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AN INEQUALITY FOR CONVEX CURVES IN THE PLANE

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Let Γ denote a simple, closed, convex curve in R^2 and let $l(\theta, a)$ denote the straight line $x\cos\theta + y\sin\theta = a$, where $0 \leqq \theta < 2\pi$ and $a \in \mathsf{R}$. We set

$$E(\theta) = \{a \in \mathsf{R} \ ; \ l(\theta, a) \cap \Gamma \neq \emptyset\} ,$$

$c(\theta) = \inf E(\theta)$ and let $k_\varkappa(\theta)$ denote the arc length of the part of Γ which lies between the lines $l(\theta, c(\theta))$ and $l(\theta, c(\theta)+\varkappa)$, $\varkappa > 0$. We shall prove the following theorem.

THEOREM 1. *Let Γ and $k_\varkappa(\theta)$ be defined as above. Then*

$$\liminf_{\varkappa \to 0} \int_0^{2\pi} k_\varkappa(\theta)\, d\theta\, (\varkappa \log 1/\varkappa)^{-1} > 0 . \tag{1}$$

In the case when Γ is piecewise linear (1) follows from an elementary computation and in this case it is also easy to see that there is a constant C such that

$$\int_0^{2\pi} k_\varkappa(\theta)\, d\theta \leqq C\varkappa \log 1/\varkappa$$

for small values of $\varkappa$. It therefore follows that the result in Theorem 1 is not true if $\varkappa \log 1/\varkappa$ is replaced by a function $\varrho(\varkappa)$ such that

$$\lim_{\varkappa \to 0} \varrho(\varkappa)^{-1} \varkappa \log 1/\varkappa = 0 .$$

Theorem 1 is a direct consequence of the following lemma.

LEMMA. *Let $I_0 = [a_0, b_0]$ be a closed interval. Assume that f is a continuous, convex and non-linear function on I_0. For $f'_+(a_0) \leqq \alpha \leqq f'_-(b_0)$ let $a(\alpha)$ be the smallest value of a for which the straight line $y = \alpha x + a$ intersects the curve $y = f(x)$ (f'_+ and f'_- denote the right- and left-hand derivatives of f). Let*

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$k_\varkappa(\alpha)$ denote the arc length of the part of the curve $y=f(x)$ which has distance less than $\varkappa$ to the line $y=\alpha x+a(\alpha)$. Then

$$\liminf_{\varkappa\to 0} \int_{f'_+(a_0)}^{f'_-(b_0)} k_\varkappa(\alpha)\,d\alpha\,(\varkappa\log 1/\varkappa)^{-1} \geqq 1/5 .$$

REMARK. It follows from the lemma that the number zero on the right hand side of (1) can be replaced by a positive number which does not depend on Γ. It also follows from a splitting of the curve Γ in Theorem 1 that if Γ is not piecewise linear then

$$\lim_{\varkappa\to\infty} \int_0^{2\pi} k_\varkappa(\theta)\,d\theta/(\varkappa\log 1/\varkappa)^{-1} = \infty .$$

We also remark that the proof of the lemma shows that $k_\varkappa(\theta)$ in (1) can be replaced by the length of the orthogonal projection of the part of Γ between the lines $l(\theta,c(\theta))$ and $l(\theta,c(\theta)+\varkappa)$ onto the line $l(\theta,c(\theta))$.

PROOF OF THE LEMMA. The lemma is easy to prove in the case when f is piecewise linear and from now on we assume that f is not piecewise linear. For $f'_+(a_0)<\alpha<f'_-(b_0)$ we set

$$\beta(\alpha) = \sup\{x\ ;\ f'(x)<\alpha\} \quad \text{and} \quad \gamma(\alpha) = \inf\{x\ ;\ f'(x)>\alpha\} .$$

Then $\beta(\alpha)=\gamma(\alpha)$ for almost every α. We choose $\psi \in C_0^\infty(\mathbf{R})$ such that $\int \psi\,dx=1$, $\psi\geqq 0$ and set $\psi_n(x)=n\psi(nx)$ for $n=1,2,3,\ldots$. Also choose a and b, $a_0<a<b<b_0$, such that f' exists and is continuous at a and b and such that $f'(a)-f'_+(a_0)>0$ and $f'_-(b_0)-f'(b)>0$. We set $I=[a,b]$ and may also assume that a and b are chosen so that there exist disjoint closed intervals J_1 and $J_2\subset(a,b)$ with

