cdot \{x_1, x_2, \ldots, x_n\}$. If we define $S: l_1^n \to E$ by:

$$S\left(\sum_{j=1}^{n} t_j e_j\right) = \sum_{j=1}^{n} t_j x_j; \qquad t_1, t_2, \ldots, t_n \in \mathbf{C};$$

then it is obvious that S has the required properties.

If $S: l_1^n \to E$ satisfies the conditions in the proposition we put

$$A = \{e_j \mid S(e_j) \in \partial_e B_E\} .$$

If $x \in \partial_e B_E$, then $S^{-1}(x) \cap B_{l_1^n}$ is a compact face of $B_{l_1^n}$ and hence it contains an extreme point, thus there is an index j and $t \in \mathbf{T}$, so that $S(te_j) = x$, but then $e_j \in A$, and $x \in \mathbf{T} \cdot S(A)$. Hence $\partial_e B_E = \mathbf{T} \cdot S(A)$.

1.2. COROLLARY. *Let E be a finite dimensional Banach space. Then E embeds isometrically into l_∞^n for some n if and only if B_{E^*} is a complex polytope.*

We are now ready to state and prove the main theorem of this section.

1.3. THEOREM. *Let X be a predual of L_1 and let F_1 and F_2 be finite dimensional subspaces of X with F_1 isometric to a subspace of $l_\infty^{k_0}$ for some k_0. Then for every $\varepsilon > 0$ there is a subspace E of X with $F_1 \subseteqq E$, $d(x, E) \leqq \varepsilon$ for every $x \in B_{F_2}$ and so that E is isometric to l_∞^m for suitable m.*

PROOF. It is enough to prove the theorem in the case $\dim F_2 = 1$, the general case will then follow by induction. Hence let $\varepsilon > 0$, $\{y_i \mid 1 \leqq i \leqq n\}$ be a unit vector basis for F_1 and z a unit vector in F_2. We define $R: B_{X^*} \to \mathbf{C} \times \mathbf{C}^n$ by

$$Rx^* = (x^*(z), x^*(y_1), \ldots, x^*(y_n)); \qquad x^* \in B_{X^*};$$

and put $W = RB_{X^*}$. Denote by D_ε the disc in $\mathbf{C}$ with radius ε and center 0 and let $P: \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ be the canonical projection. Since by our assumptions and corollary 1.2 $P(W)$ is a complex polytope and $\partial_e W$ is totally bounded we can find mutually non T-equivalent extreme points $w_1, w_2 \ldots, w_m$ of W, so that if $W' = \operatorname{conv}(\mathbf{T} \cdot \{w_1, \ldots, w_m\})$, then $P(W) = P(W')$ and

$$\{z \in \mathbf{C} \mid (z, v) \in W\} \subseteqq \{z \in \mathbf{C} \mid (z, v) \in W'\} + D_\varepsilon . \tag{1}$$

Define $S: l_1^{m+1} \to \mathbf{C} \times \mathbf{C}^n$ by

(2)
$$S(e_j) = w_j; \quad 1 \leqq j \leqq m$$
$$S(e_{m+1}) = (\varepsilon, 0, \ldots, 0).$$

Let B be the unit ball of $\mathbf{C} \oplus_\infty l_1^n(\mathbf{C})$, then by (1)

$$S(B) \supseteqq W.$$

Let $\bar{c}(l_1^{m+1})$ denote the set of all closed convex subsets of l_1^{m+1} and define $\psi, \psi_1, \psi_2 \colon B_{X^*} \to \bar{c}(l_1^{m+1})$ by:

(3)
$$\psi_1(x^*) = S^{-1}(Rx^*)$$
$$\psi_2(x^*) = \begin{cases} \{te_k\} & \text{if } Rx^* = tw_k,\ t \in \mathbf{T},\ k \leqq m \\ B & \text{else} \end{cases}$$
$$\psi(x^*) = \psi_1(x^*) \cap \psi_2(x^*).$$

It is easy to see that ψ is convex and $\mathbf{T}$-symmetric. We wish to show that ψ is lower semicontinuous when B_{X^*} is equipped with the w^*-topology. If U is an open subset of l_1^{m+1}, then the sets

$$A_j = \{t \in \mathbf{T} \mid te_j \notin U\}, \quad 1 \leqq j \leqq m$$

are compact, and since l_1^{m+1} is finite dimensional, the set $R^{-1}(SU)$ is a w^*-open subset of B_{X^*}; therefore the set

$$\{x^* \mid \psi(x^*) \cap U \neq \emptyset\} = R^{-1}(SU) \setminus \bigcup_{j=1}^{n} \{tR^{-1}(w_j) \mid t \in A_j\}$$

is w^*-open in B_{X^*}. This proves that ψ is lower semicontinuous. By the complex analogy of Lazar's selection theorem [26, theorem 4.2] ψ admits an affine, $\mathbf{T}$-symmetric and w^*-continuous selection φ. For $k = 1, 2, \ldots, m+1$ we define $x_k \in X$ by

(4)
$$x^*(x_k) = e_k^*(\varphi(x^*)), \quad x^* \in B_{X^*}.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbf{C}$. By the definition of ψ_2 we now get:

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i x_i \right\| &= \sup\left\{ \left| x^*\left(\sum_{i=1}^m \alpha_i x_i \right) \right| \;\middle|\; x^* \in B_{X^*} \right\} \\ &= \sup\left\{ \left| \sum_{i=1}^m \alpha_i e_i^*(\varphi(x^*)) \right| \;\middle|\; x^* \in B_{X^*} \right\} \\ &= \sup\left\{ \left| \sum_{i=1}^m \alpha_i e_i^*(e) \right| \;\middle|\; e \in B_{l_1^m} \right\} \\ &= \sup\{ |\alpha_i| \mid 1 \leqq i \leqq m \}. \end{aligned}$$

This gives that the linear span of $\{x_k\}_{k=1}^m$ is isometric to l_∞^m. By (4) and the definition of ψ_1 we get for all $x^* \in B_{X^*}$:

$$(5) \qquad Rx^* = S(\varphi(x^*)) = \sum_{k=1}^{n} x^*(x_k)w_k + x^*(x_{m+1})(\varepsilon, 0, \ldots, 0)\,.$$

If we put $w_k = (w_k^j)_{j=1}^{n+1}$, we get by looking on (5) coordinatewise

$$(6) \qquad y_k = \sum_{j=1}^{m} w_j^{k+1} x_j, \quad 1 \leqq k \leqq n$$

and

$$(7) \qquad \left\| z - \sum_{j=1}^{m} w_j^1 x_j \right\| \leqq \varepsilon;$$

so the proof is complete.

