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ON THE RADIAL BOUNDARY VALUES OF SUBHARMONIC FUNCTIONS

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1. Introduction.

The maximum principle says that if u is subharmonic in the unit disc D and if

$$\limsup_{z \to w} u(z) \leqq 0 \quad \text{for all } w \in \mathsf{T} ,$$

where $\mathsf{T}=\{z : |z|=1\}$, then $u \leqq 0$. In applications it is sometimes desirable to relax this condition to the weaker assumption

$$\limsup_{r \to 1} u(rw) \leqq 0 \quad \text{for all } w \in \mathsf{T} \tag{1.1}$$

and still get the conclusion $u \leqq 0$. As the harmonic function $v(re^{i\theta}) = \sum_1^\infty nr^n \sin n\theta$ shows, for which $\lim_{r \to 1} v(rw)=0$ for all $w \in \mathsf{T}$, condition (1.1) alone is not sufficient to give that $u \leqq 0$. The object of this paper is to discuss the kind of restrictions of the growth which together with (1.1) imply that $u \leqq 0$.

We shall use the following notation. If u is subharmonic in D we put

$$u^*(w) = \limsup_{r \to 1} u(rw), \qquad u_*(w) = \liminf_{r \to 1} u(rw)$$

and

$$M(r,u) = \max \{u(rw) : w \in \mathsf{T}\} .$$

We let

$$P(r,\theta) = (2\pi)^{-1} + \pi^{-1} \sum_1^\infty r^n \cos n\theta$$

denote the Poisson kernel. If $f \in L^1(\mathsf{T})$ we put

$$Pf(re^{i\theta}) = \int_0^{2\pi} f(e^{i\varphi}) P(r, \theta - \varphi)\, d\varphi .$$

We start by discussing the following special case of our main result.

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THEOREM 1. *Let u be subharmonic in D and suppose that*

$$u^*(w) < \infty \text{ for all } w \in \mathsf{T}, \tag{1.2}$$

$$\text{there is a } g \in L^1(\mathsf{T}) \text{ such that } u_* \leqq g \text{ a.e. on } \mathsf{T}, \tag{1.3}$$

$$M(r,u) = o[(1-r)^{-2}] \text{ as } r \to 1. \tag{1.4}$$

Then $u^* \in L^1(\mathsf{T})$ and $u_* = u^*$ a.e. on T. In addition $u \leqq P(u^*)$.

Considering the function $u(re^{i\theta}) = \sum_1^\infty nr^n \sin n\theta$ again, for which $M(r,|n|) = O[(1-r)^{-2}]$ we see that condition (1.4) can not be weakened to $O[(1-r)^{-2}]$.

Professor H. S. Shapiro has asked the following question: Let f be analytic in D and assume $\lim_{r\to 1} f(rw) = g(w)$ exists for all $w \in \mathsf{T}$ and $g \in L^1(\mathsf{T})$. What growth conditions should one impose on f in order to deduce that $f \in H^1$? In this direction we have the following consequence of Theorem 1.

COROLLARY 1. *Let f be analytic in D. Suppose that* $\limsup_{r\to 1} |f(rw)| < \infty$ *for all $w \in \mathsf{T}$ and there exists a $g \in L^1(\mathsf{T})$ such that* $\lim_{r\to 1} f(rw) = g(w)$ a.e. *If* $\log M(r,|f|) = o[(1-r)^{-2}]$ *as $r \to 1$ then $f \in H^1$.*

PROOF. Putting $u = \log|f|$ we have that $u^*(w) < \infty$ for all $w \in \mathsf{T}$ and $u_*(w) \leqq \log^+|g| \in L^1(\mathsf{T})$. Since u is subharmonic in D and satisfies (1.4) it follows from Theorem 1 that $\log|g| \in L^1(\mathsf{T})$ and $u \leqq P(\log|g|)$. It follows from Jensen's inequality that $|f| = \exp u \leqq P|g|$. Hence $f \in H^1$ which proves the Corollary.

We remark that we cannot weaken the growth condition of the Corollary. For the function

$$f(z) = \exp(iz(1-z)^{-2})$$

satisfies $\limsup_{r\to 1} |f(rw)| = 1$ for all $w \in \mathsf{T}$, $\lim_{r\to 1} f(rw) = g(w)$ exist a.e. on T and $g \in L^\infty(\mathsf{T})$. In addition $\log M(r,|f|) = O[(1-r)^{-2}]$ as $r \to 1$ but $f \notin H^1$.

Diederich [3] has considered a variant of Theorem 1 where radial boundary values are replaced by certain averages.

We shall use a growth condition, which is more general than (1.4). Let $\varrho > 0$, $w \in \mathsf{T}$ and put $\sigma(z,w) = (1-|z|)|z-w|^{-1}$. Suppose u is subharmonic in D. We say that *u is of type $G(w,\varrho)$* if there are functions a and b, both non-increasing and non-negative such that $a(t) = o(1)$ as $t \to 0$,

$$\int_0^1 \log(b(t)+1)\,dt < \infty$$

and

$$u(z) \leqq a(|z-w|)\,b(\sigma(z,w))|z-w|^{-\varrho} . \tag{1.5}$$

We have the following relation between conditions of the form (1.4) and (1.5).

PROPOSITION 1. *Let u be subharmonic in D and suppose $M(r,u)=o[(1-r)^{-\varrho}]$ for some $\varrho>0$. Then u is of type $G(w,\varrho)$ for all $w \in \mathsf{T}$.*

PROOF. There is no loss in generality by assuming $M(r,u)\geqq 0$ for $0\leqq r<1$. Put

$$a(t)=\sup\{s^{\varrho}M(1-s,u) \;:\; 0<s\leqq t\} .$$

Then $a(t)=o(1)$ as $t \to 0$. If $w \in \mathsf{T}$ and $z \in D$ then

$$\begin{aligned} u(z) \leqq M(|z|,u) &= M(1-|z-w|\sigma(z,w)) \\ &\leqq (\sigma(z,w))^{-\varrho}|z-w|^{-\varrho}a(|z-w|) . \end{aligned}$$

Choosing $b(t)=t^{-\varrho}$ we see that condition (1.5) is satisfied. Since w was arbitrary the proposition follows.