(2) $$f'(J_i) > 0, \quad i=1,2 .$$

We set $f_n=f*\psi_n$ and it follows that $f_n\in C^\infty(I)$ if n is large enough and that each f_n is convex, i.e., $f_n''\geqq 0$. It also follows that f_n tends to f uniformly on every interval $[a_0+\varrho,b_0-\varrho]$, $\varrho>0$, as n tends to infinity and that

$$\lim_{n\to\infty} f_n'(a) = f'(a), \quad \lim_{n\to\infty} f_n'(b) = f'(b) .$$

Then define numbers $\delta_i=\delta_i(\alpha)=\delta_i(\alpha,\varkappa)$, $i=1,2$, for $f'(a)\leqq\alpha\leqq f'(b)$ and $\varkappa\leqq\varkappa_0$ by the equalities

$$f(\gamma+\delta_2) = f(\gamma)+\alpha\delta_2+\varkappa/\cos\theta$$

and

$$f(\beta-\delta_1) = f(\beta)-\alpha\delta_1+\varkappa/\cos\theta ,$$

where $\theta = \arctan \alpha$ and $\gamma = \gamma(\alpha)$, $\beta = \beta(\alpha)$. We assume that $\varkappa_0$ is so small that $\gamma + \delta_2$ and $\beta - \delta_1 \in$ I', where I' is an interval of the form $[a_0 + \varrho, b_0 - \varrho]$ for some fixed $\varrho > 0$. From the convexity of f it follows that δ_1 and δ_2 are uniquely determined. We set $\delta = \delta(\alpha) = \delta(\alpha, \varkappa) = \max(\delta_1, \delta_2)$ and observe that $k_\varkappa(\alpha)$ equals the arc length of the curve $y = f(x)$, $\beta - \delta_1 \leqq x \leqq \gamma + \delta_2$. Hence $k_\varkappa(\alpha) \geqq \delta(\alpha)$. Analogously define β_n, γ_n, $\delta_{1,n}$, $\delta_{2,n}$, δ_n and $k_{\varkappa,n}$ (with f replaced by f_n) for $f'_n(a) \leqq \alpha \leqq f'_n(b)$, $\varkappa \leqq \varkappa_0$ and $n \geqq n_0$. We assume that $\varkappa_0$ and n_0 are chosen so that $\beta_n - \delta_{1,n}$ and $\gamma_n + \delta_{2,n} \in I'$ for all values of α and $\varkappa$. We shall prove that if $\varkappa_0$ is chosen small enough, then

$$\int_{f'_n(a)}^{f'_n(b)} \delta_n(\alpha)\, d\alpha \geqq \tfrac{1}{5}\varkappa \log 1/\varkappa, \qquad n \geqq n_0,\ \varkappa \leqq \varkappa_0\ , \tag{3}$$

and that

$$\delta_n(\alpha) \to \delta(\alpha), \qquad n \to \infty\ , \tag{4}$$

almost everywhere on $(f'(a), f'(b))$. It then follows from Lebesgue's theorem on dominated convergence that

$$\int_{f'(a)}^{f'(b)} \delta(\alpha)\, d\alpha \geqq \tfrac{1}{5}\varkappa \log 1/\varkappa$$

for $\varkappa \leqq \varkappa_0$, which yields the lemma. We first prove (4). Fix α such that $\beta(\alpha) = \gamma(\alpha)$. It is then clear that $\gamma_n \to \gamma$ as $n \to \infty$. We have

$$f_n(\gamma_n + \delta_{2,n}) = f_n(\gamma_n) + \alpha\delta_{2,n} + \varkappa/\cos\theta\ . \tag{5}$$

We assume that $\delta_{2,n} \not\to \delta_2$ and shall prove that this leads to a contradiction. Choosing a convergent subsequence we may assume that $\delta_{2,n} \to \delta^* \neq \delta_2$ as $n \to \infty$. It follows that

$$f_n(\gamma_n + \delta_{2,n}) \to f(\gamma + \delta^*) \quad \text{and} \quad f_n(\gamma_n) \to f(\gamma)$$

as $n \to \infty$. Hence (5) yields

$$f(\gamma + \delta^*) = f(\gamma) + \alpha\delta^* + \varkappa/\cos\theta\ ,$$

which shows that $\delta^* = \delta_2$. This is a contradiction and hence $\lim_{n\to\infty} \delta_{2,n} = \delta_2$. In the same way it can be proved that $\lim_{n\to\infty} \delta_{1,n} = \delta_1$ and therefore (4) is proved.