As a corollary of the above theorem we can take out the next result, proved in the real case by Lazar and Lindenstrauss [16]. This is a slightly stronger version of the main result of Michael and Pelczynski in [23].

1.4. THEOREM. *Let X be a separable predual of L_1 and let $F \subseteq X$ be a finite dimensional space which embeds isometrically into l_∞^k for some k. Then there exists an increasing sequence $(E_n)_{n=1}^\infty$ of finite dimensional subspaces of X with $X = \overline{\bigcup_{n=1}^\infty E_n}$ and so that $F \subseteq E_1$, $\dim E_{n+1} = 1 + \dim E_n$ and E_n isometric to $l_\infty^{\dim E_n}$.*

PROOF. The result can be proved as in [16] by using our theorem 1.3 instead of their theorem 3.1. The fact that the E_n's can be chosen to satisfy $\dim E_{n+1} = 1 + \dim E_n$ follows from [23].

We pass now to give an alternative proof of a functional representation theorem for complex preduals of L_1 whose unit ball has an extreme point, due to Hirsberg and Lazar [10]. A simpler proof than the original one was given by Lima [17].

1.5. THEOREM. *Let X be a predual of L_1, whose unit ball has an extreme point e. Let*

$$S = \{x^* \in X^* \mid x^*(e) = 1 = \|x^*\|\}$$

be equipped with the w^*-topology. If $\psi\colon X \to C(S)$ is the natural map defined by $\psi(x)(x^*) = x^*(x)$, $x^* \in S$, then ψ is an isometry, which maps X onto the space of w^*-continuous complex affine functions on S and $\psi(e) = 1$.

PROOF. Clearly $\psi(e)=1$ and $\|\psi(x)\| \leqq \|x\|$ for all $x \in X$. Let $y \in X$ with $\|y\| = 1$. We wish to show that $\|\psi(y)\| \geqq 1$. If $\varepsilon > 0$ is arbitrary, then by theorem 1.3 we can find a finite dimensional space $E \subseteqq X$ so that $e \in E$ and $d(y,E) < \varepsilon$ and E isometric to l_∞^n for some n. Let $(x_j)_{j=1}^n$ be a basis for E isometrically equivalent to the unit vector basis of l_∞^n and let $(x_j^*)_{j=1}^n \subseteqq X^*$ be a sequence biorthogonal to $(x_j)_{j=1}^n$ with $\|x_j^*\| = 1$, $1 \leqq j \leqq n$. By the above there is an $x \in E$ with $\|x\| = 1$ and $\|y-x\| \leqq 2\varepsilon$. Let j be chosen so that $|x_j^*(x)| = 1$, since e is an extreme point of B_X $|x_j^*(e)| = 1$ and therefore $\overline{x_j^*(e)}x_j^* \in S$, moreover:

$$|\overline{x_j^*(e)}x_j^*(y)| \geqq |x_j^*(x)| - \|x_j^*\| \, \|x-y\| \geqq 1-2\varepsilon ,$$

hence $\|\psi(y)\| \geqq 1$. An argument of [26] gives that ψ is onto.

We want to thank A. Lazar for suggesting this proof.

1.6. COROLLARY. *Let X be a predual of L_1 and $e \in X$ with $\|e\| = 1$. Put*

$$S = \{x^* \in X^* \mid x^*(e) = 1 = \|x^*\|\} .$$

Then the following statements are equivalent:

(i) *e is an extreme point of B_X.*
(ii) *S is an maximal face of B_{X^*}.*
(iii) *e considered as an element of $B_{X^{**}}$ is an extreme point.*

PROOF. Assume (i) and that S is not a maximal face in B_{X^*}. Then there exist $y^* \in B_{X^*}$ such that $y^* \notin \overline{\mathrm{conv}}[tS \mid t \in \mathsf{T}\}$. By Hahn–Banach there exist a w^*-continuous functional x, that is $x \in X$, such that

$$\begin{aligned} y^*(x) = 1 &> \sup\{\operatorname{Re} x(x^*) \mid x^* \in \overline{\mathrm{conv}}\{tS \mid t \in \mathsf{T}\}\} \\ &\geqq \sup\{|x^*(x)| \mid x^* \in S\} \end{aligned}$$

which contradicts the fact that the map ψ in the preceeding theorem is an isometry.

(ii) $\Rightarrow$ (iii). Assume S is a maximal face in B_{X^*}. We may identify X^* with $L_1(Q,\mathscr{B},m)$ for some positive measure space $(Q,\mathscr{B},m)$. First we observe that for any $B \in \mathscr{B}$ there is $f \in S$ with $\|f \cdot \chi_B\| > 0$. If not the norm would be additive on the set $\mathrm{conv}\,(S \cup \{m(B)^{-1}\chi_B\})$ so by [2, lemma 2.1] we get a contradiction to the maximality of S. Assume there is $B \in \mathscr{B}$ with $m(B) > 0$ and $|e(q)| < 1$ a.e. on B. By the above observation there is $f \in S$ with $\|f \cdot \chi_B\| > 0$. Since S is a face $\|f \cdot \chi_B\|^{-1} f \cdot \chi_B \in S$. But

$$|e(\|f \cdot \chi_B\|^{-1} f \cdot \chi_B)| < 1$$

contradicting the definition of S.

