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THE VARIETY COVERING THE VARIETY OF ALL MODULAR LATTICES

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It was first noted by Kirby Baker that the join of two finitely based lattice varieties need not be finitely based. His example has not appeared in print, but later I discovered and published, in [3], a different example. The principal purpose of that paper, however, was to prove the following positive result.

THEOREM 1. *If $\mathscr{V}$ and $\mathscr{V}'$ are finitely based varieties, and if $\mathscr{V} \subseteqq \mathscr{M}$ and $(\mathscr{V}')^3 = \mathscr{N}^3$, then $\mathscr{V} + \mathscr{V}'$ is finitely based.*

Here $\mathscr{M}$ is the variety of all modular lattices, $\mathscr{N}$ is the variety generated by the pentagon (the five-element non-modular lattice), and $(\mathscr{V}')^3$ is the variety defined by all the 3-variable identities that hold in $\mathscr{V}'$. In particular, therefore, this shows that $\mathscr{M}^+ = \mathscr{M} + \mathscr{N}$, the unique lattice variety that covers $\mathscr{M}$, is finitely based.

As was pointed out in [3], one could theoretically find a basis for $\mathscr{V} + \mathscr{V}'$, using the bases for $\mathscr{V}$ and $\mathscr{V}'$, by means of the techniques developed by Baker in [1], but even in the simplest cases this would be hopelessly inefficient, and the axioms would be far too complicated to be of any interest. We have therefore undertaken to find, by more special methods, a more manageable basis for the particular variety $\mathscr{M}^+$. We are indebted to G. A. Grätzer and R. Padmanabhan for suggestions that led to a simplification of the final steps in the proof of our principal result, Theorem 12, and to the elimination of a redundant identity from that theorem.

We write $u \prec v$ if v covers u. By a *critical edge* in a subdirectly irreducible lattice L we mean a non-trivial quotient v/u that is collapsed by the minimal non-zero congruence relation on L. The maps $t \to tx$ and $t \to t+x$ are called *weak transpositions*, and a composition of weak transpositions is called a *weak projection*. An interval v/u is said to *transpose weakly down*, respectively *up*, onto v'/u' if there exists a weak transposition $t \to tx$, respectively $t \to t+x$,

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that takes v into v' and u into u'. If there exists a weak projection that takes v into v' and u into u', then we say that v/u *projects weakly onto* v'/u'.

Throughout this paper we fix the lattice polynomials

$$a = pq+pr, \quad b = q, \quad c = p(q+pr)$$

and hence either $a=c$, or else a, b and c generate a pentagon. (We freely identify the variables and polynomials with their values under an assignment.) Our first result is motivated by the observation that for $L \in \mathcal{M}^+$ the congruence relation con (a,c) generated by identifying a and c has a very simple form. First consider the case when L is subdirectly irreducible. If $a=c$, then con (a,c) is of course trivial, but if $a<c$, then L is simply the pentagon generated by a, b and c, and con (a,c) identifies no two distinct elements other than a and c. Using these observations one can prove that in an arbitrary member L of $\mathcal{M}^+$,

$$\text{con}\,(a,c) = \{\langle u,v\rangle \in L\times L : u+v \leqq uv+c \wedge (u+v)(uv+a)=uv\}.$$

The proof of this assertion is quite easy. Call the right hand side of the equation θ. It is obvious that $\theta \subseteq \text{con}\,(a,c)$ and $a\theta c$, and thus we need only prove that θ is a congruence relation. Represent L as a subdirect product of lattices L_i ($i \in I$) and denote the image of an element $u \in L$ in L_i by u_i. We claim that θ is in fact the congruence relation on L induced by the congruence relations

$$\theta_i = \text{con}\,(a_i, c_i)$$

on the lattices L_i, i.e., that $u\,\theta\,v$ iff $u_i\,\theta_i\,v_i$ for all $i \in I$. Since the conditions $u\,\theta\,v$ and $uv\,\theta\,(u+v)$ are clearly equivalent, it suffices to consider the case when $u \leqq v$. In this case we have $u\,\theta\,v$ iff $v \leqq u+c$ and $v(u+a)=u$, i.e., iff

$$(1) \qquad v_i \leqq u_i+c_i \quad \text{and} \quad v_i(u_i+a_i) = u_i$$

for all $i \in I$. If $a_i=c_i$, then these conditions hold just in case $u_i=v_i$, but if $a_i<c_i$, and hence L_i is a pentagon with c_i/a_i as its critical edge, then there is one more solution, namely $u_i=a_i$ and $v_i=c_i$. Thus in either case, (1) holds iff $u_i\,\theta_i\,v_i$.

