MATH. SCAND. 41 (1977), 79–89

SOME RESULTS ON EIGENFUNCTIONS ON SYMMETRIC SPACES AND EIGENSPACE REPRESENTATIONS

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1. Introduction.

This note contains three simple results on the topics of the title. The first concerns the space $\mathscr{H}$ of harmonic functions on $\mathbf{R}^n$. It was proved in [2d] that the isometry group of $\mathbf{R}^n$ acts irreducibly on an eigenspace of the Laplacian if and only if the eigenvalue is $\neq 0$. For the eigenvalue 0 there is a much bigger group acting, namely the conformal group (or rather its Lie algebra), and we show an irreducibility property for this action.

Our second result concerns a symmetric space G/K of the noncompact type, G being any noncompact connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $E \subset G/K$ be a flat totally geodesic submanifold of maximal dimension and let $\boldsymbol{D}(G/K)$ denote the set of G-invariant differential operators on G/K. We determine explicitly the joint eigenfunctions of these operators, constant on each geodesic perpendicular to E.

The third result is an integral representation of the joint eigenfunctions of the invariant differential operators on a symmetric space U/K of the compact type. The formula fits in the framework of Sherman's formulation of Fourier analysis on U/K (cf. [5a, b]) and is analogous to recent integral representations for the noncompact space G/K ([3], [2b II, Corollary 7.4]). The proof involves only standard techniques from the theory of spherical functions and spherical representations.

If M and N are manifolds and $\varphi \colon M \to N$ a diffeomorphism we write $f^{\varphi} = f \circ \varphi^{-1}$ for a function f on M. If D is a differential operator on M we define the differential operator D^{φ} on N by

$$D^{\varphi} \colon g \to (Dg^{\varphi^{-1}})^{\varphi}$$

g being a differentiable function on N. If $M = N$, D is called *invariant* under φ if $D^{\varphi} = D$.

Supported in part by the National Science Foundation MCS76-21044.
Received February 3, 1977.

2. Conformal groups and harmonic functions.

The group $G = SL(2, \mathbf{C})$ acts transitively on the one-point compactification of the plane by means of the maps

$$g: z \to \frac{az+b}{cz+d} \qquad z \in \mathbf{C},$$

if g is the complex matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of determinant one. The Laplacian L on $\mathbf{R}^2$ can be written

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A simple computation shows that

$$\left(\frac{\partial}{\partial z}\right)^g = (cz-a)^2\frac{\partial}{\partial z}, \quad \left(\frac{\partial}{\partial \bar{z}}\right)^g = (\bar{c}\bar{z}-\bar{a})^2\frac{\partial}{\partial \bar{z}},$$

whence

$$L^g = |cz-a|^4 L. \tag{1}$$

Consider now the Lie algebra of $SL(2, \mathbf{C})$ as a six-dimensional Lie algebra $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ over $\mathbf{R}$. This Lie algebra acts continuously on the space $C^\infty(\mathbf{R}^2)$ as follows. Let $X \in \mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ and put $g_t = \exp tX$ ($t \in \mathbf{R}$). For $u \in C^\infty(\mathbf{R}^2)$ we put

$$(Xu)(x, y) = \left\{\frac{d}{dt}(u^{g_t}(x, y))\right\}_{t=0} \qquad (x, y) \in \mathbf{R}^2.$$

Then (1) implies that Xu is harmonic if u is harmonic. Let $\mathscr{H}(\mathbf{R}^2)$ denote the space of harmonic functions on $\mathbf{R}^2$ with the topology induced by $C^\infty(\mathbf{R}^2)$.

LEMMA 2.1. *The action of* $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ *on* $\mathscr{H}(\mathbf{R}^2)$ *is "scalar irreducible", that is, the only continuous operators on* $\mathscr{H}(\mathbf{R}^2)$ *commuting with the action are the scalar multiples of the identity.*

PROOF. Suppose $A: \mathscr{H} \to \mathscr{H}$ is a continuous linear mapping such that $AXu = XAu$ for all $X \in \mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$. Taking X as

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}$$

we see that A commutes with the partial derivatives $\partial/\partial x$ and $\partial/\partial y$ and therefore also with $\partial/\partial z$ and $\partial/\partial \bar{z}$. In particular it maps the subspace $\mathfrak{a} \subset \mathscr{H}$ of holomorphic functions, and the subspace $\bar{\mathfrak{a}} \subset \mathscr{H}$ of antiholomorphic functions, into itself. Taking X as