(iii) $\Rightarrow$ (i). Trivial.

REMARKS. Functional representations of the type in theorem 1.5 were investigated and studied by Kadison (see [1, p. 78]) who represented the self adjoint part of a C*-algebra $\mathscr{A}$ isometrically as the real affine w^*-continuous functions on the state space. In this situation one can no longer represent $\mathscr{A}$ isometrically as complex affine functions on the state space unless $\mathscr{A}$ is commutative. This is probably well known, but since we are unable to give a reference to this fact, we shall give a proof which was shown to us by Christian Skau.

Let $a \in \mathscr{A}$. By assumption there is a pure state p such that $\|a\| = |p(a)|$. Let π_p be the corresponding representation with cyclic vector ξ. Then we have

$$\|a\| = |p(a)| = |\langle \pi_p(a)\xi, \xi\rangle| \leqq \|\pi_p(a)\xi\| \, \|\xi\| \leqq \|a\| .$$

By Schwartz's equality $\pi_p(a)\xi = p(a)\xi$. If $b \in \mathscr{A}$, then

$$p(ba) = \langle \pi_p(ba)\xi, \xi\rangle = \langle \pi_p(b)\pi_p(a)\xi, \xi\rangle = p(a)\langle \pi_p(b)\xi, \xi\rangle = p(a)p(b) .$$

Similarly we get $p(ab) = p(a)p(b)$. It follows that the spectral radius coincides with the norm on $\mathscr{A}$, so [4, theorem 4.7] gives that $\mathscr{A}$ is commutative. (The above result is incorrect for Banach algebras, consider the bounded operators on a predual of L_1 [4, p. 87]).

On the other hand functional representations of Banach algebras will always be isomorphisms due to the Bohnenblust–Karlin theorem [4], and for C*-algebras the onto argument is still valid, in fact this gives a characterization of the C*-algebras among the Banach algebras. This is just a restatement of the Azimov–Ellis geometric interpretation of the Vidav–Palmer theorem [3].

Let $\mathscr{A}$ be a Banach algebra with unit and assume $\mathscr{A}$ is complex predual of L_1. Then the map ψ of theorem 1.5 is onto, so $\mathscr{A}$ is a C*-algebra. Since ψ is an isometry, $\mathscr{A}$ is commutative, so $\mathscr{A}$ is isometric to $C(S)$ for some compact Hausdorff space S. This result was proved by Hirsberg and Lazar for function algebras [10] and in general by Ellis [8].

2. Norm preserving compact extensions.

In the real theory of norm preserving extensions of compact operators the subspaces of the spaces l_∞^n play a central role. The same is the case in the complex theory, as our results in section 1 indicate; however, there is one major difference. In the real theory the subspaces of the l_∞^n's are exactly the spaces, whose unit ball is a polytope (this follows for example from corollary 1.2 and the fact that the unit ball of a real Banach space is a polytope if and only if the unit ball of the dual space is a polytope [13]); this is not so in the complex case as the example l_∞^n shows.

We recall that a function f on a circled subset K of a Banach space is called T-*homogeneous* if $f(tx)=tf(x)$ for all $x \in K$, $t \in \mathsf{T}$.

Before we can prove our main results we need the following:

2.1. PROPOSITION. *Let K be a compact metric space, $x_0, x_1, \ldots, x_n \in K$; $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{C}$; so that $\sum_{j=1}^n |\lambda_j| \leqq 1$. Then the subspace X of $C(K)$ consisting of those $f \in C(K)$ for which $f(x_0)=\sum_{j=1}^n \lambda_j f(x_j)$ is a predual of L_1.*

PROOF. We shall assume $|\lambda_j|<1$ for all j, since else X is a G-space and hence is a predual of L_1 [25]. It is immediate that

$$\partial_e B_{X^*} = \{t\delta_x \mid x \in K,\ x \neq x_0,\ t \in \mathsf{T}\}\ .$$

Let $\varphi\colon \mathsf{T} \times K \to \overline{\partial_e B_{X^*}}^{w^*}$ be the onto map defined by $\varphi(t,x)=t\delta_x$, $t \in \mathsf{T}$, $x \in K$; and let μ and ν be two boundary measures on B_{X^*} with the same barycenter. According to a theorem of Effros [7], it is enough to show $\mu(f)=\nu(f)$ for all T-homogeneous $f \in C(B_{X^*})$. By the Hahn–Banach theorem there exist Radon probabilities μ_1 and ν_1 on $\mathsf{T} \times K$ so that $\varphi(\mu_1)=\mu$, $\varphi(\nu_1)=\nu$. By maximality $\mu(\mathsf{T}\{\delta_{x0}\})=\nu(\mathsf{T}\{\delta_{x0}\})=0$, hence μ_1 and ν_1 are concentrated on $\mathsf{T} \times (K\setminus\{x_0\})$. Let $f \in C(B_{X^*})$ be T-homogeneous, and let $\varepsilon>0$. By regularity we can find a compact subset $K_1 \subseteqq K\setminus\{x_0\}$ containing $x_1, \ldots, x_n$, so that

$$|\mu_1-\nu_1|(\mathsf{T} \times K_1) \geqq \|\mu_1-\nu_1\|-\varepsilon\ .$$

By Tietze's extension theorem we can find $\tilde{f} \in C(K)$ so that $\tilde{f}(x)=f(\delta_x)$, $x \in K_1$, $\|\tilde{f}\|=\|f\|$, and $\tilde{f}(x_0)=\sum_{j=1}^n \lambda_j f(\delta_{x_n})$; clearly $\tilde{f} \in X$ and hence:

$$\begin{aligned} |u(f)-\nu(f)| &= \left|\int_{\mathsf{T} \times K} f(t\delta_x)-t\tilde{f}(x)\, d(u_1-\nu_1)(t,x)\right| \\ &\leqq \int_{\mathsf{T} \times (K\setminus K_1)} |f(\delta_x)-\tilde{f}(x)|\, d|u_1-\nu_1|(t,x) \leqq 2\|f\|\varepsilon\ . \end{aligned}$$

Since ε was arbitrary $\mu(f)=\nu(f)$.

