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A DESCRIPTION OF DISCRETE SERIES USING STEP ALGEBRAS

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1. Introduction.

In this paper we study the discrete series of an arbitrary complex finite dimensional Lie algebra $\mathfrak{g}$ with respect to a reductive subalgebra $\mathfrak{k}$ in $\mathfrak{g}$, $\operatorname{rank}\mathfrak{k} = \operatorname{rank}\mathfrak{g}$. Because our notion of discrete series differs slightly from the usual one even when $\mathfrak{g}$ is semi-simple we shall introduce some notation in order to explain this difference.

So let $\mathfrak{g}$ first be semi-simple. Let $\mathfrak{h}\subset\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$, Ψ the system of roots for $(\mathfrak{g},\mathfrak{h})$ and $\Delta_k\subset\Psi$ a positive system for $(\mathfrak{k},\mathfrak{h})$. Let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$ and let $(\cdot,\cdot)\colon \mathfrak{h}^*\times\mathfrak{h}^*\to \mathbf{C}$ be the dual of the Killing form of $\mathfrak{k}$ restricted to $\mathfrak{h}$. The set Λ of integral weights consists of those $\lambda\in\mathfrak{h}^*$ for which

$$\langle\lambda,\alpha\rangle = 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\in \mathbf{Z} \quad \forall\,\alpha\in\Delta_k\,,$$

and the set Λ^+ of dominant integral elements is

$$\Lambda^+ = \{\lambda\in\mathfrak{h}^* \mid \langle\lambda,\alpha\rangle\in\mathbf{Z}_+ \ \forall\,\alpha\in\Delta_k\}$$

where $\mathbf{Z}_+=\{0,1,2,\ldots\}$. Next we set

$$\Lambda^+_{\mathrm{reg}} = \{\lambda\in\Lambda^+ \mid \langle\lambda,\alpha\rangle\neq 0 \ \forall\,\alpha\in\Psi\}\,.$$

The elements of Λ^+_{reg} are called regular weights. If $\lambda\in\Lambda^+_{\mathrm{reg}}$ then one can define a positive system Δ^λ for $(\mathfrak{g},\mathfrak{h})$ by

$$\Delta^\lambda = \{\alpha\in\Psi \mid \langle\lambda,\alpha\rangle>0\}\,.$$

Clearly $\Delta_k\subset\Delta^\lambda$. The discrete representations $D_{\Delta^\lambda,\lambda-\delta_k+\delta_n}$ are parametrized by regular λ, where

$$\delta_k = \tfrac{1}{2}\sum_{\alpha\in\Delta_k}\alpha, \qquad \delta_n = \tfrac{1}{2}\sum_{\alpha\in\Delta^\lambda\setminus\Delta_k}\alpha\,.$$

The discrete representations have the following three properties:

$$(1)\qquad D_{\Delta^\lambda,\lambda-\delta_k+\delta_n} = \sum_\mu\oplus\, m_\lambda(\mu)X_\mu$$

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where X_μ is the irreducible finite dimensional $\mathfrak{k}$-module with highest weight μ and the $m_\lambda(\mu)$'s are integers, $0 \leqq m_\lambda(\mu) < \infty$.

(2) $$m_\lambda(\lambda - \delta_k + \delta_n) = 1 .$$

(3) If $m_\lambda(\mu) \neq 0$ then $\mu = \lambda - \delta_k + \delta_n + \nu$, where ν is a sum of elements in Δ^λ.

For more information, see [1], [2] and [9].

In our approach we choose a basis $\{\alpha_1, \ldots, \alpha_l\}$ in Δ_k and define a lexicographical ordering in Λ with respect to this basis (section 2). If $\Omega \subset \Lambda^+$ is any subset then there exists a minimal element in Ω. Let V be a $\mathfrak{k}$-finite $\mathfrak{g}$-module, that is, V is a direct sum of the $\mathfrak{k}$-modules X_μ,

$$V = \sum_\mu \oplus\, n(\mu) X_\mu .$$

We set $V_\mu = n(\mu) X_\mu$. We say that V_λ is a minimal component of V if $V_\lambda \neq 0$ and $V_\mu = 0$ for $\mu < \lambda$. The ordering "$<$" is total when $\mathfrak{k}$ is semi-simple, therefore the minimal component is unique for semi-simple $\mathfrak{k}$ (and for any $\mathfrak{g}$). The motivation for our choice of the ordering "$<$" is the fact that for any $\mathfrak{k}$-finite $\mathfrak{g}$-module V there exists a minimal component V_λ and that it is compatible with the standard partial ordering on $\mathfrak{h}^*$ defined by the choice of Δ_k: if $\lambda - \lambda'$ is a sum of elements of Δ_k then $\lambda > \lambda'$. Set

$$\Delta = \{\alpha \in \Psi \mid \alpha > 0\} .$$

Then Δ is a positive system for $(\mathfrak{g}, \mathfrak{h})$ (when $\mathfrak{g}$ is semi-simple) and $\Delta_k \subset \Delta$. If V_λ is a minimal component for V then $V_{\lambda - \alpha} = 0$ for any $\alpha \in \Delta$. On the other hand, if $\Delta' \subset \Psi$ is an arbitrary positive system for $(\mathfrak{g}, \mathfrak{h})$ such that $\Delta_k \subset \Delta'$ (e.g. $\Delta' = \Delta^\nu$ for a regular ν) then there does not always exist $0 \neq V_\mu \subset V$ such that $V_{\mu - \alpha} = 0$ for all $\alpha \in \Delta'$.

For a certain subset Λ_0^+ of Λ^+ we show that for each $\lambda \in \Lambda_0^+$ there exists a unique equivalence class $[V]$ of irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules V with minimal component V_λ. For each $\lambda \in \Lambda_0^+$ there is an irreducible $\mathfrak{g}$-module V^λ with the three properties

(1)′ $$V^\lambda = \sum_\mu \oplus\, n_\lambda(\mu) X_\mu, \quad 0 \leqq n_\lambda(\mu) < \infty .$$

(2)′ $$n_\lambda(\lambda) = 1 .$$

(3)′ If $n_\lambda(\mu) \neq 0$ then $\mu = \lambda + \nu$, where ν is a sum of elements in $\Delta \backslash \Delta_k$.

The set Λ_0^+ plays more or less the role of Λ_{reg}^+ in the earlier approach. We shall call here the set of modules V^λ as the discrete series.

The method applied here is the same which was used in [8] for describing the irreducible $\mathfrak{gl}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C})$-finite $\mathfrak{gl}(4, \mathbf{C})$-modules. The irreducible modules with minimal $\mathfrak{k}$-type λ are parametrized by the action of certain algebra D_λ, associated to the step algebra $S(\mathfrak{g},\mathfrak{k})$, on the minimal component. If $\lambda \in \Lambda_0^+$ then $D_\lambda \cong \mathbf{C}$. In an earlier paper we gave a sufficient condition for $\lambda \in \Lambda^+$ in order that a $\mathfrak{g}$-module with minimal component V_λ belongs to the discrete series ([7, theorem 4.9]). However, the condition in [7] is unnecessarily severe.