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

we see that A commutes with the operator

$$u(x,y) \to x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \qquad u \in \mathscr{H}(\mathbf{R}^2)\,.$$

But if $u \in \mathfrak{a}$,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = z\frac{\partial u}{\partial z}$$

so A, restricted to $\mathfrak{a}$, commutes with the operator $u \to z\partial u/\partial z$. Putting $f_n = A(z^n)$ ($n \in \mathbf{Z}^+$) we therefore deduce $z\partial f_n/\partial z = nf_n$ whence $f_n = c_n z^n$ ($c_n \in \mathbf{C}$). But since A commutes with $\partial/\partial z$ we get for $n \geqq 1$, $f'_n = A(nz^{n-1})$, whence $c_n = c_{n-1}$. By continuity, A is a scalar c on $\mathfrak{a}$. Similarly, A is a scalar c' on $\bar{\mathfrak{a}}$. But $\mathfrak{a} \cap \bar{\mathfrak{a}} \neq 0$ so $c = c'$; since $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ span $\mathscr{H}(\mathbf{R}^2)$ the lemma follows.

According to Ørsted [10], if X is a conformal vector field on $\mathbf{R}^n$ ($n>1$) then the operator

$$\eta(X)f = Xf - \frac{n-2}{2n}(\operatorname{div} X)f \qquad f \in C^\infty(\mathbf{R}^n)$$

satisfies

$$L\eta(X)f - \eta(X)Lf = -\frac{2}{n}(\operatorname{div} X)Lf \tag{2}$$

and $X \to \eta(X)$ is a representation of the Lie algebra $\mathfrak{c}$ of conformal vector fields on $\mathbf{R}^n$ on $C^\infty(\mathbf{R}^n)$. If $X \in \mathfrak{c}$, it is clear from (2) that $\eta(X)$ maps the space $\mathscr{H}(\mathbf{R}^n)$ of harmonic functions on $\mathbf{R}^n$ into itself.

THEOREM 2.2. *The representation $X \to \eta(X)\,|\,\mathscr{H}$ of $\mathfrak{c}$ on $\mathscr{H}(\mathbf{R}^n)$ is scalar irreducible.*

PROOF. Suppose $A: \mathscr{H} \to \mathscr{H}$ is a continuous linear transformation commuting with all $\eta(X)$, $X \in \mathfrak{c}$. Since the vector field $X = \partial/\partial x_j$ has divergence 0 it is clear that A commutes with it and thus maps the space $H(\mathbf{R}^n)$ of harmonic polynomials into itself. Taking for X the generator of the one-parameter group

$$\varphi_t: (x_1, \ldots, x_n) \to (e^t x_1, \ldots, e^t x_n)$$

we see that A maps the space $H^m(\mathbf{R}^n)$ of harmonic polynomials of degree m into itself. The above properties imply that A maps $(x_1 + ix_2)^m$ into a constant multiple of itself so by the argument of Lemma 2.1 A restricted to $\mathscr{H}(\mathbf{R}^2)$ is a scalar. Now a vector field generated by a one-parameter group of rotations in $\mathbf{R}^n$ has divergence 0 so A commutes with it. Now $H^m(\mathbf{R}^2)$ is spanned by the polynomials $(a_1x_1 + a_2x_2)^m$, $(a_1^2 + a_2^2 = 0)$ and $H^m(\mathbf{R}^n)$ is spanned by the polynomials

$$(a_1x_1 + \ldots + a_nx_n)^m, \quad \sum_1^m a_i^2 = 0 .$$

This implies easily that the set of rotated polynomials P^k ($P \in H^m(\mathbf{R}^2)$, $k \in \mathbf{O}(n)$) span $H^m(\mathbf{R}^n)$, whence by the above, A is a scalar c on $H^m(\mathbf{R}^n)$, c independent of m.

Finally, if $f \in \mathscr{H}(\mathbf{R}^n)$ we expand into a convergent series $f = \sum_\delta f_\delta$ where $f_\delta \in \mathscr{H}(\mathbf{R}^n)$ transform under $\mathbf{O}(n)$ according to an irreducible representation δ. Then $f_\delta \,|\, S^{n-1}$ is the restriction to S^{n-1} of a homogeneous harmonic polynomial so f_δ must equal this polynomial. By continuity, $Af = cf$ as desired.