We do not make direct use of the above observations, but rather we need to find an identity that holds in $\mathcal{M}^+$ and is such that the above characterization of con (a,b) holds in every lattice L that satisfies this identity. With θ defined as before, we therefore look for properties that imply that θ is a congruence relation. Since θ is obviously reflexive and $u\,\theta\,v$ iff $uv\,\theta\,(u+v)$, we can use the Grätzer–Schmidt Criterion (cf. [2], p. 149), which tells us that θ is a congruence relation iff

(2) $u \leqq v \leqq w$ and $u \,\theta\, v \,\theta\, w$ imply $u \,\theta\, w$,

(3) $u \leqq v$ and $u \,\theta\, v$ imply $(u+w)\,\theta\,(v+w)$,

(4) $u \leqq v$ and $u \,\theta\, v$ imply $uw \,\theta\, vw$.

The first implication holds in any lattice, for if $u \leqq v \leqq w$ and $u \,\theta\, v \,\theta\, w$, then

$$v \leqq u+c, \qquad v(u+a) = u\,,$$

$$w \leqq v+c, \qquad w(v+a) = v\,,$$

and therefore $w \leqq u+c$ and

$$w(u+a) = w(v+a)(u+a) = v(u+a) = u\,,$$

yielding $u \,\theta\, w$. For (3) and (4) we need to know that if $u \leqq v \leqq u+a$ and $v(u+a) = u$, then

$$v+w \leqq u+w+c, \qquad (v+w)(u+w+a) = u+w\,,$$

$$vw \leqq uw+c, \qquad vw(uw+a) = uw\,.$$

Of these four conditions, the first and the last hold in any lattice. To transform the implications

$$(u \leqq v) \wedge (u \,\theta\, v) \rightarrow (v+w)(u+w+a) = u+w\,,$$

$$(u \leqq v) \wedge (u \,\theta\, v) \rightarrow vw \leqq uw+c$$

into identities, we consider arbitrary lattice elements u and v and let

$$u' = v(u+a), \qquad v' = v(u+c)\,.$$

Then $u' \leqq v'$ and $v'(u'+a) = v(u+a) = u'$, and the inclusion

$$v'(u'+c) \leqq v'(u'+a)+c$$

holds in $\mathscr{M}^+$, so that $u' \,\theta\, v'$. In $\mathscr{M}^+$ we therefore have

(5) $$(v(u+c)+w)(v(u+a)+w+a) = v(u+a)+w\,,$$

(6) $$v(u+c)w \leqq v(u+a)w+c\,.$$

On the other hand, if these identities hold, then the two implications are also satisfied, for if $u \leqq v$ and $u \,\theta\, v$, then $u' = u$ and $v' = v$. Actually, we can use in place of (5) and (6) the single identity

(7) $$(v(u+c)+w)(u+w+a) = v(u+a)+w\,.$$

It is obvious that (7) implies (5), and to derive (6) we replace v by vw and w by c. Furthermore, (7) obviously holds when $a = c$, so that in order to show that

(7) holds in $\mathcal{M}^+$ we need only consider the case when c/a is a critical edge in a pentagon. If $u \nleqq a$, then $a+u=c+u$, and (7) holds, but if $u \leqq a$, then (7) reduces to $(vc+w)(w+a)=va+w$. To verify this equation consider two cases, $c \leqq v$ and $c \nleqq v$, noting that in the latter case $cv=av$.

We now summarize our results up to this point.

LEMMA 2. *The identity*

$$\text{(I)} \qquad (v(u+c)+w)(u+w+a) \;=\; v(u+a)+w$$

holds in $\mathcal{M}^+$. *In any lattice L that satisfies* (I),

$$\operatorname{con}(a,c) \;=\; \{\langle u,v\rangle \in L\times L \;:\; u+v \leqq uv+c \;\wedge\; (u+v)(uv+a)=uv\}\;.$$

We next look for another identity (II) which also holds in $\mathcal{M}^+$, and such that in any subdirectly irreducible lattice L that satisfies (I) and (II), the condition $a < c$ implies that c covers a and that c/a is a critical edge of L. First observe that if (I) holds, then con (a,c) does not collapse any interval v/u with $c \leqq u < v$ or $u < v \leqq a$. In fact, if $c \leqq u < v$, then $v \nleqq u+c$, and if $u < v \leqq a$, then $v(u+a) = v \neq u$. If now $a < d < c$, then a, d and b generate a pentagon, and we can apply the above observation with c replaced by d to infer that con (c,d) does not collapse d/a, and that d/a is therefore not a critical edge. Similarly c/d cannot be a critical edge. From this it follows that if some subinterval c'/a' of c/a is a critical edge of L, then $a=a'$, $c'=c$, and $a \prec c$.