We are now able to prove the following theorem on complex polytopes.

2.2. THEOREM. *Let E be a finite dimensional Banach space whose unit ball is a complex polytope, and let $x_1, x_2, \ldots, x_n$ be the extreme points modulo* T. *If $x_0 \in B_E$ with $x_0=\sum_{j=1}^n \lambda_j^0 x_j$, $\sum_{j=1}^n |\lambda_j^0| \leqq 1$, then there exist functions $\lambda_1, \lambda_2, \ldots, \lambda_n \in C(B_E)$ so that $\sum_{j=1}^n |\lambda_j(x)| \leqq 1$, $x=\sum_{j=1}^n \lambda_j(x)x_j$ for all $x \in B_E$, and $\lambda_j(x_0)=\lambda_j^0$, $1 \leqq j \leqq n$.*

PROOF. Let X be the subspace of $C(B_E)$ consisting of those f for which $f(x_0)$

$=\sum_{j=1}^n \lambda_j^0 f(x_j)$ and let $S\colon l_1^n \to E$ be the operator from Proposition 1.1. Further let $I\colon E^* \to C(B_E)$ be the canonical embedding; clearly $I(E^*) \subseteqq X$. Since E^* is isometric to a subspace of l_∞^n (via S^*) it follows from theorem 1.3 and proposition 2.1 that there is an m and a subspace F of X isometric to l_∞^m with $I(E^*) \subseteqq F$. Since F is a $\mathscr{P}_1$-space there is a norm one operator $\tilde{I}\colon l_\infty^n \to F$ so that $I = \tilde{I}S^*$. Put $\lambda_j = \tilde{I}e_j^*$, $1 \leqq j \leqq n$. If $x \in B_E$, then

(1) $$\sum_{j=1}^n |\lambda_j(x)| = \left\| \sum_{j=1}^n e_j^*(\tilde{I}^*\delta_x)e_j \right\|_{l_1^n} = \|\tilde{I}^*\delta_x\|_{l_1^n} \leqq 1 .$$

For all $y^* \in E^*$ we obtain:

$$\begin{aligned} y^*\left(\sum_{j=1}^n \lambda_j(x)x_j\right) &= (S^*y^*)(\tilde{I}^*\delta_x) \\ &= (\tilde{I}S^*y^*)(x) = y^*(x) \end{aligned}$$

and hence

(2) $$x = \sum_{j=1}^n \lambda_j(x)x_j .$$

Since tx_j is an extreme point for every $t \in \mathrm{T}$, we have $\lambda_j(tx_j) = t$ and

$$\lambda_j(x_0) = \sum_{k=1}^n \lambda_k^0 \lambda_j(x_k) = \lambda_j^0, \quad 1 \leqq j \leqq n .$$

2.3. Corollary. *Under the same conditions as in 2.2 there exist* T-*homogeneous functions* $f_1, \ldots, f_n \in C(B_E)$ *so that* $x = \sum_{j=1}^n f_j(x)x_j$ *and* $\sum_{j=1}^n |f_j(x)| \leqq 1$ *for all* $x \in B_E$.

Proof. Let λ_j, $1 \leqq j \leqq n$ be as in theorem 2.2. Define for each j, $1 \leqq j \leqq n$,

$$f_j(x) = \int_{\mathrm{T}} t^{-1}\lambda_j(tx)\,dt, \quad x \in B_E ,$$

where dt is the normalized Haar measure on T. It is easily checked that the f_j's satisfy the requirements.

Remark. A slightly weaker form of corollary 2.3 was proved in the real case by Kalman [12] using geometric arguments. The real version of theorem 2.2 was proved by Lazar [15] by modification of Kalmans proof. Note that our proof of 2.2 and 2.3 with obvious changes gives an alternative proof in the real case, without using theorem 1.3, in fact it is easy to see that a $C(K)$-space has the finite binary intersection property, then argue as in [20, proof of theorem 5.5].

Recall that a linear subspace N of a Banach space W is called an *L-ideal* if there exist a subspace N' and W such that $W = N \oplus N'$ and $\|w\| = \|x\| + \|y\|$ for all $w = (x, y) \in N \oplus N'$. The reader is refered to [2], [9, theorem 1.2] and [17] for results on L-ideals.

The proof of the next proposition was suggested to us by Å. Lima.

2.4. PROPOSITION. *Let $(\Omega, \mathscr{B}, \mu)$ be a measure space and let F be a closed face of $B_{L_1(\mu)}$ and put $E = \operatorname{span}(F)$. Then:*

(i) *E is an L-ideal*
(ii) $E \cap B_{L_1(\mu)} = \overline{\operatorname{conv}}\,(\mathsf{T} \cdot F)$
(iii) *If $L_1(\mu)$ is a dual space and F is w^*-closed, then E is w^*-closed.*

PROOF. Since F is contained in a maximal proper face whose cone defines an order in $L_1(\mu)$ which makes it an abstract L-space, it is no loss of generality to assume that F is contained in the positive cone of $L_1(\mu)$.

