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RINGS WITH AN ALMOST NOETHERIAN RING OF FRACTIONS

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Assume R is a commutative ring with 1. We shall call an R-module M an *almost Noetherian R-module* if each proper submodule of M is a finitely generated R-module. The p-quasicyclic group $\mathbf{Z}_{p^\infty}$ is an example of an almost Noetherian Z-module which is not Noetherian; in fact any almost Noetherian Z-module is either Noetherian or isomorphic to $\mathbf{Z}_{p^\infty}$ for a suitable prime p [1, Theorem 2.2]. On the other hand it is well-known that the quotient field of a discrete valuation domain R has each of its proper submodules isomorphic to ideals of R; more generally Shores and Lewis have shown that for any valuation domain R, the proper submodules of the quotient field of R are isomorphic to ideals of R [3, Proposition 2.2]. Hence in case R is a discrete valuation domain its quotient field K is an almost Noetherian R-module. These observations lead us to pose the question: Which rings have an almost Noetherian ring of fraction?

The purpose of this paper is to provide an answer and this is given by Theorem 2.1:

A ring R has an almost Noetherian ring of fractions if and only if R is either

(i) *a Noetherian ring in which each maximal ideal has nonzero annihilator*; *or*

(ii) *a 1-dimensional local Noetherian domain whose integral closure is a discrete valuation ring which is module-finite over R.*

1. Preliminaries.

In this section we will provide some basic properties of almost Noetherian modules. The proof of the first proposition is essentially the same as that of [1, Lemma 2.1] and so will be omitted.

Proposition 1.1. *Let M be an almost Noetherian module which is not Noetherian.*

(a) *For each proper submodule H of M, M/H is an almost Noetherian module which is not Noetherian.*

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(b) *Each proper submodule of M is small*; *i.e., if H, K are submodules of M and $M = H + K$ then $M = H$ or $M = K$.*

One immediate observation which follows from part (a) of the preceding is that if R is a ring for which all modules have maximal submodules, e.g., perfect rings, then any almost Noetherian module is Noetherian.

PROPOSITION 1.2. *Suppose M is an almost Noetherian module which is not Noetherian.*

(a) *For any $x \in R$ either $xM = 0$ or $xM = M$.*

(b) Ann (M) *is a prime ideal of R and M is either a torsion or torsionfree divisible module over the integral domain* $R/\text{Ann}(M)$.

PROOF. (a) Each element x of R induces an endomorphism of M via multiplication; its kernel is

$$\text{Ann}_M(x) = \{u \in M : xu = 0\} .$$

If $xM \neq 0$ then $\text{Ann}_M(x) \neq M$ so by Proposition 1.1 (a), $xM \approx M/\text{Ann}_M(x)$ is not Noetherian. Hence $xM = M$.

(b) If $x, y \in R$ with $x \notin \text{Ann}(M)$ and $y \notin \text{Ann}(M)$ then by (a), $xM = M = yM$, hence $M = xyM$. Thus $xy \notin \text{Ann}(M)$ showing that $\text{Ann}(M)$ is a prime ideal of R. By passing to $R/\text{Ann}(M)$ we may assume that R is a domain and $\text{Ann}(M) = 0$. Then for $x \in R$ with $x \neq 0$ we have $xM = M$ so that M is a divisible R-module. Now suppose M is not torsion-free and select $x \in R$, $x \neq 0$, such that $\text{Ann}_M(x) \neq 0$. Then

$$\text{Ann}_M(x) \subsetneqq \text{Ann}_M(x^2) \subsetneqq \dots$$

is an ascending chain of submodules of M. Further,

$$\text{Ann}_M(x) \neq M \text{ and } M/\text{Ann}_M(x) \approx M$$

so that the chain is strictly increasing. This means that $\bigcup_{i=1}^{\infty} \text{Ann}_M(x^i)$ cannot be a proper submodule of M and so $\bigcup_{i=1}^{\infty} \text{Ann}_M(x^i) = M$. It follows that M is a torsion R-module.

For the next result the blanket assumption that R be a commutative ring is not necessary.

PROPOSITION 1.3. *If M is an almost Noetherian R-module which is not Noetherian then $E = \text{End}_R(M)$ has no nonzero zero-divisors.*

PROOF. As in Proposition 1.2(a) if $\alpha \in E$ with $\alpha \neq 0$ then $M\alpha = M$. Thus if $\alpha, \beta \in E$ and $\alpha \neq 0$, $\beta \neq 0$ then $M(\alpha\beta) = M\beta = M$ so $\alpha\beta \neq 0$.