Assuming still that L satisfies (I), suppose v/u is a critical edge of L. Then con (u,v) identifies the elements $a'=(u+a)c$ and $c'=(v+a)c$, and hence either $a'=c'$ or else c'/a' is a critical edge of L. Thus if we can exclude the case $a'=c'$, we will be able to infer that $a \prec c$ and that c/a is a critical edge of L.

To say that $a' < c'$ is equivalent to the assertion that con (a',c') identifies u and v, and since we are assuming that (I) holds, this is true just in case

$$v \leqq u+c' \qquad \text{and} \qquad v(u+a') = u\;.$$

The second equation does in fact hold, for

$$v(u+a') \;=\; v(u+c(u+a)) \;=\; v(u+a) \;=\; a\;.$$

The first condition can be written $v \leqq u+c(v+a)$ or, since $v \leqq u+c$,

$$v(u+c) \;\leqq\; u+c(v+a)\;.$$

In this form it is actually an identity that can easily be seen to hold in $\mathcal{M}^+$, for it is obviously satisfied when $a=c$, and we therefore need only consider the case of a pentagon with c/a as its critical edge. We have therefore found the required identity (II).

LEMMA 3. *The identity*

$$\text{(II)} \qquad v(u+c) \leqq u+c(v+a)$$

holds in $\mathcal{M}^+$. *In any subdirectly irreducible lattice that satisfies* (I) *and* (II), *if* $a<c$, *then* $a \prec c$ *and* c/a *is a critical edge of* L.

We next add an identity that will guarantee that if v/u is a quotient such that con (u, v) collapse a non-trivial subinterval of c/a, then either con (bu, bv) or con (cu, cv) collapses a non-trivial subinterval of c/a. We only consider the case when the subinterval of c/a can be reached from v/u by a short sequence of weak transposition, since later identities will guarantee that a longer sequence can always be shortened.

LEMMA 4. *The identity*

$$\text{(III)} \qquad ((t+x)y+a)c = ((ct+x)y+a)c+((bt+x)y+a)c$$

holds in $\mathcal{M}^+$.

PROOF. Since the identity obviously holds when $a=c$, we need only consider the case of a pentagon with c/a as its critical edge. If (III) fails, then t must be strictly larger than bt and ct, and must therefore be the top element of the pentagon. Thus it suffices to show that

$$(y+a)c = (by+a)c+(cy+a)c\ .$$

If $y=b$ or $y \geqq c$, then both sides of this equation are equal to c, but if $y \leqq a$, then both sides have the value a.

Suppose L satisfies (III), and consider the weak projectivity

$$f(t) = ((t+x)y+a)c\ .$$

If f maps an interval v/u onto a non-trivial subinterval of c/a, then f maps either bv/bu or cv/cu onto a non-trivial subinterval of c/a. I.e., if $f(u)<f(v)$, then $f(bu)<f(bv)$ or $f(cu)<f(cv)$. Assuming first that f maps cv/cu onto a non-trivial subinterval of c/a, we want to find a simpler weak projectivity with the same property. We try

$$g(t) = (t+x)c+a\ .$$

This will work in any lattice in which $f(ct)$ can be expressed as a function of $g(ct)$. We try

$$f(ct) = ((g(ct)+xy)y+a)c\ .$$

If $a=c$, then both sides equal a, so in order to show that this holds in $\mathscr{M}^+$ we need only verify that it holds in a pentagon with c/a as its critical edge. This is of course a simple matter.

LEMMA 5. *The identity*

$$\text{(IV)} \qquad ((ct+x)y+a)c = (((ct+x)c+a+xy)y+a)c$$

holds in $\mathscr{M}^+$. *If* L *is any lattice in which* (IV) *holds, and if the weak projectivity*

$$t \to ((t+x)y+a)c$$

maps the interval cv/cu *of* L *onto a non-trivial subinterval of* c/a, *then so does the weak projectivity*

$$t \to (t+x)c+a\ .$$

We now consider the case when $f(bu)<f(bv)$. Here we try the weak projectivity

$$h(t) = (t+a)c\ .$$

This will work provided $f(bt)$ can be expressed as a function of $h(bt)$. The identity that works here is somewhat more involved, but as with the others, it is easy to check that it holds in $\mathscr{M}^+$.