3. Geodesically invariant eigenfunctions.

As in the introduction, let $X = G/K$ be a symmetric space of the noncompact type, K compact. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{p}$ being the orthogonal complement to $\mathfrak{k}$, the Lie algebra of K, with respect to the Killing form B of $\mathfrak{g}$. The restriction of B to $\mathfrak{p}$ defines a G-invariant Riemannian structure on G/K. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, let $A = \exp \mathfrak{a}$ and let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Let 0 denote the origin in G/K. Then the curves

$$\gamma_Z: t \to a \exp tZ \cdot 0, \qquad (a \in A,\ Z \in \mathfrak{q})$$

constitute the geodesics in G/K perpendicular to the submanifold $E = A \cdot 0$ and according to Mostow [4], G/K is their disjoint union. We shall now determine the joint eigenfunctions of the operators in $\mathbf{D}(G/K)$ constant on each of these geodesics.

Let log: $A \to \mathfrak{a}$ be the inverse of exp and let W denote the Weyl group of G/K, acting on A, $\mathfrak{a}$, the dual $\mathfrak{a}^*$, and its complexification $\mathfrak{a}_c^*$. For each $\lambda \in \mathfrak{a}_c^*$ let W_λ denoting the subgroup of W leaving λ fixed.

If U is a compact group of rotations of a real vector space V a polynomial function on V is called *U-harmonic* if it is annihilated by all the U-invariant constant-coefficient differential operators on V without constant term.

THEOREM 3.1. *The joint eigenfunctions of the operators $D \in \mathbf{D}(G/K)$, constant on each orthogonal geodesic γ_Z ($Z \in \mathfrak{q}$) are precisely the functions*

$$\psi_\lambda(a \exp Y \cdot 0) = \sum_{s \in W} P_s(\log a) e^{is\lambda(\log a)}, \qquad a \in A,\ Y \in \mathfrak{q}\,,$$

where $\lambda \in \mathfrak{a}_c^*$ is arbitrary and P_s is an arbitrary $W_{s\lambda}$-harmonic polynomial on $\mathfrak{a}$.

If D is a differential operator on X its *projection* on E, in the sense of [2c], Chapter I, is a differential operator D' on E satisfying

$$(D'F)(a \cdot 0) = (D\tilde{F})(a \cdot 0)$$

if $F \in C^\infty(E)$ and $\tilde{F}$ its extension to a C^∞ function on X constant on each geodesic γ_Z ($Z \in \mathfrak{q}$). As proved in [2c], Chapter I], for a general Riemannian manifold, the Laplace–Beltrami operators L_X and L_E satisfy $L'_X = L_E$. Identifying A and E by means of the mapping $a \to a \cdot 0$ we now prove an extension.

LEMMA 3.2. *The projection $D \to D'$ is a bijection of $\mathbf{D}(G/K)$ onto the set of W-invariant differential operators on A with constant coefficients.*

PROOF. Suppose $g \in G$ maps E into itself. Then for some $a \in A$, $g \cdot 0 = a \cdot 0$, so $a^{-1}g$ belongs to the normalizer M' of $\mathfrak{a}$ in K. Thus $M'A$ is the subgroup of G leaving E invariant. Hence if $D \in \mathbf{D}(G/K)$, D' is a W-invariant differential operator with constant coefficients.

Let λ: $S(\mathfrak{g}) \to \mathbf{D}(G)$ be the canonical mapping of the symmetric algebra $S(\mathfrak{g})$ onto the set $\mathbf{D}(G)$ of left invariant differential operators on G. If $Z_1, \ldots, Z_n$ is a basis of $\mathfrak{g}$ this mapping satisfies

$$(1) \qquad (\lambda(P)f)(g) = \{P(\partial_1, \ldots, \partial_n) f(g \exp(z_1 Z_1 + \ldots + z_n Z_n))\}(0)\,,$$

where $\partial_i = \partial/\partial z_i$, $f \in C^\infty(G)$, $P \in S(\mathfrak{g})$. The centralizer $\mathbf{D}_K(G)$ of $\mathfrak{k}$ in $\mathbf{D}(G)$ has the direct decomposition

$$\mathbf{D}_K(G) = \lambda(I(\mathfrak{p})) \oplus (\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k})\,,$$

where $I(\mathfrak{p})$ is the space of $\mathrm{Ad}_G(K)$ invariant in $S(\mathfrak{p})$, Ad_G denoting as usual the adjoint representation of G. Let $(H_i)_{1 \leq i \leq l}$, $(X_j)_{1 \leq j \leq q}$, and $(T_k)_{1 \leq k \leq p}$ be bases of