In section 3 we first give a general but rather complicated description of the set Λ_0^+. The structure of Λ_0^+ is worked out more explicitely for the following three classes of pairs $(\mathfrak{g},\mathfrak{k})$:

$$(\mathfrak{gl}\,(p+q,\mathbf{C}),\ \mathfrak{gl}\,(p,\mathbf{C})\oplus\mathfrak{gl}\,(q,\mathbf{C})), \qquad (C_{p+q}, C_p \oplus C_q)$$
$$\text{and} \qquad (D_{p+q}, D_p \oplus D_q)$$

where C_l and D_l are classical simple Lie algebras of rank l. In all cases we have studied so far it is found that

$$\Lambda_0^+ = \delta + \Lambda^+ ,$$

where $\delta = 1/N \sum_{\alpha \in \Delta \setminus \Delta_k} \alpha$ and N is an integer depending on the pair $(\mathfrak{g},\mathfrak{k})$.

To get a better idea of the methods used in the present work, the reader is recommended to look at the thesis of van den Hombergh, [4]. There all non-decomposable Harish–Chandra modules for certain real rank one pairs $(\mathfrak{g},\mathfrak{k})$ were classified using the step algebra $S(\mathfrak{g},\mathfrak{k})$.

2. Properties of step algebras.

Let $\mathfrak{k}$ be a reductive subalgebra in a complex finite dimensional Lie algebra $\mathfrak{g}$. Thus the adjoint representation $\operatorname{ad}\mathfrak{k}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is completely reducible and there is an $\operatorname{ad}\mathfrak{k}$-invariant complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ and fix a positive system Δ_k for $(\mathfrak{k},\mathfrak{h})$. Let $\{\alpha_1,\ldots,\alpha_l\}$ be a basis of Δ_k. We define $(\cdot,\cdot)$, $\langle\cdot,\cdot\rangle$, Λ and Λ^+ as in introduction. Next we define

$$\mathfrak{k}_s = [\mathfrak{k},\mathfrak{k}], \qquad \mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{k}_s .$$

For $\lambda \in \mathfrak{h}^*$ we define $\lambda^s \in \mathfrak{h}_s^*$ as the restriction of λ to the subspace $\mathfrak{h}_s \subset \mathfrak{h}$. If $\lambda \in \Lambda$ then

$$\lambda^s = \sum_{i=1}^{l} r_i \alpha_i^s \tag{1}$$

where each r_i is real and rational. If $\lambda \in \Lambda^+$ then $r_i \geqq 0$, $1 \leqq i \leqq l$. If $\lambda,\lambda' \in \Lambda$ and $(\lambda-\lambda')^s \neq 0$ then we define $\lambda > \lambda'$ if the first non-zero number in the row $r_1-r_1', r_2-r_2',\ldots$ is positive. This ordering on Λ is total if and only if $\mathfrak{k}$ is semi-simple, $\mathfrak{h}_s=\mathfrak{h}$. We define $\lambda \gg \lambda'$ if $\lambda-\lambda'$ is a sum of the simple roots α_i. Clearly $\lambda \gg \lambda'$ implies $\lambda > \lambda'$. The set $\{\lambda^s \mid \lambda \in \Lambda^+\}$ can be identified through (1) with a subset of $\mathbf{R}^l$ which is known to be nowhere dense (in the ordinary topology of

$\mathbf{R}^l$) and is bounded below by the vector 0. It follows that any subset $\Omega \subset \Lambda^+$ has a minimal element and that is unique if $\mathfrak{k}$ is semi-simple.

Let $\{t_1, \ldots, t_n\}$ be a basis in $\mathfrak{p}$ consisting of weight vectors,

$$[h, t_i] = \mu_i(h) t_i, \quad h \in \mathfrak{h}, \ 1 \leqq i \leqq n .$$

We can assume that $\mu_1^s \geqq \mu_2^s \geqq \ldots \geqq \mu_n^s$. The choice of Δ_k defines the splitting $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{h} \oplus \mathfrak{k}_-$. We denote by $U(\mathfrak{a})$ the enveloping algebra of a Lie algebra $\mathfrak{a}$. We define

$$S'(\mathfrak{g}, \mathfrak{k}) = \{u \in U(\mathfrak{g}) \mid \mathfrak{k}_+ u \subset U(\mathfrak{g})\mathfrak{k}_+\}$$

and we set $S(\mathfrak{g}, \mathfrak{k}) = S'(\mathfrak{g}, \mathfrak{k})/U(\mathfrak{g})\mathfrak{k}_+$, the step algebra of the pair $(\mathfrak{g}, \mathfrak{k})$. For each sequence $(i) = (i_1, \ldots, i_n) \in \mathbf{Z}_+^n$ we put $t(i) = t_1^{i_1} \ldots t_n^{i_n} \in U(\mathfrak{g})$. Consider the subspace $U_1 \subset U(\mathfrak{g})$,

$$U_1 = \sum_{(i)} t(i) U(\mathfrak{h}) .$$

We can split

$$U(\mathfrak{g}) = U_1 \oplus U(\mathfrak{g})\mathfrak{k}_+ \oplus U(\mathfrak{k}_-)\mathfrak{k}_- U_1 ,$$

$$U(\mathfrak{g}) = U_1 \oplus U(\mathfrak{g})\mathfrak{k}_+ \oplus U_1 U(\mathfrak{k}_-)\mathfrak{k}_- .$$

Let P' denote the projection onto the first summand in the first formula, and Q' the corresponding projection in the second formula. We define projections P, Q: $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}_+ \to U_1$ by

$$P(u + U(\mathfrak{g})\mathfrak{k}_+) = P'(u) \quad \text{and} \quad Q(u + U(\mathfrak{g})\mathfrak{k}_+) = Q'(u) .$$

For each $i \in \{1, 2, \ldots, n\}$ there exists $s_i \in S(\mathfrak{g}, \mathfrak{k})$ of the form

$$s_i = t_i p_i + \sum_{\mu_j \gg \mu_i} u_j t_j p_j ,$$

where $u_j \in U(\mathfrak{k}_-)$, $p_j \in U(\mathfrak{h})$ and $p_i \in U(\mathfrak{h})$ has the following property:

$$p_i(\lambda) \neq 0 \quad \text{if } \lambda + \mu_i + \delta_k \in \Lambda^+ ,$$

where $\delta_k = \frac{1}{2} \sum \Delta_k$; see proposition I. 1.8, page 18 in [4] and [3, proposition 1]. If $p \in U(\mathfrak{h})$ we denote by $p(\lambda)$ the value of a polynomial on $\mathfrak{h}^*$ obtained via the replacement $h \mapsto \lambda(h)$.

For each $\lambda \in \mathfrak{h}^*$ we define the left ideal

$$I_\lambda = U(\mathfrak{g})\{h - \lambda(h) \cdot 1 \mid h \in \mathfrak{h}\}$$

and for each $\lambda \in \Lambda^+$ let J_λ be the left ideal in $U(\mathfrak{k})$ which annihilates the vector with highest weight in the finite dimensional $\mathfrak{k}$-module X_λ. Let π_λ: $U(\mathfrak{g}) \to U(\mathfrak{g})/I_\lambda$ be the projection and set $P_\lambda = \pi_\lambda \circ P$, $Q_\lambda = \pi_\lambda \circ Q$. Then

$$P_\lambda(s_i) = t_i p_i(\lambda) \neq 0$$

if $\lambda+\mu_i+\delta_k \in \Lambda^+$. We say that $s \in S(\mathfrak{g},\mathfrak{k})$ has weight μ if $s=s'+U(\mathfrak{g})\mathfrak{k}_+$, where $s' \in S'(\mathfrak{g},\mathfrak{k})$ such that $[h,s']=\mu(h)s'$ for all $h \in \mathfrak{h}$. The step s_i has the weight μ_i ([4, proposition I.1.8]).