We can now formulate our "radial" maximum principle.

THEOREM 2. *Let u be subharmonic in D and let $E\subset\mathsf{T}$ be countable. Suppose*

(1.6) *for all* $w \in \mathsf{T}$ *u is of type* $G(w,2)$,

(1.7) *there is a* $g \in L^1(\mathsf{T})$ *such that* $u_* \leqq g$ *a.e.*

(1.8) $u^*(w) < \infty$ *for* $w \in \mathsf{T}-E$,

(1.9) $u^+(rw) = o[(1-r)^{-1}]$ *for* $w \in E$.

Then $u^* \in L^1(\mathsf{T})$ and $u_*=u^*$ a.e. In addition we have $u\leqq Pu_*$.

It follows from the example given in [13, p. 640] that E can not be taken to be an uncountable Borelset if the other assumptions are unchanged. But if we make further restrictions of the growth we can allow larger exceptional sets E, see Theorem 4.

The question when a harmonic function in D is determined by its radial limits has been extensively studied. For the most general results we refer to Wolf [16]. However, using Theorem 2 we get the following new result on this question.

THEOREM 3. *Let u be a realvalued harmonic function in D and let $E\subset\mathsf{T}$ be countable. Suppose u satisfies* (1.6) *and* (1.7). *If in addition*

(1.10) $-\infty < u^*(w) < \infty$ *for* $w \in E$

(1.11) $|u(rw)| = o[(1-r)^{-1}]$ *for* $w \in \mathsf{T}$

then $u^* \in L^1(\mathsf{T})$, $u_* = u^*$ *a.e. and* $u = Pu_*$.

We would like to point out that Theorem 3 neither implies nor is implied by the results of [16]. However, if we assume that $M(r,u) = O[(1-r)^{-2}]$ we get overlap with the results in [16]. The special cases when $M(r,|u|) = o[(1-r)^{-2}]$ and $M(r,u) = O[(1-r)^{2-\varepsilon}]$ have been treated by Shapiro [13, 14] with other methods.

For the case when the exceptional set is no longer assumed to be countable we have the following result.

THEOREM 4. *Let u be subharmonic in D and let* $0 < \alpha < 1$. *Let* $E \subset \mathsf{T}$ *be the countable union of closed sets of finite α-dimensional Hausdorff measure. Suppose*

(1.12) $M(r,u) = o[(1-r)^{-1+\alpha}]$ *as* $r \to 1$,

(1.13) $u^*(w) < \infty$ *for* $w \in \mathsf{T} - E$,

(1.14) *there is a* $g \in L^1(\mathsf{T})$ *such that* $u_* \leqq g$ *a.e.*.

Then $u^* \in L^1$ *and* $u_* = u^*$ *a.e. In addition* $u \leqq Pu_*$.

Theorem 3 is sharp for if $E \subset \mathsf{T}$ is a closed set of positive α-dimensional Hausdorff measure then there exists by [2, p. 7] a probability measure μ concentrated on E such that $\mu\{\zeta : |\zeta - z| < r\} \leqq Cr^\alpha$ for all z. If $v = P\mu$, then an integration by parts shows that $M(r,v) = O[(1-r)^{-1+\alpha}]$ as $r \to 1$ and hence condition (1.10) can not be relaxed to $O[(1-r)^{-1+\alpha}]$ as $r \to 1$.

As an application of Theorem 2 we have the following result on the "pointwise" normal derivatives.

THEOREM 5. *Let u be harmonic in D and let* $E \subset \mathsf{T}$ *be countable. Suppose*

(1.15) *u is of type* $G(w,1)$ *for all* $w \in \mathsf{T}$,

(1.16) *there is a realvalued function f on* T *such that*

$$\limsup_{r \to 1} |f(w) - u(rw)|(1-r)^{-1} < \infty \quad \text{for } w \in \mathsf{T} - E,$$

(1.17) *there is a* $g \in L^1(\mathsf{T})$ *such that*

$$\lim_{r \to 1} (f(w) - u(rw))(1-r)^{-1} = g(w) \text{ a.e.},$$

(1.18) $u(rw) = o[\log 1 - r)]$ *as* $r \to 1$ *for* $w \in E$.

Then $f \in L^1(\mathsf{T})$, $u = Pf$ *and* $r\,\partial u/\partial r = Pg$.

This theorem generalizes the pointwise saturation theorem of Berens [1] and Hedberg [7]. Their result is about the case when f is assumed to be in $L^1(\mathrm{T})$ and $u=Pf$. We observe that in this case $M(r,|u|)=o[(1-r)^{-1}]$ and hence we have from proposition 1 that u satisfies condition (1.15). Notice that we don't assume f to be integrable.

2. The refined maximum principle.

We start with the following estimate for the growth of subharmonic functions.