It remains to prove (3). We fix n and $\varkappa$ and write δ_i and γ instead of $\delta_{i,n}$ and γ_n. We observe that

$$f'_n(b) - f'_n(a) \geqq m > 0 \tag{6}$$

for some constant m. If $f'_n(a) \leqq \alpha \leqq f'_n(b)$ and $\gamma = \gamma(\alpha)$ it is clear that $f'_n(\gamma) = \alpha$. From the formula defining δ_2 it therefore follows that

$$\int_{\gamma}^{\gamma+\delta_2}\left(\int_{\gamma}^{x} f_n''(t)\,dt\right)dx = \varkappa/\cos\theta$$

and hence

$$\delta_2\int_{\gamma}^{\gamma+\delta_2} f_n''(t)\,dt \geqq \varkappa\,.$$

Defining $\varepsilon_2=\varepsilon_2(\alpha)$ by the equality

$$\varepsilon_2\int_{\gamma}^{\gamma+\varepsilon_2} f_n''(x)\,dx = \varkappa$$

we conclude that $\varepsilon_2\leqq\delta_2$. We also define ε_1 by

$$\varepsilon_1\int_{\beta-\varepsilon_1}^{\beta} f_n''(x)\,dx = \varkappa$$

and set $\varepsilon=\varepsilon(\alpha)=\max(\varepsilon_1,\varepsilon_2)$. It follows that $\varepsilon_1\leqq\delta_1$ and $\varepsilon\leqq\delta$. Let $(I_k)_1^\infty$ denote the component intervals of the set $\{x\in I\,;\,f_n''(x)>0\}$. Since $[f_n'(a),f_n'(b)]\setminus\bigcup_1^\infty f_n'(I_k)$ has Lebesgue measure zero we obtain

$$\text{(7)}\qquad \int_{f_n'(a)}^{f_n'(b)}\delta(\alpha)\,d\alpha \geqq \int_{f_n'(a)}^{f_n'(b)}\varepsilon(\alpha)\,d\alpha = \sum_1^\infty\int_{f_n'(I_k)}\varepsilon(\alpha)\,d\alpha\,.$$

From a change of variable $\alpha=f_n'(x)$ it follows that the last integral equals

$$\text{(8)}\qquad \int_{I_k}\varepsilon(f_n'(x))\,f_n''(x)\,dx\,.$$

For $x\in I$ we define $\varepsilon_1(x)$, $\varepsilon_2(x)$ and $\varepsilon(x)$ by setting

$$\varepsilon_1(x)\int_{x-\varepsilon_1(x)}^{x} f_n''(t)\,dt = \varepsilon_2(x)\int_{x}^{x+\varepsilon_2(x)} f_n''(t)\,dt = \varkappa$$

and $\varepsilon(x)=\max(\varepsilon_1(x),\varepsilon_2(x))$. It follows that the integral (8) equals

$$\int_{I_k}\varepsilon(x)\,f_n''(x)\,dx$$

and hence (7) yields

$$\text{(9)}\qquad \int_{f_n'(a)}^{f_n'(b)}\delta(\alpha)\,d\alpha \geqq \int_{I}\varepsilon(x)\,f_n''(x)\,dx\,.$$

We now write φ instead of f_n'' and then have $\varphi\in C^\infty(I')$, $\varphi\geqq 0$ and $\int_I\varphi\,dx\geqq m$. Since φ is continuous ε is continuous on I. We choose $p_0\in I$ such that $\varepsilon(p_0)=\max_{x\in I}\varepsilon(x)$ and let μ be a number satisfying $0<\mu<1$. If $p_0\mp b$ succesively choose points $(p_i)_1^N$ and intervals $(\omega_i)_0^{N-1}$ such that $p_0 < p_1 < p_2 < \ldots < p_N = b$, $\omega_i = [p_i, p_{i+1}]$,

$$\varepsilon(p_i) \int_{\omega_i} \varphi \, dx = \mu\varkappa, \quad i = 0, 1, \ldots, N-2,$$

and

$$\varepsilon(p_{N-1}) \int_{\omega_{N-1}} \varphi \, dx \leqq \mu\varkappa .$$

If $p_0 \neq a$ we also choose points $(p_i')_1^M$ and intervals $(\omega_i')_0^{M-1}$ such that $a = p_M' < p_{M-1}' < \ldots < p_1' < p_0' = p_0$, $\omega_i' = [p_{i+1}', p_i']$,

$$\varepsilon(p_i') \int_{\omega_i'} \varphi \, dx = \mu\varkappa, \quad i = 0, 1, \ldots, M-2,$$

and

$$\varepsilon(p_{M-1}') \int_{\omega_{M-1}'} \varphi \, dx \leqq \mu\varkappa .$$