If we let C denote the cone generated by F, then it is readily verified that $E = (C - C) + i(C - C)$, and since C is heriditary [2, lemma 2.7] it follows that if $f \in E$ then $f \geqq 0$ if and only if $f \in C$. Since a face cone in an L-space is a lattice cone we get by the above

$$E = \{ f \in L_1(\mu) \mid |f| \in C \} . \tag{1}$$

Since C is closed and the lattice operations are continuous, E is closed by (1). This relation also gives that E is an L_1-space under the induced order. To prove (i) we first observe that by (1) E is a solid subspace of $L_1(\mu)$ in the sense that $f \in E$, $|g| \leqq |f|$ implies $g \in E$. Since E is an L_1-space under the induced order, monotone, norm bounded nets in E converge [24], [27], and hence E is a band in $L_1(\mu)$. It follows that E is an L-ideal.

(ii) will follow from

$$B_{L_1(\mu)} \cap E = \{ f \mid |f| \in \operatorname{conv}(F, \{0\}) \} = \overline{\operatorname{conv}}\,(\mathsf{T} \cdot F) . \tag{2}$$

The first equality in (2) is obvious by (1). If $f \in L_1(\mu)$ with $|f| \in \operatorname{conv}(F \cup \{0\})$ and $\varepsilon > 0$, then we can find a simple function g with $|g| \leqq |f|$ and $\|g - f\| < \varepsilon$. Hence $g \in E$. If $g = \sum_{j=1}^{m} \alpha_j \chi_{A_j}$ with $A_j \cap A_i = \emptyset$, $i \neq j$, then $\|\chi_{A_j}\|^{-1} \chi_{A_j} \in F$, $1 \leqq j \leqq n$. Furthermore

$$1 \geqq \|g\| = \sum_{j=1}^{m} |\alpha_j| \, \|\chi_{A_j}\|$$

and thus $g \in \operatorname{conv}(\mathsf{T} \cdot F)$. The inclusion $\overline{\operatorname{conv}}\,(\mathsf{T} \cdot F) \subseteq B_{L_1(\mu)} \cap E$ is trivial.

Finally assume that $L_1(\mu)$ is a dual space and F is w^*-closed. According to the Banach–Dieudonné theorem it is enough to prove that $E \cap B_{L_1(\mu)}$ is w^*-

closed. It follows immediately that $C \cap B_{L_1(\mu)}$ is w^*-compact. If $(f_t) \subseteqq E \cap B_{L_1(\mu)}$ is a w^*-convergent net with limit f, then by a compactness argument we may assume that the nets $((\operatorname{Re} f_t)^+)$, $((\operatorname{Re} f_t)^-)$, $((\operatorname{Im} f_t)^+)$ and $((\operatorname{Im} f_t)^-)$ all converges to elements in $C \cap B_{L_1(\mu)}$. Thus $f \in E$ and trivially $\|f\| \leqq 1$.

The next lemma is actually one of the implications in our main theorem, but we have taken it out separately of technical reasons.

2.5. LEMMA. *Let X be a predual of L_1. If B_{X^*} has an infinite dimensional w^*-closed face, then X contains a subspace isometric to c.*

PROOF. Let F be an infinite dimensional w^*-closed face of B_{X^*} and put $N = \operatorname{span}(F)$. By proposition 2.4, N is a w^*-closed L-ideal of X^* with F as a maximal proper face of B_N. If $Z = X/N^0$ then $Z^* = N$, and since F is split in $\operatorname{conv}(F \cup -iF)$ every $f \in A(F)$ (here $A(F)$ denotes the complex, affine, w^*-continuous functions on F) can be extended to an element in Z [25]; hence the map $\psi: Z \to A(F)$ defined by $(\psi z)(x^*) = x^*(z)$ is an isometry onto. By Zippin's result [28] there is an isometric embedding $U: \operatorname{Re} c \to \operatorname{Re} A(F)$. If W: $c \to A(F)$ is defined by $W(x+iy) = Ux + iUy$ $x, y \in \operatorname{Re} c$ then W is an isometric embedding. In fact let $s \in F$ with

$$\|W(x+iy)\| = |W(x+iy)(s)|$$

and choose $t \in \mathrm{T}$, $t = u+iv$, $u, v \in \mathrm{R}$ so that

$$\begin{aligned} \|W(x+iy)\| &= t((Ux)(s) + i(Uy)(s)) = u(Ux)(s) - v(Uy)(s) \\ &= U(\operatorname{Re} t(x+iy))(s) \leqq \|U(\operatorname{Re} t(x+iy)\| \leqq \|x+iy\| . \end{aligned}$$

In a similar manner we get $\|W(x+iy)\| \geqq \|x+iy\|$. (The last argument was shown to us by Å. Lima.) Hence we have shown that Z contains a subspace Y isometric to c.

If $(x_n) \subseteqq Y$ is a dense sequence, then we can define a metric on Y^* with the aid of this sequence, so that it generates the w^*-topology on B_{Y^*}. Let us denote Y^*'s completion in this metric by $\hat{Y}^*$, clearly B_{Y^*} can be considered topologically as a subset of $\hat{Y}^*$.

Let $\varphi: B_N \to B_{Y^*}$ be defined by:

$$(\varphi x^*)(y) = x^*(y), \qquad y \in Y,\ x^* \in N .$$

From [26, theorem 4.2] there is an affine, T-symmetric w^*-continuous map $\Phi: B_{X^*} \to B_{Y^*}$, so that $\Phi | B_N = \varphi$. Define $S: Y \to X$ by

$$x^*(Sy) = (\Phi x^*)(y), \qquad y \in Y,\ x^* \in B_{X^*} .$$

Clearly S is an isometry and hence c embeds isometrically into X.