LEMMA 1. *Let* $B(\varepsilon)=\{z : |z-1|<\varepsilon, \operatorname{Im} z>0\}$ *and let* u *be subharmonic in* $\Omega=D\cup B(\varepsilon)$. *Suppose* u *is of type* $G(1,\varrho)$ *for some* $\varrho>0$ *and* $u^+(z)\leqq u^+(z^*)$ *for* $z\in\Omega-D$, *where* $z^*=z|z|^{-2}$. *Then*

$$\sup\{u^+(z) : z\in\Omega, \operatorname{Im} z>0, |z-1|=r\} = o(r^{-\varrho}) \quad \textit{as } r\to 0 . \tag{2.1}$$

PROOF. Let $\sigma(z)=(1-|z|)|z-1|^{-1}$ and put

$$D_r = \{z : \operatorname{Im} z>0, 2^{-1}r<|z-1|<2r, |\sigma(z)|<\tfrac{1}{2}\} .$$

It is easily seen there is a constant $C>1$ such that if $0<|z-1|<\frac{1}{2}$ then

$$|z-1| \leqq C|z^*-1| \quad \text{and} \quad |\sigma(z)| \leqq C|\sigma(z^*)| . \tag{2.2}$$

From (1.5) and (2.2) we now have if $z\in\Omega-D$ and $|z-1|<\frac{1}{2}$

$$\begin{aligned} u^+(z) \leqq u^+(z^*) &\leqq a(|z^*-1|)\, b(\sigma(z^*))|z^*-1|^{-\varrho} \\ &\leqq C^\varrho a(C^{-1}|z-1)\, b(C^{-1}|\sigma(z)|)|z-1|^{-\varrho} . \end{aligned}$$

Putting $a_1(t)=C^\varrho a(C^{-1}t)$ and $b_1(t)=b(C^{-1}t)$ we therefore have

$$u^+(z) \leqq a_1(|z-1|)\, b_1(|\sigma(z)|) \quad \text{for } z\in\Omega\cap B(\tfrac{1}{2}) . \tag{2.3}$$

Let $d(r)=\sup\{a_1(z) : \frac{1}{2}r\leqq t\leqq 2r\}$ and put $v_r(z)=(d(r))^{-1}u^+(z)$ for $z\in D_r$. If

$$F_r(z)=\sigma(z)+i2r^{-1}|z-1|$$

then F_r is a diffeomorphism of D_r onto

$$R = \{x+iy : |x|<\tfrac{1}{2} \text{ and } 1<y<4\} .$$

Letting $u_r=v_r\circ F_r^{-1}$ we claim there is a number C independent of r such that

$$u_r(z) \leqq Cs^{-2}\int_{B(z,s)} u_r(\xi,\eta)\, d\xi\, d\eta, \quad z\in R , \tag{2.4}$$

whenever $B(z,s)=\{\zeta : |\zeta-z|<s\}$ is contained in R. To prove (2.4) we first observe

$$|\operatorname{grad} F_r(z)| \leqq Cr^{-1} \quad \text{for } z \in D_r\,, \tag{2.5}$$

and if $J_r(z)$ denotes the Jacobian of the mapping F_r then

$$|J_r(z)| = r^{-1}|z-1|^{-2}|z|^{-1}y, \quad z \in D_r\,.$$

If $z=x+iy \in D_r$ then $|\sigma(z)|<\frac{1}{2}$ and multiplying both sides of this inequality with $|z-1|$ we have

$$|2(1-x)|z-1|^{-1}-|z-1|| < \tfrac{1}{2}(|z|+1)\,.$$

Hence if r is small enough $|x-1|\leqq c|z-1|$ where $c<1$. Consequently $|J_r(z)| \geqq cr^{-2}$ for $z \in D_r$ and some number c independent of r. To prove (2.4) pick $z' \in D_r$ such that $F_r(z')=z$, and if c is small enough we have from (2.5) that $F_r(B(z',crs) \subset B(z,s)$. This gives

$$\begin{aligned}\int_{B(z,s)} u_r(\xi,\eta)\,d\xi\,d\eta &\geqq \int_{B(z',crs)} v_r(\xi,\eta)|J_r(\xi+i\eta)|\,d\xi\,d\eta \\ &\geqq cr^{-2}\int_{B(z',crs)} v_r(\xi,\eta)\,d\xi\,d\eta\,.\end{aligned}$$

Since v_r is subharmonic we therefore have $u_r(z)\geqq cs^2 v_r(z')=cs^2 u_r(z)$. It follows from (2.3) that $u_r(\xi,\eta)\leqq b_1(|\xi|)$ and from [4, Theorem 3] we have $u_r(\xi,2)\leqq C$ for $|\xi|\leqq\frac{1}{3}$ with C independent of r. This means

$$\begin{aligned}&\sup\{u^+(z) : |z-1|=r, \operatorname{Im} z>0\} \\ &\quad\leqq Cd(r)+\sup\{u^+(z) : |z-1|=r, \operatorname{Im} z>0, 1>|\sigma(z)|\geqq\tfrac{1}{3}\}\,.\end{aligned}$$

Since the both last terms are $o(r^{-\varrho})$ as $r \to 0$ the conclusion of the lemma follows.

PROOF OF THEOREM 2. Let u fulfil the assumptions of Theorem 2. If we put $f=(u^+)_*$ then from (1.7) $f \in L^1(\mathsf{T})$. Let $v=u^+-Pf$ and define

$$\Omega = \left\{w \in \mathsf{T} : \limsup_{z\to w} v(z)\leqq 0\right\}.$$

We first note that Ω is open in T. To this end we use the following fact: If a function u is subharmonic in D and bounded from above in

$$S(I) = \{rw : 0<r<1, w \in I\}\,,$$

where $I\subset\mathsf{T}$ is an open arc, then the condition $u_*(w)\leqq 0$ a.e. in I implies

$$\limsup_{z\to w} u(z) \leqq 0 \quad \text{for all } w \in \mathsf{T} \tag{2.6}$$

Let h be the harmonic function in $S(I)$ with boundary values zero on $\bar{I}$ and boundary values $u^+(z)$ for $z \in \partial S(I)-\bar{I}$. Define $v(z)=(u(z)-h(z))$ for $z \in S(I)$ and zero otherwise. Then v is bounded and subharmonic in D. For some $F \in L^\infty(\mathrm{T})$ PF is the least harmonic majorant of v in D. Littlewoods theorem [15, p. 172] gives $F \leqq 0$ a.e. in I. Therefore $\lim_{z\to w} v(z)=0$ for all $w \in I$, which gives (2.6).