Since $\varepsilon(x) \leqq C$ it follows that the equality $\varepsilon(p_i) \int_{\omega_i} \varphi \, dx = \mu\varkappa$ implies

$$\int_{\omega_i} \varphi \, dx \geqq \mu\varkappa / C ,$$

which ensures that in the above construction we reach the points b and a in a finite number of steps. Set

$$\mathscr{F} = \{\omega_0, \omega_1, \ldots, \omega_{N-2}, \omega_0', \omega_1', \ldots, \omega_{M-2}'\}$$

and for $\omega \in \mathscr{F}$ set $\varepsilon(\omega) = \varepsilon(p_i)$ if $\omega = \omega_i$ and $\varepsilon(\omega) = \varepsilon(p_i')$ if $\omega = \omega_i'$. Hence

$$\varepsilon(\omega) \int_{\omega} \varphi \, dx = \mu\varkappa, \quad \omega \in \mathscr{F} . \tag{10}$$

From (2) and the definition of $\mathscr{F}$ it is easy to see that there exist constants m_0 and m_1 (independent of n) such that

$$\sum_{\omega \in \mathscr{F}} |\omega| \geqq m_0 > 0 \tag{11}$$

and

$$\sum_{\omega \in \mathscr{F}} \int_{\omega} \varphi \, dx \geqq m_1 > 0 . \tag{12}$$

We shall now prove that

$$\varepsilon(\omega) \geqq |\omega|, \quad \omega \in \mathscr{F} . \tag{13}$$

Assume $\omega=\omega_i$ for some i (the proof is similar if $\omega=\omega_i'$ for some i). Then

$$\varepsilon(\omega) = \varepsilon(p_i) \quad \text{and} \quad \varepsilon(p_i)\int_\omega \varphi\, dx = \mu\varkappa .$$

Assume $\varepsilon(p_i)<|\omega|$. Then $\varepsilon_2(p_i)<|\omega|$ and hence $p_i+\varepsilon_2(p_i) \in \omega$. It follows that

$$\varkappa = \varepsilon_2(p_i)\int_{p_i}^{p_i+\varepsilon_2(p_i)} \varphi\, dx \leqq \varepsilon(p_i)\int_\omega \varphi\, dx = \mu\varkappa ,$$

which gives a contradiction and proves (13).

We now prove that if $x \in \omega \in \mathscr{F}$, then

$$\varepsilon(x) \geqq \varepsilon(\omega)/3 . \tag{14}$$

We may assume that $\omega=\omega_i$ for some i. Assume (14) is not true, i.e. there exists $x \in \omega$ such that $\varepsilon(x)<\varepsilon(p_i)/3$ and hence $\varepsilon_j(x)<\varepsilon(p_i)/3, j=1,2$. We consider two cases.

CASE 1. One of the points $x\pm\varepsilon(p_i)/3 \in \omega$.

Assume for instance that $x+\varepsilon(p_i)/3 \in \omega$ (the proof is similar if $x-\varepsilon(p_i)/3 \in \omega$). It follows that $x+\varepsilon_2(x) \in \omega$ and hence we have

$$\varkappa = \varepsilon_2(x)\int_x^{x+\varepsilon_2(x)} \varphi\, dt \leqq 3^{-1}\varepsilon(p_i)\int_\omega \varphi\, dt = \mu\varkappa/3 ,$$

which gives a contradiction.

CASE 2. $x\pm\varepsilon(p_i)/3 \notin \omega$.

It follows that $\varepsilon(p_i)/3>|\omega|/2$ and hence

$$\varepsilon(p_i) > 3|\omega|/2 . \tag{15}$$

We first treat the case $\varepsilon(p_i)=\varepsilon_2(p_i)$. (15) implies that $[x,x+\varepsilon(p_i)/3]\subset[p_i, p_i+\varepsilon(p_i)]$ and it follows that

$$\int_x^{x+\varepsilon(p_i)/3} \varphi\, dt \leqq \int_{p_i}^{p_i+\varepsilon(p_i)} \varphi\, dt .$$

We conclude that

$$\tfrac{1}{3}\varepsilon(p_i)\int_x^{x+\varepsilon(p_i)/3} \varphi\, dt \leqq \tfrac{1}{3}\varepsilon(p_i)\int_{p_i}^{p_i+\varepsilon(p_i)} \varphi\, dt = \varkappa/3 .$$