$\mathfrak{a}$, $\mathfrak{q}$ and $\mathfrak{k}$, respectively, orthonormal with respect to $-B(X,\theta Y)$. Let $P \in I(\mathfrak{p})$ be homogeneous of degree m. Writing

$$N = (n_1,\ldots,n_l), \quad |N| = n_1+\ldots+n_l\,,$$
$$M = (m_1,\ldots,m_q), \quad |M| = m_1+\ldots+m_q$$

we have

$$P = \sum_{|N|+|M|=m} a_{N,M} H_1^{n_1}\ldots H_l^{n_l} X_1^{m_1}\ldots X_q^{m_q}\,.$$
$$= P_{\mathfrak{a}} + \sum_{N,|M|>0} a_{N,M} H_1^{n_1}\ldots H_l^{n_l} X_1^{m_1}\ldots X_q^{m_q}\,,$$

where $P_{\mathfrak{a}} \in S(\mathfrak{a})$. Since the restriction map $f \to f\,|\,\mathfrak{a}$ induces an isomorphism of $I(\mathfrak{p})$ onto $I(\mathfrak{a})$, the set of W-invariant in $S(\mathfrak{a})$, we see that $P_{\mathfrak{a}}$ is W-invariant of degree m. Writing $\tilde{Z}=\lambda(Z)$ for $Z \in \mathfrak{g}$ we have

$$\lambda(P) = \lambda(P_{\mathfrak{a}}) + \sum_{N,|M|>0} a_{N,M} \tilde{H}_1^{n_1}\ldots H_l^{n_l} \tilde{X}_1^{m_1}\ldots \tilde{X}_q^{m_q} + Q\,, \tag{2}$$

where Q has order $<m$ and $|N|+|M|=m$. By Theorem 5 in Mostow [4], G has the topological decomposition $G=\exp\mathfrak{a}\exp\mathfrak{q}K$. If μ denotes the canonical homomorphism of $\boldsymbol{D}_K(G)$ onto $\boldsymbol{D}(G/K)$ we put $D_Q=\mu(\lambda(Q))$ for $Q \in I(\mathfrak{p})$. Then if $F \in C^\infty(E)$ and $f \in C^\infty(G)$ is determined by

$$F(a\cdot 0) = f(a\exp Xk) \qquad a \in A,\ X \in \mathfrak{q},\ k \in K$$

we have

$$(D'_P F)(a\cdot 0) = (\lambda(P)f)(a) \qquad a \in A\,. \tag{3}$$

But if $a(h)=a\exp(h_1H_1+\ldots+h_lH_l)$ we have

$$(\tilde{H}_1^{n_1}\ldots \tilde{H}_l^{n_l}\tilde{X}_1^{m_1}\ldots \tilde{X}_q^{m_q} f)(a) = \left\{\frac{\partial^{|N|}}{\partial h_1^{n_1}\ldots \partial h_l^{n_l}}(\tilde{X}_1^{m_1}\ldots \tilde{X}_q^{m_q} f)(a(h))\right\}_{h_i=0}$$

and

$$\tilde{X}_1^{m_1}\ldots \tilde{X}_q^{m_q} = \lambda(X_1^{m_1}\ldots X_q^{m_q}) + T\,,$$

where $T \in \boldsymbol{D}(G)$ has order $<|M|$. But by (1)

$$(\lambda(X_1^{m_1}\ldots X_q^{m_q})f)(a(h)) = 0 \qquad \text{if } |M|>0\,.$$

Thus we conclude from (2) and (3) that for a certain $R \in \boldsymbol{D}(G)$ of order $<m$,

$$(D'_P F)(a\cdot 0) = (P_{\mathfrak{a}}F)(a\cdot 0) + (Rf)(a)\,,$$

for all $a \in A$ and all $F \in C^\infty(A\cdot 0)$. If the differential operator R is expressed in terms of the coordinate system

$$\exp(h_1H_1+\ldots+h_lH_l)\exp(x_1X_1+\ldots+x_qX_q)\exp(t_1T_1+\ldots+t_pT_p)$$
$$\to (h_1,\ldots,h_l,x_1,\ldots,x_q,t_1,\ldots,t_p)$$