From (2.6) follows now that Ω is open since $v_* \leqq 0$ a.e. in T. Define $R=\mathrm{T}-\Omega$. We want to show $R=\emptyset$. We assume now $R \neq \emptyset$. Let $E=\{e_j\}$ and let

$$F_j = \{w \in \mathrm{T} : v(rw) \leqq j \text{ for } 0<r<1\} .$$

We claim F_j is closed for all j. Let w_0 be a limit point of F_j and let

$$S=\{rw_0 : 0<r<1\} .$$

We notice that it follows from the Wiener criterion [8, p. 220] that F_j is not thin [8, p. 209] at any point of S. Hence $v(z) \leqq j$ for $z \in S$ and therefore F_j is closed. It follows from (1.8) and (1.9) that $\mathrm{T}=[\bigcup_{j=1}^\infty F_j] \cup E$. From the Baire category theorem follows the existence of an open arc I and an integer j such that $I \cap R \neq \emptyset$ and $I \cap R \subset \{e_j\}$ or $I \cap R \subset F_j$. We will now show that in each case there is a contradiction.

Let $I \cap R \subset \{e_j\}$. Pick an open arc J such that $e_j \subset J \subset \bar{J} \subset I$ and let the endpoints of J be b_1, b_2. Let

$$P_j(z) = \varepsilon \operatorname{Re}[(e_j+z)(e_j-z)^{-1}]$$

where $\varepsilon>0$. Then there is a number $M>0$ such that $v(re_j) \leqq P_j(re_j)+M$ and $v(rb_k) \leqq M$ for $k=1,2$, and $0<r<1$. If we define $h(z)$ as $(v(z)-M-P_j(z))^+$ in $S(J)$ and zero otherwise then h is subharmonic in D. From Lemma 1 and (1.6) follows

$$\sup\{h^+(z) : |z-e_j|=r\}=o(r^{-2}) \quad \text{as } r \to 0 .$$

Pick a point $e \neq e_j$ in T and let J_1 and J_2 be the two arcs in T with endpoints e and e_j. Mapping $S(J_1)$ and $S(J_2)$ respectively on the upper halfspace such that e_j corresponds to ∞ then we find from the Phragmén–Lindelöf Theorem [5, p. 104] that $h \leqq 0$. Put

$$m(r) = \max\{v^+(z) : |z-e_j|=r,\ z \in D\} .$$

Then $\limsup_{r\to 0} rm(r) \leqq C\varepsilon$. Since ε was arbitrary this gives $m(r)=o(r^{-1})$ as $r \to 0$ and a Phragmén–Lindelöf argument now gives $\limsup_{z\to e_j} v(z) \leqq 0$ that is, $e_j \in \Omega$ which is a contradiction.

Let $I \cap R \subset F_j$ and $I \cap R = I$. This means v is bounded from above in $S(I)$ and from (2.6) we have that this is a contradiction.

Let $I \cap R \subset F_j$ and $I \cap R \neq I$. Without loss of generality we may assume the endpoints of I are in Ω — otherwise we shrink I. We can write $I - I \cap R$ as a union of at most countably many pairwise disjoint open arcs I_n. Our assumptions now imply the existence of a number $M \geqq j$ such that $u(rw) \leqq M$ whenever $0 < r < 1$ and $w \in F_j$ or w is an endpoint of some I_n. From Lemma 1 follows

$$\sup\{u^+(z) : |z-a_n| = r \text{ or } |z-b_n| = r\} = o(r^{-2}) \quad \text{as } r \to 0 .$$

A Phragmén–Lindelöf argument gives now $u \leqq M$ in $S(I_n)$. Hence $u \leqq M$ in $S(I)$ which in view of (2.6) is a contradiction.

We have now proved $\Omega = \mathsf{T}$, that is, $u^+ \leqq Pf$. Hence there is a measure with nonpositive singular part such that $P\mu$ is the least harmonic majorant of u in D. The Littlewood theorem gives now $u_* = u^*$ a.e. in T and $d\mu = u_* dw + \mu_s$ and consequently $u \leqq Pu_*$. Theorem 2 is proved.

We shall now prove Theorem 3.

Proof of Theorem 3. From the proof of Theorem 2 we know $u = Pu_* - P\lambda$ where λ is a nonnegative singular measure. It is sufficient to show $\lambda = 0$. Let $I(w,r)$ be the open arc on T with center w and length $2r$. Putting $dm = (|u_*|+1)dw$ one finds in the same way as [10, p. 159] that

$$\lim_{r \to 0} \lambda(I(w,r))/m(I(w,r)) = \infty \quad \text{a.e. } [\lambda] .$$

From [10, p. 226] follows

$$\liminf_{r \to 1} (-u(rw)) \geqq \liminf_{r \to 0} (2r)^{-1}[\lambda(I(w,r) - m(I(w,r))]$$

and consequently $u^*(w) = -\infty$ a.e. $[\lambda]$. Now (2.7) gives that λ is concentrated on the countable set E and (2.8) gives $\lambda = 0$. The Theorem is proved.

3. Exceptional sets.

Theorem 4 will be a consequence of the following lemma.

Lemma 2. *Suppose u is subharmonic in D, $0 < \alpha < 1$ and $E \subset \mathsf{T}$ is a closed set of finite α-dimensional Hausdorff measure. If* $\limsup_{z \to w} u(z) \leqq 0$ *for $w \in \mathsf{T} - E$ and*

$$M(r,u) = o[(1-r)^{\alpha-1}] \quad \textit{as } r \to 1$$

then $u \leqq 0$.