We recall that a Banach space X is called an $\mathscr{L}_{\infty,\lambda}$-space, if for every finite dimensional subspace $E \subseteqq X$ there is a finite dimensional subspace $F \subseteqq X$ with $E \subseteqq F$ and $d(F, l_\infty^{\dim F}) \leqq \lambda$. It is well-known that a Banach space X is a predual of L_1 if and only if it is an $\mathscr{L}_{\infty,1+\varepsilon}$-space for all $\varepsilon > 0$ [22]. The Banach space c_0 is an $\mathscr{L}_{\infty,1}$-space, while c is not, as it is seen from:

2.6. LEMMA. *There is a two dimensional subspace E of c, which does not embed isometrically into l_∞^n for any n.*

PROOF. Let

$$x^1 = (\cos k^{-1})_{k=1}^\infty, \qquad x^2 = (\sin k^{-1})_{k=1}^\infty$$

and put $E = \operatorname{span}(x_1, x_2)$. For the element $x_k = \cos k^{-1} x^1 + \sin k^{-1} x^2$, $k \in \mathsf{N}$, we get

$$x_k(n) = \cos(n^{-1} - k^{-1}), \qquad n, k \in \mathsf{N};$$

and hence $x_k(k) = 1$, $|x_k(n)| < 1$ when $n \neq k$. This shows that $\delta_n \in E^*$ defined by $\delta_n(x) = x(n)$ for all $x \in E$, $n \in \mathsf{N}$ is an extreme point. Corollary 1.2 now completes the proof.

In [15] Lazar characterized those real Banach spaces X which have the property that every compact operator with image in X can be extended preserving the norm. A similar result is true in the complex case; the proof of it goes along the lines of [15, proof of theorem 3].

2.7. THEOREM. *Let X be a predual of L_1. The following statements are equivalent:*

(i) *X is a $\mathscr{L}_{\infty,1}$-space.*

(ii) *No subspace of X is isometric to c.*

(iii) *B_{X^*} has no infinite dimensional w^*-closed faces.*

(iv) *For all Banach spaces Y and Z with $Y \subseteqq Z$ and every compact operator $S: Y \to X$, there is a compact extension $\tilde{S}: Z \to X$ with $\|\tilde{S}\| = \|S\|$.*

(v) *For all Banach spaces Y and Z with $Y \subseteqq Z$ and every operator $S: Y \to X$ with* $\dim S(Y) \leqq 2$, *there is a compact extension $\tilde{S}: Z \to X$ with $\|S\| = \|\tilde{S}\|$.*

PROOF. (i) $\Rightarrow$ (ii): follows from lemma 2.6.

(ii) $\Rightarrow$ (iii): is lemma 2.5.

(iii) $\Rightarrow$ (iv): Assume that (iii) holds and let $S: Y \to X$ be compact with $\|S\| = 1$. It follows that S^* is continuous from B_{X^*} to B_{Y^*}, when the first ball is equipped with the w^*-topology and the latter with the norm topology. We wish to construct an affine, T-symmetric map $\varphi: B_{X^*} \to B_{Y^*}$, continuous when the sets are equipped with the w^*-topology, respectively the norm topology, so that

$$\varphi x^* \mid Y = S^* x^* \quad \text{for all } x^* \in B_{X^*} .$$

Put $K = S^* B_{X^*}$. Arguing as Lazar [15, p. 360] we get that there are finitely many non T-equivalent extreme points $u_1^*, u_2^*, \ldots, u_n^*$ of K such that

$$\partial_e K \cap \{y^* \in Y^* \mid \|y^*\| = 1\} = \mathrm{T} \cdot \{u_1^*, \ldots, u_n^*\} .$$

We also get that there is a β $0 < \beta < 1$, so that if $y^* \in \partial_e K$ with $\|y^*\| > \beta$, then $\|y^*\| = 1$. Put

$$K_\beta = \{y^* \in K \mid \|y^*\| \leqq \beta\} . \tag{1}$$

The Krein–Milman theorem gives that

$$K = \operatorname{conv}(K_\beta \cup \mathrm{T} \cdot \{u_1^*, \ldots, u_n^*]) \tag{2}$$

and

$$K \cap \{y^* \in Y^* \mid \|y^*\| = 1\} \subseteqq \operatorname{conv}(\mathrm{T} \cdot \{u_1^*, \ldots, u_n^*\}) = K_1 . \tag{3}$$

Let for $j = 1, 2, \ldots, n$; $z_j^* \in Z^*$ be Hahn–Banach extensions of u_j^*. We define a map ψ of K into the closed convex subsets of B_{Z^*} by

$$\psi(y^*) = \{z^* \in B_{Z^*} \mid z^* \mid Y = y^*\} \quad \text{for } y^* \in K \text{ and } \|y^*\| < 1 \tag{4}$$

$$\psi(y^*) = \left\{ \sum_{j=1}^n \lambda_j z_j^* \;\middle|\; y^* = \sum_{j=1}^n \lambda_j u_j^*,\ \sum_{j=1}^n |\lambda_j| = 1 \right\} \tag{5}$$

$$\text{for } y^* \in K,\ \|y^*\| = 1 .$$

Clearly $\psi(y^*) \neq \emptyset$ for all $y^* \in K$ and it is readily verified that ψ is convex and T-symmetric. We shall prove ψ is lower semicontinuous, when K and B_{Z^*} are equipped with the norm topologies. Hence let U be an open subset of Z^* and let

$$y_0^* \in \{y^* \in K \mid \psi(y^*) \cap U \neq \emptyset\} = K_2 .$$

If $\|y_0^*\| < 1$, then we argue like Lazar [15] to get that y_0^* is an interior point of K_2. Next suppose that $\|y_0^*\| = 1$ and let $z_0 \in \psi(y_0^*) \cap U$ with $z_0 = \sum_{j=1}^n \lambda_j^0 z_j^*$, where $y_0^* = \sum_{j=1}^n \lambda_j^0 u_j^*$ and $\sum_{j=1}^n |\lambda_j^0| = 1$. By theorem 2.2 there are functions $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathrm{C}(\mathrm{K}_1)$ so that $y^* = \sum_{j=1}^n \lambda_j(y^*) u_j^*$, $\sum_{j=1}^n |\lambda_j(y^*)| \leqq 1$ for all $y^* \in K_1$ and $\lambda_j(y_0^*) = \lambda_j^0$, $1 \leqq j \leqq n$. Let $\varepsilon > 0$ be given so that the ball with center z_0^* and radius ε is contained in U, and let W_1 be a neighborhood of y_0^* so that

$$\left\| \sum_{j=1}^n \lambda_j(y^*) z_j^* - z_0^* \right\| \leqq 3^{-1}\varepsilon \quad \text{for } y^* \in W_1 \cap K_1 . \tag{6}$$