LEMMA 2.1. *Suppose $s \in S(\mathfrak{g},\mathfrak{k})$ has weight μ, and $\lambda \in \Lambda^+$. If $\lambda+\mu \notin \Lambda^+$, then $s \in U(\mathfrak{g})J_\lambda/U(\mathfrak{g})\mathfrak{k}_+$.*

PROOF. Consider the $\mathfrak{g}$-module $V=U(\mathfrak{g})/U(\mathfrak{g})J_\lambda$. It contains a finite dimensional $\mathfrak{k}$-module with highest weight λ and highest vector $x=\mathbf{1}+U(\mathfrak{g})J_\lambda$. Clearly $V=U(\mathfrak{g})x$. From [6, proposition 4.2], it follows that V is $\mathfrak{k}$-finite. The elements of $S(\mathfrak{g},\mathfrak{k})$ act in a natural way on $\mathfrak{k}_+$-extreme vectors in V. Consider the vector $y=sx$. Then

$$\mathfrak{k}_+y = 0, \quad hy = (\lambda+\mu(h))y \quad \forall h \in \mathfrak{h}\,.$$

Therefore $y=sx=0$ if $\lambda+\mu \notin \Lambda^+$. But the annihilator of x is $U(\mathfrak{g})J_\lambda$ and the assertion follows.

We say that the pair $(\mathfrak{g}, \mathfrak{k})$ is of type (A) if

$$|\langle\mu_i,\alpha_j\rangle| \leqq 1 \quad \text{for } 1\leqq i\leqq n \text{ and } 1\leqq j\leqq l\,.$$

Here again $\langle\mu,\alpha\rangle=2(\mu,\alpha)/(\alpha,\alpha)$.

LEMMA 2.2. *Let $(\mathfrak{g},\mathfrak{k})$ be of type (A) and let $\lambda \in \Lambda^+$ such that $\lambda+\mu_{i0} \in \Lambda^+$ for some $1\leqq i_0\leqq n$. Then for t_j such that $\mu_j=\mu_{i0}$ there exists*

$$r_j = \sum_{\mu_i=\mu_{i0}} a_i s_i \in S(\mathfrak{g},\mathfrak{k}) \qquad (a_i \in \mathbf{C})$$

such that $Q_\lambda(r_j)=t_j$.

PROOF. (1) Let N_λ be the Verma module for $\mathfrak{k}$ with highest weight λ. First we show that $N_{\lambda-\nu+\mu}\not\subset N_\lambda$ when $\mu=\mu_{i0}$ and $\nu\neq\mu$, $\nu \in \{\mu_1,\ldots,\mu_n\}$. Namely, if $N_{\lambda-\nu+\mu}\subset N_\lambda$ then $\lambda-\nu+\mu+\delta_k=w(\lambda+\delta_k)$ for some w in the Weyl group of $\mathfrak{k}$. Now $\lambda \in \Lambda^+$ so $w(\lambda+\delta_k) \notin \Lambda^+$ for $w\neq 1$ and $\langle w(\lambda+\delta_k), \alpha_i\rangle<0$ for some $1\leqq i\leqq l$. But $\langle\lambda+\mu,\alpha_i\rangle\geqq 0$ so

$$\begin{aligned} 0 > \langle w(\lambda+\delta_k),\alpha_i\rangle &= \langle\lambda+\mu,\alpha_i\rangle+\langle\delta_k,\alpha_i\rangle-\langle\nu,\alpha_i\rangle \\ &\geqq 1-\langle\nu,\alpha_i\rangle \geqq 0\,, \end{aligned}$$

a contradiction.

(2) Let $S_\mu\subset S(\mathfrak{g},\mathfrak{k})$ be the subspace spanned by the vectors s_i with $\mu_i=\mu_{i0}=\mu$.

If $s \in S_\mu$, we can write

$$(*) \qquad s = \sum_{\mu_i=\mu} a_i s_i = \sum_{\mu_i=\mu} a_i t_i p_i + \sum_{\mu_i \gg \mu} v_i t_i p_i$$

where $p_i \in U(\mathfrak{h})$ such that $p_i(\lambda) \neq 0$ for $\mu_i = \mu$ and $v_i \in U(\mathfrak{k}_-)\mathfrak{k}_-$. Thus

$$Q_\lambda(S_\mu) \subset \mathfrak{p}_\mu ,$$

where $\mathfrak{p}_\mu$ consists of vectors with weight μ in $\mathfrak{p}$. Because $\dim S_\mu = \dim \mathfrak{p}_\mu$, we only have to show that the mapping Q_λ: $S_\mu \to \mathfrak{p}_\mu$ is injective. If the second term in (*) is in I_λ then

$$Q_\lambda(s) = \sum_{\mu_i=\mu} a_i t_i p_i(\lambda)$$

and $Q_\lambda(s) = 0$ implies that all $a_i = 0$, and thus $s = 0$. In other cases we write

$$s = \sum_{\mu_i=\mu} t_i p_i' + \sum_{\substack{\mu_i \gg \mu \\ i \neq j_0}} t_i v_i' p_i' + t_{j_0} v_{j_0} p_{j_0}$$

where again $p_i' \in U(\mathfrak{h})$, $v_i' \in U(\mathfrak{k}_-)\mathfrak{k}_-$ and μ_{j_0} is a minimal weight such that $v_{j_0} p_{j_0} \notin I_\lambda$. If now

$$Q_\lambda(s) = \sum_{\mu_i=\mu} t_i p_i'(\lambda) = 0$$

then $s \in U(\mathfrak{g})J_\lambda$ by [7, lemma 4.4], or [4, proposition II.2.12]. From $\mathfrak{k}_+ s \subset U(\mathfrak{g})\mathfrak{k}_+$ it follows that

$$\mathfrak{k}_+ v_{j_0} \subset U(\mathfrak{k})\mathfrak{k}_+ + I_\lambda .$$

But $\operatorname{ad} h(v_{j_0}) = (\mu - \mu_{j_0})(h)$ for $h \in \mathfrak{h}$ and this implies $N_{\lambda - \mu_{j_0} + \mu} \subset N_\lambda$, a contradiction with (1). Thus $Q_\lambda(s) \neq 0$ for $s \neq 0$, $s \in S_\mu$.