PROOF. Let L be the class nonnegative subharmonic functions in D vanishing in a neighbourhood of the origin. For $v \in L$ we define

$$H_1 v(rw) = \int_0^r t^{-1} v(rw)\, dt, \qquad 0 \leq r < 1,\ w \in \mathsf{T},$$

and $Hv = H_1(H_1 v)$. If $v \in L$ and $v \in C^2(D)$ then

$$\Delta Hv(rw) = r^{-2} \int_0^r t^{-1} \left(\int_0^t s \Delta v(sw)\, ds \right) dt .$$

If $\varphi \in C^\infty(\mathsf{T})$ we therefore have

$$\int_{\mathsf{T}} Hv(rw)\, \varphi''(w)\, dw = r^2 \int_{\mathsf{T}} \Delta Hv(rw)\, \varphi(w)\, dw - \int_{\mathsf{T}} r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} Hv(rw) \right) \varphi(w)\, dw$$

$$= \int_{|\zeta| < r} \Delta v(\zeta)\, \varphi(\zeta|\zeta|^{-1}) \log (r|\zeta|^{-1})\, d\xi\, d\eta - \int_{\mathsf{T}} v(rw)\, \varphi(w)\, dw ,$$

where $\zeta = \xi + i\eta$. If $v \in L$ and is not assumed to be in $C^2(D)$ we can by [8, p. 114] find a sequence $\{v_n\}_1^\infty$ of twice continuously differentiable subharmonic functions such that $v_n \downarrow v$ and $\Delta v_n dxdy$ tends weakly to the Riesz measure μ associated to v. Hence we have

(3.1)
$$\int_{\mathsf{T}} Hv(rw)\, \varphi''(w)\, dw = \int_{|\zeta| < r} \log (r|\zeta|^{-1})\, \varphi(\zeta|\zeta|^{-1})\, d\mu(\zeta) - \int_{\mathsf{T}} v(rw)\, \varphi(w)\, dw ,$$

where μ is the Riesz measure associated to v.

Let

$$L_\alpha = \{ v \in L :\ M(r,v) = o[(1-r)^{-1+\alpha}] \ \text{as}\ r \to 1 \} ,$$

where $0 < \alpha < 1$. Since $H_1 v \in L^\infty(D)$ if $v \in L_\alpha$ it follows

$$\lim_{r \to 1} Hv(rw) = Kv(w)$$

exists for all $w \in \mathsf{T}$ and

$$\sup_{w \in \mathsf{T}} |Kv(w) - Hv(rw)| = O(1-r) \quad \text{as}\ r \to 1 . \tag{3.2}$$

Hence Kv is upper semicontinuous if $v \in L_\alpha$.

Let u fulfil the assumptions of Lemma 2. Since $E \neq \mathsf{T}$ there is a point $w_0 \in \mathsf{T} - E$. Therefore there is an open arc I and a number M such that $w_0 \in I$ and $u(rw) \leq M$ if $0 < r < 1$ and $w \in I$ and $u(z) \leq M$ if $|z| \leq \frac{1}{2}$. Define $v = (u - M)^+$. Then $v \in L_\alpha$ and

$$Kv(w) = 0 \quad \text{for}\ w \in I . \tag{3.3}$$

If $\varphi \in C^\infty(\mathsf{T})$, $\varphi \geqq 0$ and the support of φ lies in $\mathsf{T}-E$ then we get from (3.1) and (3.2):

$$\int_{\mathsf{T}} Kv(w)\,\varphi''(w)\,dw = \lim_{r\to 1} \int_{\mathsf{T}} Kv(rw)\,\varphi''(w)\,dw$$

$$\geqq \limsup_{r\to 1} -\int_{\mathsf{T}} v(rw)\,\varphi(w)\,dw = 0$$

and therefore Kv is convex in $\mathsf{T}-E$. We will show that Kv is convex in T.

There is a constant C such that for all $\varepsilon > 0$ there are finitely many open arcs $I_j \subset \mathsf{T}$, $j=1,\ldots,N$ with length $\varepsilon_j < \varepsilon$, $\bigcup_{j=1}^N I_j \supset E$ and $\sum_{j=1}^N \varepsilon_j^\alpha \leqq C$. By [6, p. 43] there are $\varphi_j \in C^\infty(\mathsf{T})$, $\varphi_j \geqq 0$ and the support of φ_j is in I_j^*, where I_j^* is the open arc with the same center as I_j and the length of I_j^* is $2\varepsilon_j$, and

$$0 \leqq \varphi_j \leqq 1, \quad \sum_{j=1}^N \varphi_j = 1 \quad \text{on } E\,,$$

$$\sup_w |\varphi_j^{(k)}(w)| \leqq C_k \varepsilon_j^{-k}\,.$$

Let $\varphi \in C^\infty(\mathsf{T})$ and $\varphi \geqq 0$. Since Kv is convex in $\mathsf{T}-E$ we have

$$\int_{\mathsf{T}} Kv\varphi''\,dw \geqq \sum_{j=1}^N \int_{\mathsf{T}} Kvg_j''\,dw$$

where $g_j = \varphi_j\varphi$. We notice

$$\sup\{|g_j^{(k)}(w)| : w \in \mathsf{T}\} \leqq C\varepsilon_j^{-k}, \quad 0 \leqq k \leqq 2\,,$$

where C only depends on φ. Since $v \in L_\alpha$ an integration by parts shows

$$\sup_{w\in\mathsf{T}} |Kv(w) - 2Hv(rw) + Hv(2r-1)w)| = A(1-r)(1-r)^{1+\alpha}$$

where $A(t)$ is an increasing function with $\lim_{t\to 0} A(t) = 0$. Hence

$$\int_{\mathsf{T}} Kvg_j''\,dw = \int_{\mathsf{T}} \big(2Hv(r_jw) - H\big((2r_j-1)w\big)\big)\,g_j''\,dw + R_j$$

where $|R_j| \leqq CA(\varepsilon_j)\varepsilon_j^\alpha$. Let S_j denote the integral on the right hand side. From (3.1) we find

$$S_j \geqq -2\int_{\mathsf{T}} v(r_jw)\,g_j(w)\,dw \geqq -CM(1-\varepsilon_j, v)\,\varepsilon_j\,.$$

Since E has finite α-dimensional Hausdorff-measure it now follows $\int Kv\varphi''\,dw \geqq o(1)$ as $\varepsilon \to 0$ and therefore Kv is convex and hence constant. From (3.3) follows $Kv = 0$ and hence $v = 0$, that is, $u \leqq M$. Using (2.6) we have $u \leqq 0$ and the lemma is proved.