It is easy to see that there is a neighborhood W of y_0^* so that if $y^* \in W \cap K$ and

$$y^* = \alpha y_1^* + (1-\alpha) y_2^* ,$$

where $y_1^* \in K_\beta$, $y_2^* \in K_1$, and $\alpha \in [0,1]$, then $\alpha < 3^{-1}\varepsilon$ and $y_2^* \in W_1$. Let $y^* \in W$,

$$y^* = \alpha y_1^* + (1-\alpha) y_2^* \quad \text{with } y_1^* \in K_\beta,\ y_2^* \in K_1$$

and let $z^* \in \psi(y_1^*)$. Put

$$v^* = \alpha z^* + (1-\alpha) \sum_{j=1}^{n} \lambda_j(y_2^*) z_j^* . \tag{7}$$

By the convexity of ψ, $v^* \in \psi(y^*)$ and furthermore

$$\|z_0^* - v^*\| \leqq \alpha \|z_0^* - z^*\| + (1-\alpha) \left\| \sum_{j=1}^{n} \lambda_j(y_2^*) z_j^* - z_0 \right\| \leqq \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \tag{8}$$

so $v^* \in \psi(y^*) \cap U$, which gives that y_0^* is an interior point of K_2, and thus we have proved that ψ is lower semicontinuous.

The map $\psi \circ S^*$ is w^*-lower semicontinuous and T-symmetric and therefore by [26] it has a w^*-continuous affine selection φ. Define $\tilde{S}: Z \to X$ by

$$x^*(\tilde{S}z) = \varphi(x^*)(z), \qquad z \in Z,\ x^* \in B_{X^*} . \tag{9}$$

By the properties of φ, $\tilde{S}$ is compact and it is an extension of S with $\|\tilde{S}\| = 1$.

(iv) $\Rightarrow$ (i). The proof of this implication is essentially the same as the proof of theorem 7.9 in [20], but let us give it for the sake of completeness. Assume (iv) and let $E \subseteqq X$ be finite dimensional. By assumption there is a compact operator S in X, so that $Sx = x$ for $x \in E$ and $\|S\| = 1$. Put for every $n \in \mathbf{N}$

$$S_n = n^{-1}\left(\sum_{k=0}^{n} S^k\right).$$

By the ergodic theorem on compact operators [6, p. 711] S_n converges to a finite dimensional projection P with $\|P\| = 1$ and with image

$$F = \{x \in X \mid Sx = x\} . \tag{10}$$

Since X^{**} is a $\mathscr{P}_1$-space and $P^{**}(X^{**}) = F$, it follows that F is a $\mathscr{P}_1$-space and hence F is isometric to $l_\infty^{\dim F}$. Clearly $E \subseteqq F$.

(iv) $\Rightarrow$ (v): is trivial.

(v) $\Rightarrow$ (ii): Assume (v). Then the proof of the implication (iv) $\Rightarrow$ (i) shows that every two dimensional subspace of X embeds isometrically into l_∞^n for suitable n, and hence according to lemma 2.6 X does not contain c isometrically.

REMARK. We do not know whether the condition (v) implies that X is a predual of L_1. Lima [17, theorem 4.10] proved that the answer is positive for real spaces, and in [18, theorem 4.1] he proved that the answer is positive in the complex case if we in (v) require $\dim S(Y) \leqq 3$ instead of $\dim S(Y) \leqq 2$.

As a corollary to theorem 2.7 we get as in the real case:

2.8. THEOREM. *If X is an $\mathscr{L}_{\infty,1}$-space, then $X^* = l_1(\Gamma)$ for some Γ.*

PROOF. Assume first that X is separable. If $\partial_e B_{X^*}$ is uncountable modulo T, then $\partial_e B_{X^*}$ has an infinite compact subset E (one may even choose E to be the Cantor set) with $E \cap tE = \varnothing$ for all $t \in \mathrm{T} \setminus \{1\}$. By [25, lemma 22] $F = \overline{\mathrm{conv}}\,(E)$ is a w^*-closed face of B_{X^*} which contradicts theorem 2.7. Hence $\partial_e B_{X^*}$ is countable modulo T and thus $X^* = l_1$.

The general case follows from this together with [14, theorem 6, p. 227] (the implication we need is also proved for the complex case, although this is not stated explicitly; it is also likely, using the result of [26], that this theorem carries over to the complex case).

The final result of this section due to W. B. Johnson and the first named author shows that every predual of l_1 is isomorphic to an $\mathscr{L}_{\infty,1}$-space.