LEMMA 2.3. *Let $s \in S(\mathfrak{g}, \mathfrak{k})$ be of weight μ and let $\lambda \in \Lambda$ such that $\lambda + \mu \in \Lambda^+$ and $P_\lambda(s) = 0$. Then $s \in I_\lambda / U(\mathfrak{g})\mathfrak{k}_+$.*

PROOF. First we write

$$s = \sum_{\mu(i)=\mu} t(i)p(i) + \sum_{\mu(i) \gg \mu} v(i)t(i)p(i)$$

where $p(i) \in U(\mathfrak{h})$, $v(i) \in U(\mathfrak{k}_-)\mathfrak{k}_-$ and $\mu(i) = i_1\mu_1 + \ldots + i_n\mu_n$. Then $p(i)(\lambda) = 0$ for $\mu(i) = \mu$. If $s \notin I_\lambda$ then we can choose a minimal $\mu(i_0) \gg \mu$ such that $p(i_0)(\lambda) \neq 0$. From $\mathfrak{k}_+ s \subset U(\mathfrak{g})\mathfrak{k}_+$ follows that

$$\mathfrak{k}_+ v(i_0) \subset U(\mathfrak{k})\mathfrak{k}_+ + I_{\lambda + \mu(i_0)} .$$

Thus $v(i_0)$ is $\mathfrak{k}_+$-extreme with weight $\lambda + \mu$ in the Verma module

$$N_{\lambda + \mu(i_0)} = U(\mathfrak{k}) / (U(\mathfrak{k})\mathfrak{k}_+ + I_{\lambda + \mu(i_0)}) .$$

It follows that

$$N_{\lambda+\mu} \subset N_{\lambda+\mu(i_0)} .$$

There is an element w in the Weyl group of $\mathfrak{k}$ such that $\lambda+\mu+\delta_k=w(\lambda+\mu(i_0)+\delta_k)$ so

$$\lambda+\mu(i_0)+\delta_k = w^{-1}(\lambda+\mu+\delta_k) \ll \lambda+\mu+\delta_k$$

because of $\lambda+\mu \in \Lambda^+$. But this inequality is in contradiction with $\mu(i_0)\gg\mu$.

Let $S_0(\mathfrak{g},\mathfrak{k})$ be the subalgebra of $S(\mathfrak{g},\mathfrak{k})$ which is generated by the s_i's and $U(\mathfrak{h})$. We define $S^k\subset S_0(\mathfrak{g},\mathfrak{k})$ as the subspace of elements which are at most of degree k in the variables s_i. We set

$$S^k(\mu) = \{s \in S^k \mid s \text{ is of weight } \mu\} ,$$

$$S_+(\mu) = S^1(\mu)+\left\{s \in S^2(\mu) \;\middle|\; s = \sum_{\mu_i^s\leqq\mu_j^s} s_is_jp_{ij};\ p_{ij} \in U(\mathfrak{h})\right\} ,$$

$$S_-(\mu) = S^1(\mu)+\left\{s \in S^2(\mu) \;\middle|\; s = \sum_{\mu_i^s\geqq\mu_j^s} s_is_jp_{ij};\ p_{ij} \in U(\mathfrak{h})\right\} ,$$

$$S_\pm(\lambda,\mu) = (S_\pm(\mu)+U(\mathfrak{g})J_\lambda)/U(\mathfrak{g})J_\lambda, \quad \lambda \in \Lambda^+ .$$

LEMMA 2.4. *Let* $(\mathfrak{g},\mathfrak{k})$ *be of type* (A), $\lambda \in \Lambda^+$, *and* $\mu \in \Lambda$ *such that* $\lambda+\mu \in \Lambda^+$. *Then* $S^2(\mu)\subset S_+(\mu)+I_\lambda$.

PROOF. Because of lemma 2.3 it is sufficient to show that for any t_it_j with $\mu_i+\mu_j=\mu$ there is $s_{ij} \in S_+(\mu)$ such that $P_\lambda(s_{ij})=t_it_j$. For any t_i with $\mu_i=\mu$ there is $s_i \in S^1(\mu)$ such that $P_\lambda(s_i)=t_ip_i(\lambda)$ where $p_i(\lambda)\neq 0$. Thus we can forget the first order terms. Now $t_it_j\equiv t_jt_i \bmod \mathfrak{g}$ so we can assume for example that $\mu_i^s\leqq\mu_j^s$. We prove the existence of s_{ij} by induction on j. The assertion is true for $j=1$ because $s_1=t_1$ and

$$P_\lambda(s_is_1) = P_{\lambda+\mu_1}(s_i)t_1 = t_it_1p_i(\lambda+\mu_1) .$$

Now $\lambda+\mu_1+\mu_i+\delta_k=\lambda+\mu+\delta_k\subset\Lambda^+$ so $p_i(\lambda+\mu_1)\neq 0$ and we can set

$$s_{i1} = (p_i(\lambda+\mu_1))^{-1}s_is_1 .$$

Suppose that the assertion is true for $j=k$. But

$$P_\lambda(s_is_{k+1}) = t_it_{k+1}p_i(\lambda+\mu_{k+1})p_{k+1}(\lambda)+ \sum_{\substack{\mu_r+\mu_s=\mu\\ \mu_s\gg\mu_{k+1}}} t_rt_sa_{rs}$$

where $a_{rs} \in \mathbf{C}$. By the induction hypothesis, there is $s \in S_+(\mu)$ such that $P_\lambda(s)$ is equal to the last term in the above formula. If $a = p_i(\lambda+\mu_{k+1})p_{k+1}(\lambda) \neq 0$ we can define

$$s_{ik+1} = a^{-1}(s_i s_{k+1} - s) .$$

The first factor $p_i(\lambda+\mu_{k+1}) \neq 0$ for the same reason as before. As for the second,

$$\langle \lambda+\mu_{k+1}+\delta_k, \alpha_m \rangle = \langle \lambda, \alpha_m \rangle + \langle \mu_{k+1}, \alpha_m \rangle + 1 \geqq \langle \lambda, \alpha_m \rangle$$

for a pair $(\mathfrak{g}, \mathfrak{k})$ of type (A), so $\lambda+\mu_{k+1}+\delta_k \in \Lambda^+$ and therefore $p_{k+1}(\lambda) \neq 0$.

LEMMA 2.5. *Let $(\mathfrak{g}, \mathfrak{k})$ be of type (A). Let $\lambda \in \Lambda^+$, $\mu \in \Lambda$ such that $\lambda+\mu \in \Lambda^+$. Let $n_+(\lambda, \mu)$ (respectively $n_-(\lambda, \mu)$) be the number of pairs (t_i, t_j) such that $\mu = \mu_i + \mu_j$, $\mu_i^s \leqq \mu_j^s$ and $\lambda+\mu_j \in \Lambda^+$ (respectively $\lambda+\mu_i \in \Lambda^+$). If $n_+(\lambda, \mu) \leqq n_-(\lambda, \mu)$ then $S_+(\lambda, \mu) = S_-(\lambda, \mu)$.*

PROOF. From lemma 2.4 it follows that $S_-(\lambda, \mu) \subset S_+(\lambda, \mu)$. All we need to show is $\dim S_-(\lambda, \mu) \geqq \dim S_+(\lambda, \mu)$.