PROOF OF THEOREM 4. Let u fulfil the assumptions of Theorem 4 and let

$$\Omega = \left\{ w \in \mathsf{T} : \limsup_{z \to w} (u(z) - Pg^+(z)) \leqq 0 \right\}.$$

Put $R = \mathsf{T} - \Omega$, and assume $R \neq \emptyset$. Assume $E = \bigcup_j E_j$, where E_j is a closed set of finite α-dimensional Hausdorff measure. Arguing as in the proof of Theorem 2 there is an open arc $I \subset \mathsf{T}$ and an integer j such that $I \cap R \neq \emptyset$ and $I \cap R \subset E_j$. We may assume the endpoints of I are in Ω, otherwise we shrink I. Therefore there is a number M such that $u(z) - Pg^+(z) \leqq M$ when $z \in \partial S(I) - I$. Let $v(z) = (u(z) - Pg^+(z) - M)^+$ when $z \in S(I)$ and zero otherwise. Then v is subharmonic in D and $\lim_{z \to w} v(z) = 0$ for $z \in \mathsf{T} - E_j$. Lemma 2 gives $v \leqq 0$, hence $u - Pg^+$ is bounded from above in $S(I)$. From (2.6) follows $I \subset \Omega$, which is a contradiction. Therefore $\Omega = \mathsf{T}$ and the conclusion follows now from Littlewood's Theorem.

4. Pointwise normal derivatives.

We will deduce Theorem 5 from the following lemma.

LEMMA 3. *Suppose u is harmonic in D and $I_0 \subset \mathsf{T}$ is an open arc and $E \subset I_0$ is countable. If*

(4.1) $|u|$ *is of type* $G(w, 1)$ *for all* $w \in I_0$,

(4.2) *there is a function* $f: \mathsf{T} \to \mathsf{R}$ *such that*

$$\limsup_{r \to 1} (1-r)^{-1} |f(w) - u(rw)| < \infty \quad \textit{for all } w \in I_0 - E,$$

(4.3) *there is a* $g \in L^1(\mathsf{T})$ *such that* $\lim_{r \to 1} (1-r)^{-1}(f(w) - u(rw)) = g(w)$ *in* I_0,

(4.4) $|u(rw)| = o(\log(1-r))$ *as* $r \to 1$ *for* $w \in E$,

then f is locally integrable in I_0 and for all $\varphi \in C^\infty(\mathsf{T})$ with support in I_0 we have

$$\lim_{r \to 1} \int \varphi(w) \frac{\partial u}{\partial r}(rw)\, dw = \int_{\mathsf{T}} \varphi(w) g(w)\, dw .$$

For the proof we will study a certain type of kernels. Let $I \subset \mathsf{T}$ be an open, nonempty arc. Put

$$D(I) = \{rw : \tfrac{1}{2} < r < 1,\ w \in I\}, \qquad D^*(I) = \{rw : \tfrac{1}{2} < r < 2,\ w \in I\}$$

and let $g(z, \zeta; I)$ be the Green function of $D^*(I)$, normalized by

$g(z,\zeta;I)+(2\pi)^{-1}\log|z-\zeta|$ is harmonic in $D^*(I)$ as a function of z. For $z\neq 0$ let $z^*=z|z|^{-2}$ and define for $z,\zeta\in D(I)$:

$$K(z,\zeta;I) = g(z,\zeta;I)+g(z^*,\zeta;I).$$

Let $\Gamma(I)=\partial D(I)-\bar{I}$, $f\in L^1(\Gamma(I),ds)$ where ds is the element of arc length, and let μ be a measure supported on $\bar{I}$. Then we put

$$N(f,\mu;I) = \int_{\Gamma(I)}\frac{\partial K}{\partial n_\zeta}(z,\zeta;I)f(\zeta)\,ds+\int_I K(z,\zeta;I)\,d\mu(\zeta),$$

where $\partial/\partial n_\zeta$ denotes differentiation with respect to the unit inward normal of $\Gamma(I)$.

We now have for all $\varphi\in C^\infty(\mathrm{T})$ with support in I:

$$\lim_{r\to 1}\int_I\frac{\partial}{\partial r}N(f,\mu;I)(rw)\varphi(w)\,dw = \int\varphi\,d\mu. \tag{4.5}$$

To see this put $u=N(f,0;I)$ and $v=N(0,\mu;I)$. Noticing u has a harmonic extension to $D^*(I)$ such that $u(z)=u(z^*)$ we have

$$\frac{\partial u}{\partial r}(w) = 0 \quad \text{for } w\in I. \tag{4.6}$$

Let $w\in I$ and put

$$g_w(z) = (2\pi)^{-1}\int_{\partial D^*(I)}\log|z-\zeta|\frac{\partial}{\partial n_\zeta}g(w,\zeta;I)\,ds.$$

Then g_w is harmonic in $D^*(I)$. If we put $h_w(z)=g_w(z)+g_w(z^*)$ then

$$K(z,w;I) = b_w(z)-(2\pi)^{-1}\log|z-w||z^*-w|.$$

If V is an open set and $\bar{V}\subset D^*(I)$ then

$$\sup\{|h_w(z)| : w\in I,\ z\in v\}=c_v<\infty.$$

Since h_w is harmonic and $h_w(z)=h_w(z^*)$ we find

$$r\frac{\partial}{\partial r}K(re^{i\theta},e^{it}) = P(r,\theta-t)+S(r,\theta,t) \tag{4.7}$$

where

$$\sup\{|S(r,\theta,t)| : e^{i\theta}\in K,\ e^{it}\in I\} = o(1-r) \quad \text{as } r\to 1$$

for all compact sets $K\subset I$. The relation (4.5) follows from (4.6) and (4.7).