LEMMA 6. *The identity*

$$\text{(V)} \qquad ((bt+x)y+a)c = ((bt+a)c+xy)((b+x)y+a)c$$

holds in $\mathscr{M}^+$. *If* L *is any lattice in which* (IV) *holds, and if the weak projectivity*

$$t \to ((t+x)y+a)c$$

maps the interval bv/bu *of* L *onto a non-trivial subinterval of* c/a, *then so does the weak projectivity*

$$t \to (t+a)c\ .$$

We are now ready to show that one need not consider long sequences of weak projectivities.

LEMMA 7. *Suppose* L *is a lattice that satisfies* (III), (IV) *and* (V) *and their duals. If* $u,v \in L$, $u<v$, *and* con (u,v) *collapses a non-trivial subinterval of* c/a, *then there exist* $x,y \in L$ *such that either*

$$(ux+y)c+a < (vx+y)c+a \quad \text{or} \quad ((u+x)y+a)c < ((v+x)y+a)c\ .$$

PROOF. There exists a sequence of intervals $v/u = v_0/u_0, v_1/u_1, \ldots, v_n/u_n$ with $a \leqq u_n < v_n \leqq c$ such that for $i = 1, 2, \ldots, n-1$, v_i/u_i transposes weakly alternatingly up and down onto v_{i+1}/u_{i+1}. We may assume that the last two maps are $t \rightarrow t+a$ and $t \rightarrow tc$ in one order or the other. Subject to this restriction, we assume that n has been chosen as small as possible, and the problem reduces to showing that in this case $n \leqq 4$.

Suppose, to the contrary, that $n \geqq 5$. By duality, we may assume that the last map is $t \rightarrow tc$. Then for some x_0, x_1, x_2, the map

$$t \rightarrow ((tx_0 + x_1)x_2 + a)c$$

maps v_{n-5}/u_{n-5} onto a non-trivial subinterval of c/a, and therefore one of the maps

$$t \rightarrow (tx_0c + x_1)c + a, \qquad t \rightarrow (tx_0b + a)c$$

maps v_{n-5}/u_{n-5} onto a non-trivial subinterval of c/a. Thus the number of steps can be reduced by at leat one.

COROLLARY 8. *For any interval v/u in a lattice that satisfies* (III), (IV) *and* (V) *and their duals, if* con (u, v) *collapses a non-trival subinterval of c/a, then so does either* con (bu, bv) *or* con (cu, cv).

COROLLARY 9. *Suppose v/u is an interval in a lattice that satisfies* (III), (IV), *and* (V) *and their duals, and suppose* con (u, v) *collapses a non-trivial subinterval of c/a. If $v \leqq c$, then*

$$(u+x)c+a \ < \ (v+x)c+a)$$

for some x. If $v \leqq b$, then

$$(u+a)c \ < \ (v+a)c\ .$$

COROLLARY 10. *For any element t in a lattice that satisfies* (III), (IV) *and* (V) *and their duals,* con $(bt+ct, t)$ *does not collapse a non-trivial subinterval of c/a.*

COROLLARY 11. *If L is a subdirectly irreducible lattice that satisfies* (I)–(V) *and the duals of* (III), (IV) *and* (V), *and if $a < c$, then $bt+ct = t$ for all $t \in L$.*

Suppose L is a subdirectly irreducible lattice that satisfies (I)–(V) and the duals of (III), (IV) and (V), and suppose $a < c$. We know that $a \prec c$, and that c/a is a critical edge. We also know that, for all $t \in L$, $t = bt + ct$ and, dually, $t = (b + t)(a + t)$. Thus $b + c$ is the largest element of L and bc the smallest. It is easy to show that $bc \prec b$. In fact, if $bc < t < b$, then both con (bc, t) and con (t, b) collapse

c/a, and hence by Corollary 9,

$$a = (bc+a)c < (t+a)c < (b+a)c = c\,,$$

contrary to the fact that $a \prec c$. Dually, $b \prec b+c$. Next note that $c \prec b+c$, for if $c \leqq t < b+c$, then $bc \leqq bt < b$, hence $bt = bc$, which implies that $t = bt+ct = c$. Dually, $bc \prec a$.

We are now ready to prove that L must be a pentagon, i.e. that the only elements of L are a, b, c, bc and $b+c$. Since every element t of L has the form $t = x+y$ with $x \leqq b$ and $y \leqq c$, and since there is no element strictly between bc and b, it suffices to show that the only element strictly between bc and c is a. So, assume that $bc < x < c$ and $x \neq a$. Then a and x must be incomparable, and hence $a+x = c$. Also, b and x must be incomparable, and therefore $b < b+x$, which implies that $b+x = b+c$. Consequently, $x = (a+x)(b+x) = c$, a contradiction.