From lemma 2.1 follows immediately the inequality

$$\dim S_+(\lambda, \mu) \leqq n_+(\lambda, \mu) + \dim \left((S^1(\mu) + U(\mathfrak{g})J_\lambda)/U(\mathfrak{g})J_\lambda \right) .$$

For each pair (t_i, t_j) such that $\mu_i + \mu_j = \mu$, $\mu_i^s \leqq \mu_j^s$ and $\lambda+\mu_i \in \Lambda^+$ we choose $r_i, r'_j \in S^1$ such that $Q_\lambda(r_i) = t_i$ and $Q_{\lambda+\mu_i}(r'_j) = t_j$ (see lemma 2.2). To show that

$$\dim S_-(\lambda, \mu) \geqq n_-(\lambda, \mu) + \dim \left((S^1(\mu) + U(\mathfrak{g})J_\lambda)/U(\mathfrak{g})J_\lambda \right) .$$

we prove that the elements $r'_j r_i$ are linearly independent in $S_-(\lambda, \mu)$. Suppose that

$$s = \sum a_{ij} r'_j r_i \in U(\mathfrak{g})J_\lambda \qquad (a_{ij} \in \mathbf{C}) .$$

Then $Q_\lambda(s) = 0$. Let $a_{i_0 j_0} \neq 0$ but $a_{ij} = 0$ when $i < i_0$. Then

$$Q_\lambda(s) = a_{i_0 j_0} t_{j_0} t_{i_0} + \sum_{\substack{\mu_j \gg \mu_{j_0} \\ \mu_i \ll \mu_{i_0}}} b_{ij} t_j t_i \neq 0 ,$$

a contradiction. Thus all $a_{ij} = 0$ and the $r'_j r_i$'s are linearly independent in $S_-(\lambda, \mu)$.

3. Discrete series.

We denote

$$\Lambda_0^+ = \{ \lambda \in \Lambda^+ \mid n_+(\lambda, \mu_i+\mu_j) \leqq n_-(\lambda, \mu_i+\mu_j) \ \forall\, i, j \text{ such that } \mu_i < 0 < \mu_j \text{ and } \lambda+\mu_i+\mu_j \in \Lambda^+ \} .$$

We say that Λ_0^+ is stable if

$$(\Lambda_0^+ + \mu_k) \cap \Lambda^+ \subset \Lambda_0^+ \qquad \forall\, \mu_k > 0 .$$

As we shall see later, in many interesting cases Λ_0^+ is in fact stable.

A $\mathfrak{g}$-module V is said to be $\mathfrak{k}$-finite if it is a sum of finite dimensional $\mathfrak{k}$-modules. If $\lambda \in \Lambda^+$ then V_λ denotes the sum of all $\mathfrak{k}$-submodules in V with highest weight λ. We set

$$V_\lambda^+ = \{x \in V_\lambda \mid \mathfrak{k}_+ x = 0\} .$$

Let D be the centralizer of $\mathfrak{h}$ in $S_0(\mathfrak{g},\mathfrak{k})$, i.e. the subalgebra of elements with weight zero. We set

$$A_{\beta,\alpha} = \{u \in U(\mathfrak{g}) \mid uV_\alpha^+ \subset V_\beta^+ \text{ for any } \mathfrak{g}\text{-module } V\} ,$$

$$M_\alpha = \sum_{\beta<\alpha} A_{\beta,\alpha} ,$$

$$D_\alpha = D/D \cap U(\mathfrak{g})M_\alpha ,$$

$$R_+ = \{\mu_i \mid \mu_i > 0\}, \qquad R_- = \{\mu_i \mid \mu_i < 0\} .$$

If V is a $\mathfrak{g}$-module such that $V_\alpha = 0$ for $\alpha < \lambda$, then V_λ^+ is in a natural way a D_λ-module. In [8, theorem 1], it was shown that the mapping $V \mapsto V_\lambda^+$ determines a bijection from the set of equivalence classes of irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules for which $V_\lambda \neq 0$ and $V_\alpha = 0$ if $\alpha < \lambda$, onto the set of equivalence classes of irreducible D_λ-modules.

THEOREM 3.1. *Let* $(\mathfrak{g},\mathfrak{k})$ *be a pair of type* (A) *such that* $\mu_i^s \neq 0\ \forall\, i \in \{1,2,\ldots,n\}$ *(if* $\mathfrak{k}$ *is semi-simple the last condition is equivalent with* $\operatorname{rank}\mathfrak{k} = \operatorname{rank}\mathfrak{g}$*). In addition, we assume that* Λ_0^+ *is stable. Then for each* $\lambda \in \Lambda_0^+$ *there is one and only one equivalence class* $[V]$ *of irreducible* $\mathfrak{k}$*-finite* $\mathfrak{g}$*-modules such that* $V_\lambda \neq 0$ *but* $V_\alpha = 0$ *for* $\alpha < \lambda$*. Furthermore,* $\dim V_\lambda^+ = 1$, *and* $\dim V_\alpha^+ \leqq$ *the number of sequences* $\{\mu_{i_1},\ldots,\mu_{i_p}\}$ *of elements in* R_+ *such that* $\mu_{i_1} + \ldots + \mu_{i_p} + \lambda = \alpha$.

PROOF. We shall show that $D_\lambda \cong \mathbf{C}$ from which the first assertion follows immediately using the remark above. A general element in D is a linear combination of vectors of type $s = s_{i_1} \ldots s_{i_p} u$, when $u \in U(\mathfrak{h})$ and

$$(*) \qquad \mu_{i_1} + \ldots + \mu_{i_p} = 0 .$$

From $\mu_i^s \neq 0$ follows that either $\mu_i > 0$ or $\mu_i < 0$. Now $I_\lambda \subset M^{\lambda\lambda}$, thus u is a complex number modulo M_λ. We shall show by induction on p that each $s_{i_1} \ldots s_{i_p} u$ is a complex number modulo M_λ. We saw already that this is the case for $p = 0$. Suppose that it is true for $p = k$ and let us consider the case $p = k+1$. If $\mu_{i_p} < 0$ then $s_{i_p} \in M_\lambda$ and $s \in D \cap U(\mathfrak{g})M_\lambda$. Suppose then that $\mu_{i_p} > 0$. By

(*) there is a last index i_m for which $\mu_{i_m}<0$. Let $\nu=\lambda+\mu_{i_p}+\ldots+\mu_{i_{m+2}}$. We may assume that $\nu\in\Lambda^+$; otherwise

$$s_{i_{m+2}}\ldots s_{i_p}\in U(\mathfrak{g})J_\lambda\subset U(\mathfrak{g})M_\lambda$$

and therefore $s\in D\cap U(\mathfrak{g})M_\lambda$. Then

$$s_{i_m}s_{i_{m+1}}=\sum_{\mu_i^s\geqq\mu_j^s}a_{ij}s_is_j+r$$

where $a_{ij}\in\mathbf{C}$ and $r\in S^1+U(\mathfrak{g})J_\nu$ (lemma 2.5) and a_{ij} can be different from zero only when $\mu_i+\mu_j=\mu_{i_m}+\mu_{i_{m+1}}$. If $v\in U(\mathfrak{g})J_\nu$ then $vs_{i_{m+2}}\ldots s_{i_p}\in U(\mathfrak{g})J_\lambda$, thus

$$s=\sum_{\mu_i^s\geqq\mu_j^s}a_{ij}s_{i_1}\ldots s_{i_{m-1}}s_is_js_{i_{m+2}}\ldots s_{i_p}+r'$$

where $r'\in S^{p-1}+U(\mathfrak{g})M_\lambda$. Consider a typical term

$$s'=s_{i_1}\ldots s_{i_{m-1}}s_is_js_{i_{m+2}}\ldots s_{i_p}.$$

If $\mu_j>0$ then $\mu_i^s\geqq\mu_j^s$ implies $\mu_i>0$ and we have reduced the number of factors s_h with negative weight μ_h by one (compared with s). If $\mu_j<0$ then we can consider $s_js_{i_{m+2}}$ instead of $s_{i_m}s_{i_{m+1}}$ and continue as above. Noting that a s_h with $\mu_h<0$ on the right gives zero modulo M_λ, we can finally write