Let $h\in L^1(\mathrm{T})$ and let h be lower semicontinuous. Since h is bounded from below by a constant we have from (4.7) that

$$\lim_{r\to 1} N(0,h;\,I)(rw) = h^*(w)$$

exists for all $w \in I$ and the monotone convergence theorem gives

$$h^*(w) = \int_I K(w,\zeta;\,I)\,h(\zeta)\,d\zeta, \qquad w \in I\,.$$

Moreover, since $K(w,\zeta;\,I) \geqq 0$, this expression for h^* shows that h^* has a lower semicontinuous extension to $\bar{I}$. Since h is lower semicontinuous we have

$$\liminf_{r\to 1} \int P(r,\theta-t)\,h(e^{it})\,dt \geqq h(e^{i\theta})\,.$$

Therefore we have from (4.7)

$$\liminf_{r\to 1}\,(h^*(w)-N(0,h;\,I)(rw))(1-r)^{-1} \geqq h(w)\,. \tag{4.8}$$

Proof of Lemma 3. Let

$\Omega = \{w \in I_0 :$ there is an open arc I, $w_0 \in I \subset I_0$ and $u = N(u,g;\,I)$ in $D(I)\}$

and put $F = I_0 - \Omega$. Then F is relatively closed in I_0 and it is sufficient to prove $F = \emptyset$. We therefore assume $F \neq \emptyset$. Let

$$F_j = \{w \in I_0 :\ |u(rw)-u(sw)| \leqq j(2-r-s) \text{ for } 0<r,s<1\}$$

and let $E = \{e_j\}$. Since F_i is closed in I_0 for all i, the Baire category theorem implies the existence of an open arc I and integer j such that $I \cap F \neq \emptyset$ and $I \cap F \subset \{e_j\}$ or $I \cap F \subset F_j$. We will show that each case leads to a contradiction.

Let $I \cap F \subset \{e_j\}$. We may without loss of generality assume the endpoints of I are in Ω, otherwise we make I smaller. Put $v = u - N(u,g;\,I)$. It follows from (4.5) and the reasoning in [11] that we can extend v to a function harmonic in the set

$$S = \{rw :\ w \in I,\ 0<r<\infty\} - \{e_j\}$$

such that $v(z) = v(z^*)$ in S. It is easy to see that $|v|$ is of type $G(w,1)$ for $w \in \bar{I}$. Putting

$$m(r,v) = \sup\{v^+(z) :\ |z-e_j| = r\}$$

it follows from Lemma 1 that $m(r,v) = o(r^{-1})$ as $r \to 1$. Since $v(ra_j) = o[\log(r-1)]$ as $r \to 1$ a Phragmén–Lindelöf argument gives

$$m(r,v) = o(\log r) \qquad \text{as } r \to 0\,. \tag{4.9}$$

The assumptions are symmetrical with respect to v and $-v$. This gives $m(r,|v|) = o(\log r)$ as $r \to 0$ and consequently the singularity at e_j is removable. We have $v(z)=0$ for $z \in \partial D^*(I)-\bar{I}$. It now follows from Lemma 1, (4.1) and the Phragmén–Lindelöf theorem that $v=0$ in $D^*(I)$. Hence $e_j \in \Omega$ which is a contradiction.

REMARK 1. Notice that in the proof of (4.9) we only used that u was of type $G(w, 1)$ for $w \in I_0$.

Let $I \cap F \subset F_j$. There is by (4.2), (4.3) and [12, p. 73] a lower semicontinuous function h in $L'(\mathrm{T})$ such that

$$\limsup_{r \to 1} (1-r)^{-1}(f(w)-u(rw)) \leqq h(w) \tag{4.10}$$

for all $w \in I_0$. In addition we can make $\int_{\mathrm{T}} |h-g|\, dw$ as small as we want. Let J be an open nonempty open arc such that $\bar{J} \subset I$. Let $v=u-N(u,h;\, J)$. It follows from the choice of h and the definition of F_j that $\lim_{r\to 1} v(rw)=H(w)$ exists for all $w \in J$. Notice also that v has a subharmonic extension across $I-I\cap F$. Therefore the restriction of H to $J-J\cap F$ is upper semicontinuous. It follows from the definition of F_j that the restriction of H to $J \cap F$ is upper semicontinuous. We now claim H is upper semicontinuous in J. To show this it is sufficient to show that if $\eta \in J\cap F$, $\{w_k\} \subset J-J\cap F$ and $w_k \to \eta$ then $\limsup_{k\to\infty} H(w_k) \leqq H(\eta)$. Let I_k be the maximal open arc in $J-J\cap F$ containing w_k. Pick $\varepsilon>0$. Then there is a $\delta>0$ and a neighbourhood V of η in T such that $v(rw)<H(\eta)+\varepsilon$ if $1-\delta<r<1$ and $w \in J\cap F \cap V$. Let

$$S_k = \{rw :\ (1-\delta)<r<(1-\delta)^{-1},\ w \in I_k\}\ ,$$

and $\gamma_{k,j}=\{ra_{k,j} : 1-\delta<r<(1-\delta)^{-1}\}$, where $a_{k,j}$, $j=1,2$, are the endpoints of I_k and $\gamma_{k,3}=\partial S_k-(\gamma_{k,1}\cup\gamma_{k,2})$. Let $\sigma_{k,j}$ be the harmonic measure of $\gamma_{k,j}$ with respect to S_k. Since v is lower semicontinuous it follows that

$$\sup\{v(z) :\ z \in \gamma_{k,j}\} = A_{k,j} < \infty \quad \text{for all } k, j\ .$$