it becomes obvious, since f in these coordinates is independent of (x_i) and (t_j), that the mapping

$$F \to (Rf)\,|\,A\cdot 0$$

is a differential operator of order less than or equal to that of R. Hence

$$\text{order}\,(D'_P - P_{\mathfrak{a}}) < m\,. \tag{4}$$

Suppose now $Q \in I(\mathfrak{a})$. We wish to find $D \in \boldsymbol{D}(G/K)$ such that $D' = Q$. Proceeding by induction let $m = \deg(Q)$ and assume statement holds for all elements of $I(\mathfrak{a})$ of degree $<m$. Decomposing Q into homogeneous components we can by the above find $P \in I(\mathfrak{p})$ such that

$$\text{degree}\,(D'_P - Q) < m\,.$$

But $D'_P - Q \in I(\mathfrak{a})$ so by the inductive hypothesis there exists an $E \in \boldsymbol{D}(G/K)$ such that $E' = D'_P - Q$. Thus $D = D_P - E$ has the desired property.

Finally, suppose $D \neq 0$ of $\boldsymbol{D}(G/K)$ of order m such that $D' = 0$. Let $P \in I(\mathfrak{p})$ be the homogeneous polynomial of degree m such that order $(D - D_P) < m$. Then $D'_P = (D - D_P)'$ has order $<m$ whereas $P_{\mathfrak{a}}$ has degree m. This contradicts (4) so the lemma is proved.

The lemma shows that a function $\psi \in C^\infty(X)$, constant on each geodesic γ_Z ($Z \in \mathfrak{q}$), is a joint eigenfunction of the $D \in \boldsymbol{D}(G/K)$ if and only if the restriction $\psi\,|\,E$ to the Euclidean space E is an eigenfunction of all the W-invariant differential operators on E with constant coefficients. Such eigenfunctions on E were found by Steinberg [7] and Harish-Chandra (cf. Warner [9, p. 316]) to be just linear combinations of exponential functions $e^{is\lambda}$ ($s \in W$), $\lambda \in \mathfrak{a}_c^*$ being fixed, with $W_{s\lambda}$-harmonic polynomials as coefficients. This proves Theorem 3.1.

4. Eigenfunctions on compact symmetric spaces.

Let U/K be a symmetric space (of the compact type) where U is a simply connected compact semisimple Lie group and K the fixed point set of an involutive automorphism θ of U. Let $\mathfrak{u}$ and $\mathfrak{k}$ denote the corresponding Lie algebras and $\mathfrak{p}_*$ the eigenspace for eigenvalue -1 of the automorphism $d\theta$ of $\mathfrak{u}$. Then if $\mathfrak{p} = i\mathfrak{p}_*$, the real subspace $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the complexification $\mathfrak{u}^c$ is a semisimple Lie algebra over $\mathbf{R}$. If U^c is the simply connected Lie group with Lie algebra $\mathfrak{u}^c$, let G be the analytic subgroup of U^c with Lie algebra $\mathfrak{g}$. Then G/K is a symmetric space of the noncompact type and we can use the notions of section 3. Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, let $\mathfrak{n}$ denote the subalgebra spanned by the root spaces for roots of $(\mathfrak{g}, \mathfrak{a})$ which are positive on $\mathfrak{a}$, and let N be the corresponding analytic subgroup of G. Given $g \in G$ let $A(g) \in \mathfrak{a}$ be determined by $g \in N \exp A(g) K$. We consider $\mathfrak{u}_c$ with the usual Hilbert space inner product $(X, Y) \to -B(X, \tau Y)$, τ being the conjugation of $\mathfrak{u}^c$ with respect to $\mathfrak{u}$. Because of the vector space direct sum of the complexifications,

$$\mathfrak{u}^c = \mathfrak{n}^c + \mathfrak{a}^c + \mathfrak{k}^c$$

the map $(X, H, T) \to \exp X \exp H \exp T$ is a holomorphic diffeomorphism of a neigborhood of $(0,0,0)$ onto a neighborhood U_0^c of e in U^c. The map

$$\exp X \exp H \exp T \to H$$

is therefore a well-defined holomorphic mapping of U_0^c into $\mathfrak{a}^c$ extending the map A. This extension, which was considered by Stanton [6] and Clerc [11], we denote also by A. We can take U_0^c as the diffeomorphic image (under exp) of a ball $B_0 \subset \mathfrak{u}^c$ with center 0. Then U_0^c is invariant under the maps $u \to kuk^{-1}$, and so is the set $U_0 = U_0^c \cap U$.