2. Principle result.

We proceed to our principle result.

THEOREM 2.1. *Let R be a ring with ring of fractions K. Then K is an almost Noetherian R-module if and only if R is either*

(i) *a Noetherian ring in which each maximal ideal has nonzero annihilator*; *or*

(ii) *a 1-dimensional local Noetherian domain whose integral closure is a discrete valuation ring module-finite over R.*

PROOF. We shall first show that rings of type (i) or (ii) have an almost Noetherian ring of fractions. First assume that R is of type (i). If $t \in R$ is not a zero-divisor then t belongs to no maximal ideal of R hence t is a unit in R. Thus in this case $R=K$ and K is, in fact, a Noetherian R-module. Next assume R is of type (ii) and let $S=$integral closure of R. Then K is the quotient field of S and as mentioned in the introduction K is an almost Noetherian S-module. Because S is a finitely generated R-module, in order to show that K is an almost Noetherian R-module it suffices to show that if A is a proper R-submodule of K then $SA \neq K$. Now

$$S = Ru_1+Ru_2+\ldots+Ru_n$$

hence there exists $b \in R$ such that $S \subseteq Rb^{-1}$. If A is an R-submodule of K such that $K=SA$ then

$$K = SA \subseteq Rb^{-1}A = Ab^{-1}$$

so that $K=Ab^{-1}$. Hence $A=Kb=K$ establishing our statement. We remark that if R is a domain whose integral closure S is a discrete valuation ring modulefinite over R then R is necessarily local 1-dimensional by [2, Theorems 44 and 48] and Noetherian by the Eakin–Nagata theorem [2, p. 54].

For the converse suppose first that $R=K$. Then R is a Noetherian ring in which each nonzero-divisor is a unit. By [2, Theorem 86] each proper ideal has nonzero annihilator; in particular each maximal ideal has nonzero annihilator, so in this case R is a ring of type (i). Thus we assume $R \neq K$. Then R is a Noetherian ring while K is not a Noetherian R-module. Because Ann $(K)=0$, R is a domain by Proposition 1.2(b) and so K is the quotient field of R. We will establish that R is local by showing that the non-units of R form an ideal. Let $x \in R$, $x \neq 0$, be a non-unit; then $R[x^{-1}]$ is an R-submodule of K. If $R[x^{-1}] \neq K$ then $R[x^{-1}]$ is a Noetherian R-module and so x^{-1} is integral over R. But then $x^{-1} \in R$, a contradiction. Thus we must have $R[x^{-1}]=K$ whenever $x \neq 0$ is a non-unit of R. Applying [2, Theorem 19] we conclude that the non-units of R coincide with the intersection of the nonzero prime ideals of R. Thus, in fact, R is local and 1-dimensional. The integral closure S of R is a proper R-submodule of K hence S is module-finite over R. Furthermore K is an almost Noetherian S-module so the preceding argument can be applied to S. Hence S is local, 1-dimensional, Noetherian and integrally closed, i.e., S is a discrete valuation ring.

By combining this theorem with Proposition 1.2 (b) we have

THEOREM 2.2. *Let R be a ring having an almost Noetherian R-module. M which is not Noetherian. If M is a torsion-free* $R/\mathrm{Ann}\,(M)$*-module then M is isomorphic to the quotient field of* $R/\mathrm{Ann}\,(M)$; *hence* $R/\mathrm{Ann}\,(M)$ *is a domain of type* (ii).

REMARK. Rings of type (i) need not be Artinian. An example of such a ring is the ring

$$R = F[[x,y]]/(x^2, xy)\,,$$

where F is any field. We should note however that if R is of type (i) then R is semi-local. This is because

$$\mathrm{Hom}_R\,(R/I, R) \approx \mathrm{Ann}\,(I)$$

for any ideal I of R. Hence each simple R-module is isomorphic to an ideal of R. Thus if T is the socle of R and J is the intersection of the maximal ideals of R then $J = \mathrm{Ann}\,(T)$. Because R is Noetherian, R satisfies the descending chain condition on annihilators so that

$$J = \mathrm{Ann}\,(Ru_1 + \ldots + Ru_k) \qquad \text{for some } u_1, \ldots, u_k \in T\,.$$

Then R/J embeds in $Ru_1 \oplus \ldots \oplus Ru_k \subseteqq T^{(k)}$ and $T^{(k)}$ is a finitely generated completely reducible R-module.

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