Put $M=\sup\{v(z) : |z|=1-\delta\}$. Then (4.1), Lemma 1 and the Phragmén–Lindelöf theorem gives:

$$v(z) \leqq A_{k,1}\sigma_{k,1}(z)+A_{k,}\sigma_{k,2}(z)+M\sigma_{k,3}(z), \qquad z \in S_k\ . \tag{4.11}$$

There are now two cases to consider. If $I_k=I_{k0}$ for all $k \geqq k_0$ then $\eta=a_{k0,j}$ for some j. In this case (4.11) gives

$$\limsup_{k\to\infty} H(w_k) \leqq A_{k0,j} \leqq H(\eta)+\varepsilon\ .$$

Otherwise $\lim_{k\to\infty} \operatorname{diam}(I_k)=0$ and $I_k \subset V$ for $k \geqq k_0$. From (4.11) follows now

$$H(w_k) \leqq H(\eta)+\varepsilon+M\sigma_{k,3}(w_k), \qquad k \geqq k_0\ .$$

Since it is straight forward to show

$$\lim_{k\to\infty} (\sup\{\sigma_{k,3}(w) : w \in I_k\}) = 0$$

it follows that H is upper semicontinuous. In the same way it follows that $\limsup_{w\to w_0} H(w) \leqq 0$ whenever w_0 is an endpoint of J. We claim $v \leqq 0$. To show this put $p(z) = v^+(z)$ for $z \in S(J)$ and zero otherwise. Then p is subharmonic in D and p is of type $G(w,1)$ for all $w \in \mathsf{T}$. Theorem 2 gives $p \leqq PH_1$, where $H_1(w) = 0$ when $w \notin J$ and $S = H^+(w)$ when $w \in J$. Since H_1 is upper semicontinuous there is a point $w_0 \in I$ such that $\max_{z\in\bar{D}} p(z) = H_1(w_0)$. Suppose $H_1(w_0) > 0$. Then it follows from [9, p. 67] that

$$u(rw_0) - N(u,h;\, J)(rw) \leqq -c(1-r) + H(w_0) \quad \text{for some } c > 0\,.$$

But this contradicts (4.8) and (4.10). Hence $v \leqq 0$. Letting $h \to g$ in L^1-norm we find

$$u \leqq N(u,g;\, J)\,. \tag{4.12}$$

Since the argument can be carried out with $-u$ as well it follows $u = N(u,g;\, J)$. This contradiction shows $\Omega = I_0$ and the lemma is proved.

REMARK 2. In the proof of (4.12) we only used that u was of type $G(w,1)$ for $w \in I_0$.

LEMMA 4. *Suppose u fulfils the assumptions of Theorem* 5. *If for some open arc* $I \subset \mathsf{T}$, $I \neq \emptyset$, *we have* $f^+ \in L^1(I)$, *then* $f \in L^1_{\mathrm{loc}}(I)$ *and* $|u|$ *us of type* $G(w,1)$ *for all* $w \in I$.

PROOF. Let h be the harmonic function in $S(I)$ with boundary values equal to $u^+(z)$ when $z \in \partial S(I) - \bar{I}$ and zero elsewhere. Put $v = (u^+ - h)^+$. Then v is subharmonic in D and of type $G(w,1)$ for all $w \in \mathsf{T}$. From Theorem 2 we now have $v \leqq PF$ for some $F \in L'(\mathsf{T})$. This means $u\,|\,S(I)$ is equal to the difference of two positive harmonic functions. Let Φ be a conformal map of D onto $S(I)$. From Fatou's theorem follows that $u \circ \Phi$ has a nontangential limit $G(w)$ a.e. in T and $G \in L^1(\mathsf{T})$. From Poisson's representation formula follows

$$M(r, |u\circ\Phi|) = O[(1-r)^{-1}] \quad \text{as } r \to 1\,.$$

Going back to u this means $f \in L^1_{\mathrm{loc}}(I)$ and $|u|$ is of type $G(w,2)$ for all $w \in I$. Let be an open arc such that $J \neq \emptyset$ and $\bar{J} \subset I$. Then $f \in L^1(J)$. Let h_1 be the harmonic function in $S(J)$ with boundary values equal to $|u|$ on $S(J) - \bar{J}$ and zero elsewhere. Arguing as in the beginning of the proof, it follows $(|u| - h_1)^+ \leqq PF$ for some $F \in L^1(\mathsf{T})$ and hence $|u|$ is of type $G(w,1)$ for all $w \in J$. Since J was arbitrary the Lemma follows.

PROOF OF THEOREM 5. Let u fulfil the assumptions of Theorem 5. Let

$$F_j = \{w \in \mathsf{T} : |u(rw)-u(sw)| \leqq j(2-r-s) \text{ for } 0<r,s<1\},$$

$$E=\{e_j\}_{j=1}^{\infty}$$

and

$$\Omega = \{w \in \mathsf{T} : \text{ for some open arc } I,\ w \in I\}$$

and let $u=N(u,g;I)$. Then as above R is closed and the Baire category theorem implies the existence of an open arc I and an integer j such that $I \cap R \neq \emptyset$ and $I \cap R \subset \{e_j\}$ or $I \cap R \subset F_j$. If $I \cap R \subset \{e_j\}$ it follows from (4.9) and Remark 1 that $f^+ \in L^1_{\mathrm{loc}}(I)$. If $I \cap R \subset F_j$ it follows from (4.12) and Remark 2 that $f^+ \in L^1_{\mathrm{loc}}(I)$. Hence we have from Lemma 4 that in both cases u fulfils the assumptions of Lemma 3 on I and consequently $I \subset \Omega$. This contradiction shows $\Omega = \mathsf{T}$ which yields Theorem 5.

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