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HARMONIC AND POLYHARMONIC DEGENERACY

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The harmonic and polyharmonic classifications of Riemannian manifolds have been largely brought to completion. In contrast, nothing seems to be known about relations between these two classifications. One might expect that a harmonic degeneracy implies or is implied by at least the corresponding polyharmonic degeneracy. The result in the present paper is that this is not so: There are no inclusion relations whatever between harmonic and polyharmonic null classes of Riemannian manifolds of any dimension.

1. Decomposition.

Let P, B, D, C be the classes of functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. Denote by H the class of harmonic functions h, $\Delta h=0$, with $\Delta = d\delta+\delta d$ the Laplace–Beltrami operator. The class of nondegenerate polyharmonic functions u, $\Delta^k u=0$, $\Delta^{k-1}u \not\equiv 0$, $k \geqq 2$, is denoted by H^k. For any function classes X, Y, set $HX = H \cap X$ and $H^kY = H^k \cap Y$, and let $O^N_{HX}, O^N_{H^kY}$ be the classes of noncompact Riemannian N-manifolds on which $HX = \mathsf{R}$ and $H^kY = \varnothing$, respectively. The complementary classes are denoted by $\tilde{O}^N_{HX}$ and $\tilde{O}^N_{H^kY}$.

THEOREM. *For $k \geqq 2$; $X = P, B, C, D$; $Y = B, D, C$; and $N \geqq 2$, the totality of Riemannian N-manifolds decomposes into four nonvoid disjoint classes*

$$O^N_{HX} \cap O^N_{H^kY},\ \tilde{O}^N_{HX} \cap O^N_{H^kY},\ O^N_{HX} \cap \tilde{O}^N_{H^kY},\ \tilde{O}^N_{HX} \cap \tilde{O}^N_{H^kY}.$$

The proof will be given in Nos. 1–4.

We know from Mirsky–Sario–Wang [1] that the class O^N_G of parabolic Riemannian N-manifolds satisfies

$$O^N_G \cap O^N_{H^kB} \cap O^N_{H^kD} \neq \varnothing, \quad O^N_G \cap \tilde{O}^N_{H^kC} \neq \varnothing$$

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for $N \geqq 2$, $k \geqq 2$. From this and the inclusion $O_G^N \subset O_{HX}^N$ (e.g., Sario–Nakai [2]) we conclude that

$$O_{HX}^N \cap O_{H^kY}^N \neq \emptyset, \quad O_{HX}^N \cap \tilde{O}_{H^kY}^N \neq \emptyset .$$

In view of the Euclidean N-ball, we have trivially $\tilde{O}_{HX}^N \cap \tilde{O}_{H^kY}^N \neq \emptyset$. Thus the only relation that needs proving is

$$\tilde{O}_{HX}^N \cap O_{H^kY}^N \neq \emptyset .$$

Consider the N-cylinder

$$R = \{|x| < \infty,\ |y^i| \leqq \pi,\ i = 1, 2, \ldots, N-1\}$$

with the metric

$$ds^2 = \varphi^2(x)dx^2 + \psi^{2/(N-1)}(x) \sum_{i=1}^{N-1} dy^{i2} ,$$

where

$$\varphi^2(x) = e^{x^2}(1+x^2)^{-1} \quad \text{and} \quad \psi^2(x) = e^{x^2}(1+x^2) .$$

We first observe that $R \in \tilde{O}_{HC}^N$. In fact, the harmonic equation

$$\Delta h(x) = -g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{xx}h')' = 0$$

has a solution

$$h(x) = \int_0^x (g^{\frac{1}{2}}g^{xx})^{-1}\,dx = \int_0^x (1+x^2)^{-1}\,dx ,$$

which is bounded and has the Dirichlet norm

$$D(h) = \int_R h'^2 * g^{xx} = c\int_{-\infty}^{\infty} (1+x^2)^{-1}\,dx < \infty .$$

Next we show that R carries no H^kB or H^kD functions. The reasoning is divided into several lemmas.

2. Auxiliary results.

Let Q be the class of quasiharmonic functions q, defined by $\Delta q = 1$. We start by proving:

LEMMA. *For $X = B, D, C$, and $N \geqq 2$,*

$$R \in O_{QX}^N .$$

PROOF. The quasiharmonic equation

$$\Delta q(x) = -g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{xx}q')' = 1$$

has a solution

$$q(x) = -\int_0^x (g^{\frac{1}{2}}g^{xx})^{-1}(t)\int_0^t e^{s^2}\,ds\,dt$$

$$= -\int_0^x (1+t^2)^{-1}\int_0^t e^{s^2}\,ds\,dt\,,$$

which is unbounded. It is also Dirichlet infinite:

$$D(q) = c\int_{-\infty}^{\infty} q'^2 g^{xx}g^{\frac{1}{2}}\,dx = c\int_{-\infty}^{\infty} q'^2(1+x^2)\,dx = \infty\,.$$