$$s=q_1+q_2$$

where $q_2\in S^{p-1}+M_\lambda$ and q_1 is a linear combination of monomials of degree p, each of them containing one less factors s_h with negative weight μ_h than the original monomial s. We can make a second induction on the number of negative factors and we arrive at $s=w+q$, where $q\in S^{p-1}+M_\lambda$ and w contains no negative factors. Because w is of weight zero it contains no positive factors either, and therefore $w\in U(\mathfrak{h})$, which implies $w\in\mathbf{C}\cdot\mathbf{1}+I_\lambda$. By the first induction, $q\in\mathbf{C}\cdot\mathbf{1}+D\cap U(\mathfrak{g})M_\lambda$, and thus $s\in\mathbf{C}\cdot\mathbf{1}$ modulo M_λ. We have now shown that $D_\lambda\cong\mathbf{C}$. From this and the fact that V_λ^+ is an irreducible D_λ-module it follows that $\dim V_\lambda^+=1$.

Let $0\neq x\in V_\lambda^+$ and $y\in V_\alpha^+$. Then by [4, corollary II.1.5 p. 29], there exists $s\in S_0(\mathfrak{g},\mathfrak{k})$ such that $y=sx$. Using the same technique as above we can eliminate all factors s_i with $\mu_i<0$ from s. Thus

$$s\equiv\sum_{p=0}^{k}\sum_{\substack{\mu_{i_1}+\ldots+\mu_{i_p}=\alpha-\lambda\\ \mu_{i_1},\ldots,\mu_{i_p}>0}}a(i_1,\ldots,i_p)s_{i_1}\ldots s_{i_p}\mod M_\lambda,$$

where $a(i_1,\ldots,i_p)\in\mathbf{C}$. This proves the last assertion.

Next we shall describe explicitely the set Λ_0^+ for three classes of classical reductive Lie algebras with respect to a reductive subalgebra of equal rank. Looking at the root space structure of the Lie algebras A_l, C_l and D_l (see e.g. [5]) it is easily seen that these pairs are of type (A). In each case we shall see that Λ_0^+ is stable so that theorem 3.1 applies.

a) $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(p+q, \mathbf{C}), \mathfrak{gl}(p, \mathbf{C}) \oplus \mathfrak{gl}(q, \mathbf{C}))$.

The Lie algebra $\mathfrak{g} = \mathfrak{gl}(p+q, \mathbf{C})$ consists of complex $(p+q) \times (p+q)$-matrices. We define e_{ij} as the matrix for which

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl},$$

where $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$. We define $\mathfrak{gl}(p, \mathbf{C})$ as the subalgebra generated by the elements e_{ij}, $1 \leqq i, j \leqq p$. The subalgebra $\mathfrak{gl}(q, \mathbf{C})$ is spanned by the elements e_{ij}, $p+1 \leqq i, j \leqq p+q$. We set $\mathfrak{k} = \mathfrak{gl}(p, \mathbf{C}) \oplus \mathfrak{gl}(q, \mathbf{C})$. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$ of $\mathfrak{g}$ is spanned by the diagonal matrices e_{ii}, $1 \leqq i \leqq p+q$. The semi-simple part $\mathfrak{k}_s$ consists of trace zero matrices, $\mathfrak{k}_s = \mathfrak{sl}(p, \mathbf{C}) \oplus \mathfrak{sl}(q, \mathbf{C})$ and $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{k}_s$. A positive system Δ_k for $(\mathfrak{k}, \mathfrak{h})$ is defined by setting

$$\mathfrak{k}_+ = \sum_{1 \leqq i < j \leqq p} \mathbf{C} \cdot e_{ij} + \sum_{p+1 \leqq i < j \leqq p+q} \mathbf{C} \cdot e_{ij}.$$

Then $\mathfrak{k}_-$ is obtained by transposing the matrices in $\mathfrak{k}_+$. The simple roots $\alpha_1, \ldots, \alpha_{p+q-2}$ correspond to the vectors $e_{12}, e_{23}, \ldots, e_{p-1,p}, e_{p+1,p+2}, \ldots, e_{p+q-1,p+q}$. If $\lambda \in \mathfrak{h}^*$, we denote $\lambda_i = \lambda(e_{ii})$. The set of weights Λ consists of those $\lambda \in \mathfrak{h}^*$ for which the numbers $\lambda_i - \lambda_j$ $(1 \leqq i, j \leqq p)$ and $\lambda_k - \lambda_l$ $(p+1 \leqq k, l \leqq p+q)$ are all real integers. The dominant integral weights are given by

$$\Lambda^+ = \{\lambda \in \Lambda \mid \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_p;\ \lambda_{p+1} \geqq \lambda_{p+2} \geqq \ldots \geqq \lambda_{p+q}\}.$$

An ad $\mathfrak{k}$-invariant complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is spanned by the vectors

$$e_{ij}, e_{ji}; \quad 1 \leqq i \leqq p,\ p+1 \leqq j \leqq p+q.$$

We define

$$\lambda^j = \frac{1}{j} \sum_{k=1}^{j} \lambda_k - \frac{1}{p-j} \sum_{k=j+1}^{p} \lambda_k \quad \text{when } 1 \leqq j \leqq p-1;$$

$$\lambda^j = \frac{1}{j-p+1} \sum_{k=p+1}^{j+1} \lambda_k - \frac{1}{q-j+p-1} \sum_{k=j+2}^{p+q} \lambda_k \quad \text{when } p \leqq j \leqq p+q-2.$$

Then

$$\lambda^s = \sum_{j=1}^{p+q-2} \lambda^j \alpha_j^s,$$

where $\lambda^s = \lambda|_{\mathfrak{h}_s}$. As before, $\lambda > \lambda'$ if the first non-zero number in the sequence $\lambda^1 - \lambda'^1, \lambda^2 - \lambda'^2, \ldots$ is positive.

We denote the root corresponding to e_{ij} by α_{ij}. Then

$$R_- = \{\alpha_{j1} \mid p+1 \leqq j \leqq p+q\} \cup \{\alpha_{ij} \mid 2 \leqq i \leqq p,\ p+1 \leqq j \leqq p+q\},$$
$$R_+ = \{\alpha_{1j} \mid p+1 \leqq j \leqq p+q\} \cup \{\alpha_{ji} \mid 2 \leqq i \leqq p,\ p+1 \leqq j \leqq p+q\}.$$

THEOREM 3.2. *For the pair* $(\mathfrak{g}, \mathfrak{k})$ *defined above,* $\Lambda_0^+ = \{\lambda \in \Lambda^+ \mid \lambda_1 - \lambda_2 > 0\}$. *This set is stable.*