Since $\varepsilon_2(x)<\varepsilon(p_i)/3$ it follows from the definition of ε_2 that the left hand side above is larger than $\varkappa$, which leads to a contradiction. We then treat the case $\varepsilon(p_i)=\varepsilon_1(p_i)$. It is clear that $p_i-\varepsilon(p_i)<x-\varepsilon(p_i)/3<p_i<x$ and we obtain

$$(16) \qquad \varkappa = \varepsilon(p_i)\int_{p_i-\varepsilon(p_i)}^{p_i} \varphi\,dt \geqq \varepsilon(p_i)\int_{x-\varepsilon(p_i)/3}^{p_i} \varphi\,dt$$

$$= \varepsilon(p_i)\int_{x-\varepsilon(p_i)/3}^{x} \varphi\,dt - \varepsilon(p_i)\int_{p_i}^{x} \varphi\,dt\,.$$

From the inequality $\varepsilon_1(x)<\varepsilon(p_i)/3$ it follows that

$$\tfrac{1}{3}\varepsilon(p_i)\int_{x-\varepsilon(p_i)/3}^{x} \varphi\,dt > \varkappa$$

and we also have

$$\varepsilon(p_i)\int_{p_i}^{x} \varphi\,dt \leqq \varepsilon(p_i)\int_{\omega} \varphi\,dt = \mu\varkappa\,.$$

Inserting these estimates in (16) we obtain

$$\varkappa \geqq 3\varkappa - \mu\varkappa \geqq 2\varkappa\,,$$

which is a contradiction and completes the proof of (14).

We need some more notation. Let d denote the number of elements in $\mathscr{F}$, d_j the number of intervals $\omega \in \mathscr{F}$ such that $2^{-j-1} \leqq \varepsilon(\omega) < 2^{-j}, j \in \mathbf{Z}$. Let j_1 be the smallest and j_0 the largest value of j for which $d_j \geqq 1$. Hence

$$d = \sum_{j=j_1}^{j_0} d_j\,.$$

It follows from (14) that

$$\varepsilon(p_{i+1}) \geqq \varepsilon(p_i)/3 \quad \text{and} \quad \varepsilon(p'_{i+1}) \geqq \varepsilon(p'_i)/3\,.$$

Since $\varepsilon(p_0)=\max_{x\in I}\varepsilon(x)$ it therefore follows from the construction of the set $\mathscr{F}$ that the number of values of j for which $d_j \geqq 1$ is not less than $(j_0-j_1)/2$ (in fact, if $2^{-j-1} \leqq \varepsilon(p_i) < 2^{-j}$ and $2^{-k-1} \leqq \varepsilon(p_{i+1}) < 2^{-k}$, then $2^{-k} > 3^{-1}2^{-j-1}$ and hence $k \leqq j+2$). We therefore have

$$(17) \qquad j_0 - j_1 \leqq 2d\,.$$

It follows from (14) and the definition of $\varepsilon(\omega)$ that

$$(18) \qquad \int_I \varepsilon\varphi\,dx \geqq \sum_{\omega\in\mathscr{F}} \int_{\omega} \varepsilon\varphi\,dx$$

$$\geqq \sum_{\omega} \tfrac{1}{3}\varepsilon(\omega)\int_{\omega} \varphi\,dx = \tfrac{1}{3}\sum_{\omega} \mu\varkappa$$

$$= \tfrac{1}{3}\mu\varkappa d\,.$$

Using (11) and (13) we also get

$$m_0 \leqq \sum_{\omega \in \mathscr{F}} |\omega| \leqq \sum_{\omega} \varepsilon(\omega) \leqq d2^{-j_1}\,. \tag{19}$$

For $\omega \in \mathscr{F}$ we have $\int_\omega \varphi dx = \mu\varkappa/\varepsilon(\omega)$ and hence, using (12), we obtain

$$\begin{aligned} m_1 &\leqq \sum_{\omega \in \mathscr{F}} \int_\omega \varphi\, dx = \sum_{\omega} \mu\varkappa\varepsilon(\omega)^{-1} \\ &\leqq \mu\varkappa d 2^{j_0+1} = 2\mu\varkappa d 2^{j_0}\,. \end{aligned} \tag{20}$$

(19) and (20) yield

$$m_0 m_1 \leqq 2\mu\varkappa d^2 2^{j_0 - j_1}$$

and invoking (17) we get $m_0 m_1 \leqq 2\mu\varkappa d^2 2^{2d}$. It follows that

$$C_1 + \log 1/\varkappa \leqq 2\log d + 2d\log 2$$

for some constant C_1. Since $2\log 2 < \frac{3}{2}$ it follows that $\log 1/\varkappa \leqq \frac{3}{2}d$ if $\varkappa \leqq \varkappa_0$ and $\varkappa_0$ is small enough. The lemma now follows from a combination of this estimate and (18) if we choose μ close to 1.

The proof of Theorem 1 is complete.

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