2.9. THEOREM. *Let X be a real or complex Banach space with $X^* = l_1$. Then there exists an $\mathscr{L}_{\infty,1}$-space Y which is isomorphic to X.*

PROOF. Let $(x_n^*) \subseteq X^*$ be a basis isometrically equivalent to the unit vector basis of l_1. Put for every natural number n $E_n = \mathrm{span}\,\{x_1^*, \ldots, x_n^*\}$ and define a new norm on X^* by:

$$(1) \qquad |||x^*||| = \|x^*\| + \sum_{n=1}^{\infty} 2^{-n} d(x^*, E_n),, \qquad x^* \in X^* .$$

(This renorming technique was used in [5].) Since the E_n's are finite dimensional the unit ball determined by $|||\cdot|||$ is w^*-closed and hence $|||\cdot|||$ is the dual norm of a norm $|||\cdot|||$ on X which is readily seen to be equivalent to $\|\cdot\|$. Put $Y = (X, |||\cdot|||)$. We shall show that Y is a $\mathscr{L}_{\infty,1}$-space. Put for every natural number n $y_n^* = |||x^*|||^{-1} x_n^*$ and let k be a natural number and $t_1, t_2, \ldots, t_k$ scalars. Then

$$(2) \qquad \left|\left|\left| \sum_{n=1}^{k} t_n y_n^* \right|\right|\right| = \left|\left|\left| \sum_{n=1}^{k} t_n |||x_n^*|||^{-1} x_n^* \right|\right|\right|$$

$$= \sum_{n=1}^{k} 2^{-1}|t_n|(1-2^{-n})^{-1}$$

$$+\sum_{n=1}^{k-1} 2^{-n}\left(\sum_{j=n+1}^{k} 2^{-1}|t_j|(1-2^{-j})^{-1}\right)$$

$$= 2^{-1}\sum_{j=1}^{k}\sum_{n=0}^{j-1} 2^{-n}|t_j|(1-2^{-j})^{-1}$$

$$= \sum_{j=1}^{k} |t_j|$$

which show $Y^* = l_1$.

If we show that every w^*-limit point of the sequence (y_n^*) has norm strictly less than 1, then it will follow from theorem 2.7 for the complex case and Lazar [15, theorem 3] for the real case that Y is an $\mathscr{L}_{\infty,1}$-space. Hence let $x^* \in X^*$ and let $(y_{n_k}^*)$ be a sequence with $y_{n_k}^* \xrightarrow{w^*} x^*$. Since $|||x_{n_k}^*||| \to 2$ we get that $\|x^*\| \leqq 2^{-1}$ and therefore for n sufficiently large $d(x^*, E_n) < 2^{-1}$. This gives

$$|||x^*||| < 2^{-1}+2^{-1}\sum_{n=1}^{\infty} 2^{-n} \leqq 1 .$$

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals* (Ergebnisse der Math. und ihrer Grenzengebiete 57), Springer-Verlag, Berlin · Heidelberg · New York, 1971.
2. E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces* I & II, Ann. of Math. 96 (1972), 98–173.
3. L. A. Azimov and A. J. Ellis, *On hermitian functionals on unital Banach algebras*, Bull. London. Math. Soc. 4 (1973), 333–336.
4. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, Lecture Notes 2, London Math. Soc., Cambridge, 1971.
5. W. J. Davis and W. B. Johnson, *A renorming of non-reflexive Banach spaces*, to appear.
6. N. Dunford and J. Schwartz, *Linear operators part* I (Pure and Applied Mathematics 7), Interscience, New York, 1958.
7. E. G. Effros, *On a class of complex Banach spaces*, Illinois J. Math. 18 (1974), 48–59.
8. A. Ellis, *Some applications of convexity theory on Banach algebras*, Math. Scand. 33 (1973), 23–30.
9. B. Hirsberg, *M-ideals in complex function spaces and algebras*, Israel J. Màth. 12 (1972), 133–146.
10. B. Hirsberg and A. Lazar, *Complex Lindenstrauss spaces with extreme points*, Trans. Amer. Math. Soc. 186 (1973), 141–150.
11. O. Hustad, *Intersection properties of balls in complex Banach spaces whose duals are L_1-spaces*, Acta Math. 132 (1974), 283–313.

12. J. A. Kalman, *Continuity and convexity of projections and barycentric coordinates in convex polyhedra*, Pacific J. Math. 11 (1961), 1017–1022.
13. V. Klee, *Polyhedral sections of convex bodies*, Acta Math. 103 (1960), 243–267.
14. H. E. Lacey, *The isometric theory of classical Banach spaces* (Die Grundlehren der Math. Wissenschaften 208), Springer-Verlag, Berlin · Heidelberg · New York, 1974.
15. A. Lazar, *Polyhedral Banach spaces and extensions of compact operators*, Israel J. Math. 7 (1969), 357–364.
16. A. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. 126 (1971), 165–193.
17. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, to appear.
18. Å. Lima, *Complex Banach spaces whose duals are L_1-spaces*, to appear in Israel J. Math.
19. Å. Lima, *An application of a theorem of Hirsberg and Lazar*, Math. Scand. 38 (1976), 325–340.
20. J. Lindenstrauss, *Extensions of compact operators*, Mem. Amer. Math. Soc. 48 (1964).
21. J. Lindenstrauss and A. Pelczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), 275–326.
22. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* (Lecture notes in Math. 338), Springer-Verlag, Berlin · Heidelberg · New York, 1973.
23. E. Michael and A. Pelczyński, *Separable Banach spaces which admit l_∞^n approximations*, Israel J. Math. 4 (1966), 189–198.
24. N. J. Nielsen, *On Banach ideals determined by Banach Lattices and their applications*, Dissertationes Math. (Rozprawy Math.) 109 (1973), 1–66.
25. G. Olsen, *On the classification of complex Lindenstrauss spaces*, Math. Scand. 35 (1974), 237–258.
26. G. Olsen, *Edwards separation theorem for complex Lindenstrauss spaces with application to selection and embedding theorems*, Math. Scand. 38 (1976), 97–105.
27. H. H. Schaefer, *Topological vector spaces*, third edition, Springer-Verlag, Berlin · Heidelberg · New York, 1970.
28. M. Zippin, *On some subspaces of Banach spaces, whose duals are L_1-spaces*, Proc. Amer. Math. Soc. 23 (1969), 378–385.

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