Our eight identities hold in the pentagon, as well as in every modular lattice, and they therefore hold in $\mathscr{M}^+$. We have now shown that the pentagon is the only subdirectly irreducible non-modular lattice in which they hold, and we conclude that they do in fact form an equational basis for $\mathscr{M}^+$.

THEOREM 12. *The following identities* (I)–(V) *and the duals of* (III), (IV) *and* (V) *form an equational basis for* $\mathscr{M}^+$.

(I) $((x+c)y+z)(x+z+a) = (x+a)y+z.$
(II) $(x+c)y \leqq x+(y+a)c.$
(III) $((t+x)y+a)c = ((ct+x)y+a)c + ((bt+x)y+a)c.$
(IV) $((ct+x)y+a)c + (((ct+x)c+a+xy)y+a)c.$
(V) $((bt+x)y+a)c = ((bt+a)c+xy)((b+x)y+a)c.$

Here $a = pq+pr$, $b = q$ *and* $c = p(q+rq)$.

COROLLARY 13. *A lattice L belongs to $\mathscr{M}^+$ iff every sublattice of L generated by a set with six elements or less belongs to $\mathscr{M}^+$.*

This follows from the fact that none of the identities (I)–(V) contains more than six variables. It is an open question whether the number six can be replaced by five, or even by four, but certainly three will not suffice. In fact, several of the lattices listed by McKenzie in [4] that generate varieties covering $\mathscr{N}$ cannot be generated by fewer than four elements, and of course every proper sublattice belongs to $\mathscr{N}$, and therefore to $\mathscr{M}^+$.

Each of the lattices in McKenzie's list must fail to satisfy one of the identities in Theorem 12, and since each lattice is generated by four elements or less, each must violate some four variable special case of one of the identities. In fact, the identity

(8) $$cx \leqq ax+b\ ,$$

which is a special case of (I), fails in P_0^d, P_1 and Q^*, and a special case of (III),

(9) $$(x+a)c = (bx+a)c+cx\ ,$$

fails in Q_0, Q_1, Q_2, Q_3, Q_4^d and N_6. Of course, the dual of (8) fails in P_0 and Q^{*^d}, and the dual of (9) fails in Q_0^d, Q_1^d, Q_2^d and Q_4. Together, the identities (8) and (9) and their duals therefore exclude all the lattices in McKenzie's list, but they still do not characterize $\mathscr{M}^+$. To see this, observe that all four identities hold in the lattices in Figs. 1 and 2, but of course these lattices do not belong to $\mathscr{M}^+$. In fact, (III) fails in both lattices with the indicated assignment of values to the variables. This yields the four-variable identities

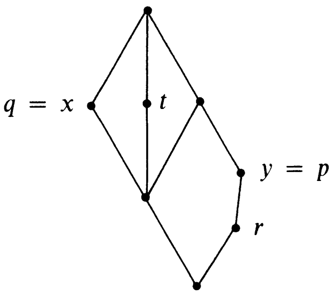

Fig. 1

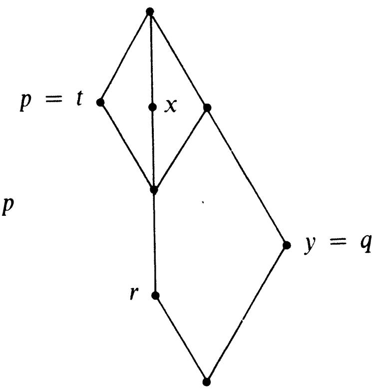

Fig. 2

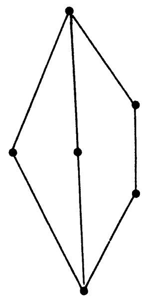

Fig. 3

(10) $$(t+b)c \leqq (ct+b)c+a\ ,$$

(11) $$((p+x)q+pr) \leqq (c+x)q+pr\ ,$$

which exclude these two lattices. On the other hand, the lattice in Fig. 3 fails to satisfy (9). We conjecture that any variety of lattices that properly contains $\mathscr{M}^+$ has as a member one of the following twenty lattices: the fifteen lattices in McKenzie's list, the lattices in Figs. 1, 2 and 3, and the duals of the lattices in Figs. 1 and 2. If this is correct, then it follows that the identities (8)–(11) and their duals form an equational basis for $\mathscr{M}^+$, and hence that the eight identities in Theorem 12 can be replaced by four, (I) and (III) and their duals. Another consequence would be that $\mathscr{M}^+$ has exactly twenty covers in the lattice of all varieties of lattices.

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