PROOF. We have to show that for each $\nu = \alpha_- + \alpha_+$, where $\alpha_- \in R_-$ and $\alpha_+ \in R_+$, $n_-(\lambda, \nu) \geqq n_+(\lambda, \nu)$ for all $\lambda \in \Lambda^+$ such that $\lambda_1 > \lambda_2$. As an example we shall consider the case $\nu = 0$. The other cases are treated in similar manner and are left to the reader. When $\nu = 0$ the number $n_-(\lambda, \nu)$ (respectively $n_+(\lambda, \nu)$) will be equal to the number $n_-(\lambda)$ (respectively $n_+(\lambda)$) of the roots $\alpha_{ij} \in R_-$ (respectively $\alpha_{ij} \in R_+$) such that $\lambda + \alpha_{ij} \in \Lambda^+$. If we denote $\lambda' = \lambda + \alpha_{ij}$, then $\lambda'_k = \lambda_k$ for $k \neq i, j$, $\lambda'_i = \lambda_i + 1$ and $\lambda'_j = \lambda_j - 1$. Thus $\lambda' \in \Lambda^+$ iff $\lambda_{i-1} > \lambda_i$ and $\lambda_j > \lambda_{j+1}$. Let $n_1(\lambda)$ be the number of indices $2 \leqq i \leqq p-1$ for which $\lambda_i > \lambda_{i+1}$ and let $n_2(\lambda)$ be the number of indices $p+1 \leqq j \leqq p+q-1$ for which $\lambda_j > \lambda_{j+1}$. It is easily seen that

$$n_+(\lambda) = (n_1(\lambda)+2)(n_2(\lambda)+1) \qquad \text{for all } \lambda \in \Lambda^+;$$
$$n_-(\lambda) = (n_1(\lambda)+2)(n_2(\lambda)+1), \qquad \lambda \in \Lambda^+,\ \lambda_1 > \lambda_2;$$
$$n_-(\lambda) = n_1(\lambda) \cdot (n_2(\lambda)+1), \qquad \lambda \in \Lambda^+,\ \lambda_1 = \lambda_2.$$

Therefore $n_-(\lambda) \geqq n_+(\lambda)$ iff $\lambda_1 > \lambda_2$. The stability of Λ_0^+ follows from the fact that $\lambda'_1 - \lambda'_2 = \lambda_1 - \lambda_2$ or $\lambda'_1 - \lambda'_2 = \lambda_1 - \lambda_2 + 1$ for any $\alpha_{ij} \in R_+$.

b) $(\mathfrak{g}, \mathfrak{k}) = (C_{p+q}, C_p \oplus C_q)$.

Let γ be the $(2p+2q) \times (2p+2q)$-matrix defined by

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i \neq -j \\ 1 & \text{if } i = -j < 0; \\ -1 & \text{if } i = -j > 0 \end{cases} \qquad i, j = \pm 1, \pm 2, \ldots, \pm(p+q).$$

Then the classical Lie algebra $\mathfrak{g} = C_{p+q}$ consists of complex $(2p+2q) \times (2p+2q)$-matrices a such that $a^t\gamma + \gamma a = 0$ (a^t is the transpose of a). A basis

$$\{f_{ij} \mid i, j = \pm 1, \ldots, \pm(p+q);\ |i| \leqq |j|\}$$

for $\mathfrak{g}$ can be chosen in such a way that

$$[f_{ij}, f_{kl}] = \gamma_{ik} f_{jl} + \gamma_{il} f_{jk} + \gamma_{jk} f_{il} + \gamma_{jl} f_{ik}$$

where we have defined the auxiliary vectors $f_{ij}=f_{ji}$ for $|i|>|j|$. A subalgebra C_p is spanned by the vectors f_{ij} where $|i|,|j|\leqq p$ and there is a subalgebra C_q spanned by the elements f_{kl}, $\cdot|k|,|l|\geqq p+1$. We define $\mathfrak{k}=C_p\oplus C_q$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{k}$ is spanned by the vectors $h_i=f_{i,-i}$, $i=1,2,\ldots,p+q$.

We denote the root corresponding to f_{ij} $(i\neq -j)$ by α_{ij}. We set

$$\alpha_1 = \alpha_{1-2},\ \alpha_2 = \alpha_{2-3},\ldots,\alpha_{p-1} = \alpha_{p-1,-p},\ \alpha_p = \alpha_{pp}\,,$$

$$\alpha_{p+1} = \alpha_{p+1,-(p+2)},\ldots,\alpha_{p+q-1} = \alpha_{p+q-1,-(p+q)},\ \alpha_{p+q} = \alpha_{p+q,p+q}\,.$$

Then $\{\alpha_1,\ldots,\alpha_{p+q}\}$ is a set of simple roots for $(\mathfrak{k},\mathfrak{h})$ and

$$\Delta_k = \{\alpha_{ij} \mid |i|\leqq|j|,\ i>0;\ |i|,|j|\leqq p \text{ or } |i|,|j|\geqq p+1\}\,.$$

Now

$$\Lambda = \{\lambda\in\mathfrak{h}^* \mid \lambda(h_i)\in\mathbf{Z}\ \forall i\}\,.$$

We define $\lambda_i=\lambda(h_i)$. Then

$$\Lambda^+ = \{\lambda\in\Lambda \mid \lambda_1\geqq\ldots\geqq\lambda_p\geqq 0;\ \lambda_{p+1}\geqq\ldots\geqq\lambda_{p+q}\geqq 0\}\,.$$

Next we set

$$\lambda^i = \sum_{k=1}^{i}\lambda_k \quad \text{for } 1\leqq i\leqq p-1; \qquad \lambda^p = \tfrac{1}{2}(\lambda_p+\lambda_{p-1});$$

$$\lambda^i = \sum_{k=p+1}^{i}\lambda_k \quad \text{for } p+1\leqq i\leqq p+q-1;$$

$$\lambda^{p+q} = \tfrac{1}{2}(\lambda_{p+q}+\lambda_{p+q-1})\,.$$

Then

$$\lambda = \sum_{i=1}^{p+q}\lambda^i\alpha_i, \qquad \lambda\in\Lambda\,,$$

and $\lambda>\lambda'$ if $\lambda\neq\lambda'$ and the first non-zero number in the sequence $\lambda_1-\lambda_1'$, $\lambda_2-\lambda_2',\ldots$ is positive. Here

$$R_+ = \{\alpha_{ij} \mid 1\leqq i\leqq p,\ |j|\geqq p+1\}\,,$$

$$R_- = \{\alpha_{ij} \mid -p\leqq i\leqq -1,\ |j|\geqq p+1\}\,.$$

Let $\varphi\colon R_+\to R_-$ be the bijection defined by $\varphi(\alpha_{1j})=\alpha_{-pj}$, $\varphi(\alpha_{ij})=\alpha_{-(i-1),j}$ for $2\leqq i\leqq p$, $|j|\geqq p+1$. If $\lambda\in\Lambda^+$, $\lambda_p>0$, then it is easily seen that $\lambda+\alpha_{ij}\in\Lambda^+$ iff $\lambda+\varphi(\alpha_{ij})\in\Lambda^+$ for any $\alpha_{ij}\in R_+$. Let $\nu=\alpha_-+\alpha_+$ where $\alpha_-\in R_-$ and $\alpha_+\in R_+$. If $\nu=0$, $\alpha_{ij}\in R_+$, then $\alpha_{kl}+\alpha_{ij}=\nu$ iff $k=-i$ and $l=-j$. In that case, for any $(\alpha_{-i,-j},\alpha_{ij})$ such that $\lambda+\alpha_{ij}\in\Lambda^+$, there is $(-\varphi(\alpha_{ij}),\varphi(\alpha_{ij}))$ with $\lambda+\varphi(\alpha_{ij})\in\Lambda^+$, where $\lambda\in\Lambda^+$, $\lambda_p>0$. Thus $n_-(\lambda,\nu)=n_+(\lambda,\nu)$ when $\nu=0$, $\lambda_p>0$. The case $\nu=\alpha_{-il}+\alpha_{ij}$ $(l\neq -j)$ is treated in the same way. If $\nu=\alpha_{kl}+\alpha_{ij}$ where