An arbitrary quasiharmonic function on R has the form

$$q(x,y) = q(x)+h(x)+h(x,y)\,,$$

where $h(x)$ and $h(x,y)$ are harmonic and $h(x,y)$ is L^2-orthogonal to any function of x. As $x \to \infty$, we can choose y depending on x such that $h(x,y)=0$. For this choice of x and y, since $|q(x)|$ will eventually dominate $|h(x)|$, we have $q(x,y) \to -\infty$. Thus $q(x,y)$ is unbounded. Moreover, $D(q(x,y)) \geqq D(q(x)) = \infty$, and the Lemma follows.

We shall later make use of the following eigenfunction expansion:

Every C^∞ function $g(x,y)$ on R can be written

$$g(x,y) = \sum_n f_n(x)G_n(y)\,,$$

where $G_n(y)=\prod_{i-1}^{N-1} g_{n_i}(y^i)$ with $g_{n_i}(y^i)=\pm\sin n_i y^i$ or $\pm\cos n_i y^i$.

3. Biharmonic degeneracy.

First we assert:

Every nonharmonic biharmonic function $u(x)$ on R is unbounded and Dirichlet infinite.

In fact, the equation

$$\Delta u(x) = -g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{xx}u'(x))' = h(x)$$

has a solution u with

$$(1+x^2)u'(x) = -\int_0^x e^{s^2}h(s)\,ds\,,$$

which gives

$$u(x) = -\int_0^x (1+t^2)^{-1}\int_0^t e^{s^2}h(s)\,ds\,dt\,.$$

This is unbounded. By an argument similar to that in No. 2, we also see that $D(u(x))=\infty$.

Let $G=G_n$ be as at the end of No. 2.

LEMMA. *If $u(x,y)=v(x)G(y)$ is nonharmonic biharmonic on R, then it is unbounded and Dirichlet infinite, and $|v(x)|$ grows at least exponentially as $x\to\infty$ or else as $x\to-\infty$.*

PROOF. We have

$$\Delta u = \Delta v\cdot G+v\Delta G\,,$$

where

$$\Delta v = -e^{-x^2}((1+x^2)v')'\,,$$

$$\Delta G = \eta e^{-x^2}(e^{x^2}[e^{x^2}(1+x^2)]^{-1/(N-1)})G\,,$$

with η a positive constant. Hence,

$$\Delta u = e^{-x^2}[(-(1+x^2)v')'+\eta e^{[(N-2)/(N-1)]x^2}(1+x^2)^{-1/(N-1)}v]G$$

$$= fG\in H$$

and therefore,

$$((1+x^2)v')' = \eta e^{[(N-2)/(N-1)]x^2}(1+x^2)^{-1/(N-1)}v-e^{x^2}f\,.$$

By the maximum principle for harmonic functions, f is monotone. Thus f is bounded away from zero as $x\to\infty$ or else $x\to-\infty$. We may assume the former, so that $|f|>c$, for sufficiently large x. If $|v|$ grows less rapidly than $e^{|x|}$, then the right-hand side $r(x)$ of the above formula is dominated by $e^{x^2}f$ as $x\to\infty$ and therefore grows faster than e^{2x}. On the other hand,

$$(1+x^2)v' = \int_0^x r(s)\,ds+c$$

and

$$v(x) = \int_0^x (1+t^2)^{-1}\int_0^t r(s)\,ds\,dt+c\int_0^x (1+t^2)^{-1}\,dt+c'\,.$$

Since $r(x)$ grows faster than e^{2x} as $x \to \infty$, $v(x)$ has to grow faster than e^x, a contradiction. The first and third assertions of the Lemma follow.

To prove the second assertion, we observe that

$$D(vG) \geqq \int_R \left(v\frac{\partial G}{\partial y^i}\right)^2 * g^{y^i y^i}$$

$$= c\int_{-\infty}^{\infty} v^2 e^{[(N-2)/(N-1)]x^2}(1+x^2)^{-1/(N-1)}\,dx\,.$$

Since $N \geqq 2$, and v grows at least exponentially, the integral is infinite. The Lemma follows.

4. Completion of proof.

Next we show:

LEMMA. *For* $Y = B, D, C$, *and* $N \geqq 2$,

$$R \in O^N_{H^2 Y}\,.$$

PROOF. Let $u(x,y) = u(x) + \sum_n v_n(x) G_n(y)$ be a nonharmonic biharmonic function. By the orthogonality of any two summands, we argue in the same manner as in the proof of the Lemma in No. 2, that $u(x,y)$ is unbounded and Dirichlet infinite.

Our Theorem has thus been proved for $k=2$.

For $k>2$, we use induction. The argument is as in Nos. 1–3, with obvious modifications.

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