$k \neq -i$ and $\lambda, \lambda+\nu \in \Lambda$ then $\lambda+\alpha_{ij} \in \Lambda^+$ iff $\lambda+\alpha_{kj} \in \Lambda^+$. It follows that for each pair $(\alpha_{kl}, \alpha_{ij})$ such that $\nu=\alpha_{kl}+\alpha_{ij}$ and $\lambda+\alpha_{ij} \in \Lambda^+$ there corresponds a pair $(\alpha_{kj}, \alpha_{il})$ such that $\nu=\alpha_{kj}+\alpha_{il}$ and $\lambda+\alpha_{kj} \in \Lambda^+$. We have now shown that $n_-(\lambda,\nu)=n_+(\lambda,\nu)$ for all $(\lambda,\nu) \in \Lambda^+ \times (R_- + R_+)$ such that $\lambda+\nu \in \Lambda^+$, $\lambda_p>0$. Noting that $\lambda'_p=\lambda_p$ or $\lambda'_p=\lambda_p+1$ when $\lambda'=\alpha_{ij}+\lambda$ and $\alpha_{ij} \in R_+$, we get the following result:

THEOREM 3.3. *For the pair* $(\mathfrak{g},\mathfrak{k})=(C_{p+q}, C_p \oplus C_q)$, Λ_0^+ *is equal to the set* $\{\lambda \in \Lambda_p^+ \mid \lambda_p>0\}$ *and it is stable.*

c) $(\mathfrak{g},\mathfrak{k})=(D_{p+q}, D_p \oplus D_q)$.

If we think of D_n as the Lie algebra of complex antisymmetric $2n \times 2n$-matrices, then we have the following subalgebras in $\mathfrak{g}=D_{p+q}$:

$$D_p = \{a \in \mathfrak{g} \mid a_{ij}=0 \text{ when } i>2p \text{ or } j>2p\},$$

$$D_q = \{a \in \mathfrak{g} \mid a_{ij}=0 \text{ when } i \leqq 2p \text{ or } j \leqq 2p\}.$$

We set $\mathfrak{k}=D_p \oplus D_q$. Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$ and let

$$\{\alpha_{ij} \mid |i|<|j|;\ i,j=\pm 1, \pm 2, \ldots, \pm(p+q)\}$$

be the set of roots for $(\mathfrak{g},\mathfrak{h})$ such that

$$\{\alpha_{ij} \mid |i|<|j| \leqq p\} \cup \{\alpha_{ij} \mid |j|>|i| \geqq p+1\}$$

is the set of roots for $(\mathfrak{k},\mathfrak{h})$. There exists a basis $\{h_1, \ldots, h_{p+q}\}$ in $\mathfrak{h}$ such that

$$\alpha_{ij}(h_k) = \delta_{ik}+\delta_{jk}-\delta_{-ik}-\delta_{-jk}.$$

As the set of simple roots for $(\mathfrak{k},\mathfrak{h})$ we take $\{\alpha_1, \ldots, \alpha_{p+q}\}$ where

$$\alpha_i = \alpha_{i,-(i+1)} \quad \text{when } 1 \leqq i \leqq p-1 \text{ or } p+1 \leqq i \leqq p+q-1,$$

$$\alpha_p = \alpha_{p-1,p}, \quad \alpha_{p+q} = \alpha_{p+q-1,p+q}.$$

We denote $\lambda_i=\lambda(h_i)$. Then

$$\Lambda = \{\lambda \in \mathfrak{h}^* \mid \lambda_i \in \mathbf{Z}\ \forall 1 \leqq i \leqq p \text{ or } \lambda_i+\tfrac{1}{2} \in \mathbf{Z}\ \forall 1 \leqq i \leqq p;$$
$$\lambda_i \in \mathbf{Z}\ \forall p+1 \leqq i \leqq p+q \text{ or } \lambda_i+\tfrac{1}{2} \in \mathbf{Z}\ \forall p+1 \leqq i \leqq p+q\},$$

$$\Lambda^+ = \{\lambda \in \Lambda \mid \lambda_1 \geqq \ldots \geqq \lambda_{p-1} \geqq \lambda_p \geqq -\lambda_{p-1};$$
$$\lambda_{p+1} \geqq \ldots \geqq \lambda_{p+q-1} \geqq \lambda_{p+q} \geqq -\lambda_{p+q-1}\}.$$

Any $\lambda \in \Lambda$ can be written in the form $\lambda=\sum \lambda^i \alpha_i$, where

$$\lambda^i = \sum_{j=1}^{i} \lambda_j, \quad 1 \leqq i \leqq p-2,$$

$$\lambda^{p-1} = \tfrac{1}{2}\left(\sum_{j=1}^{p-1} \lambda_j - \lambda_p\right), \quad \lambda^p = \tfrac{1}{2}\sum_{j=1}^{p} \lambda_j ,$$

$$\lambda^i = \sum_{j=p+1}^{i} \lambda_j, \quad p+1 \leqq i \leqq p+q-2 ,$$

$$\lambda^{p+q-1} = \tfrac{1}{2}\left(\sum_{j=p+1}^{p+q-1} \lambda_j - \lambda_{p+q}\right), \quad \lambda^{p+q} = \tfrac{1}{2}\sum_{j=p+1}^{p+q} \lambda_j .$$

We define again a lexicographical ordering "$<$" in Λ with respect to the basis $\{\alpha_1,\ldots,\alpha_{p+q}\}$. Now

$$R_+ = \{\alpha_{ij} \mid 1 \leqq i \leqq p-1 \text{ or } i=-p;\ |j| \geqq p+1\} ,$$

$$R_- = \{\alpha_{ij} \mid 1-p \leqq i \leqq -1 \text{ or } i=p;\ |j| \geqq p+1\} .$$

The proof of the following theorem is a simple counting of different types of pairs $(\alpha_-,\alpha_+) \in R_- \times R_+$.

THEOREM 3.4. *For the pair* $(\mathfrak{g},\mathfrak{k}) = (D_{p+q}, D_p \oplus D_q)$, Λ_0^+ *is equal to the set* $\{\lambda \in \Lambda^+ \mid \lambda_{p-1} > \lambda_p\}$ *and it is stable.*

REMARK. Let $N=2q$ for the cases a), b) and let $N=4q$ for c). Set

$$\delta = \frac{1}{N} \sum_{\alpha \in R_+} \alpha .$$

Then $\Lambda_0^+ = \Lambda^+ + \delta$. This kind of rule seems to be more generally valid; for example, when $\mathfrak{g} = G_2$ (exceptional simple algebra of rank 2) and $\mathfrak{k} = A_2$, then it is found that

$$\Lambda_0^+ = \Lambda^+ + \delta \quad \text{for } \delta = \tfrac{1}{2} \sum_{\alpha \in R_+} \alpha .$$

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