Let $\langle , \rangle$ denote the bilinear form on $\mathfrak{a}^*$ induced by the Killing form, let Σ^+ be the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$, and 2ϱ their sum with multiplicity. Finally, let M be the centralizer of A in K and dk_M the normalized K-invariant measure on K/M.

THEOREM 4.1. *Each joint eigenfunction of all $D \in \mathbf{D}(U/K)$ has the form*

$$f(uK) = \int_{K/M} e^{-\mu(A(k^{-1}uk))} F(kM)\, dk_M, \qquad u \in U_0, \tag{1}$$

where $\mu \in \mathfrak{a}^*$ and F satisfy

$$F \in C^\infty(K/M) \text{ and } \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}^+ \text{ for } \alpha \in \Sigma^+ . \tag{2}$$

Conversely, if μ and F satisfy (2) *then the function f defined by* (1) *extends uniquely to an analytic functions on U/K and this function is a joint eigenfunction of all $D \in \mathbf{D}(U/K)$.*

First we note that if $\lambda \in \mathfrak{a}_c^*$ the spherical function φ_λ on G can be written

$$\varphi_\lambda(g \cdot 0) = \int_K e^{(i\lambda + \varrho)(A(kg))}\, dk \tag{3}$$

([1, p. 261], [2bI, p. 94]) and moreover

$$\varphi_\lambda(h^{-1} g \cdot 0) = \int_K e^{(i\lambda + \varrho)(A(k^{-1}gk))} e^{(-i\lambda + \varrho)(A(k^{-1}hk))}\, dk , \tag{4}$$

[2bI, p. 116]. Extending A to U_0^c we can extend φ_λ by the formula

$$\varphi_\lambda(uK) = \int_K e^{(i\lambda+\varrho)(A(k^{-1}uk))} dk \qquad u \in U_0^c$$

(cf. [6] and [11]) and then (4) holds by analytic continuation for $g, h, g^{-1}h \in U_0^c$. This extension of [4] was proved by Sherman [5b], even for nonsymmetric spaces (where a different proof is of course required).

For the proof of Theorem 4.1, let f be a joint eigenfunction, and let $\chi(D)$ be determined by $Df = \chi(D)f$ for $D \in \mathbf{D}(U/K)$. The joint eigenspace

$$\{\varphi \in C^\infty(U/K) \mid D\varphi = \chi(D)\varphi \quad \text{for } D \in \mathbf{D}(U/K)\} \tag{5}$$

is finite-dimensional and irreducible under U ([8], [2a, p. 454]). Hence we have

$$f(uK) = \langle \pi(u)v_0, v \rangle , \tag{6}$$

where π is an irreducible representation of U on a finite-dimensional Hilbert space V with inner product $\langle , \rangle$, the vectors v_0 and v belong to V and v_0 is a unit vector fixed under $\pi(K)$.

Choose $u_0 \in U$ such that $f(u_0K) \neq 0$ and put

$$\varphi(uK) = c \int_K f(u_0kuK)\, dk ,$$

where the constant c is chosen such that $\varphi(0) = 1$. Since the fixed vector v_0 is unique up to a constant multiple we derive from (6)

$$\varphi(uK) = \langle \pi(u)v_0, v_0 \rangle \qquad u \in U .$$

Now extending, as we can, π to a representation π_c of U^c, φ extends to the function

$$\tilde{\varphi}(u) = \langle \pi_c(u)v_0, v_0 \rangle \qquad u \in U^c .$$

A simple direct proof (Harish-Chandra [1, Lemma 5]) shows that on G,

$$\tilde{\varphi}(g) = \int_K e^{-\mu(A(kg^{-1}))} dk \qquad g \in G ,$$

where $\mu \in \mathfrak{a}^*$ is the highest weight of π restricted to $\mathfrak{a}$. Writing $\mu = i\lambda - \varrho$ we have since $\varphi_\lambda = \varphi_{s\lambda}$ $(s \in W)$ and since $\varphi_\lambda(gK) = \varphi_{-\lambda}(g^{-1}K)$,

$$\tilde{\varphi}(g) = \varphi_\lambda(gK) = \int_K e^{s^*\mu)(A(kg))} dk \qquad g \in G ,$$

where s^* is the Weyl group element mapping $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The vector v is a linear combination,

$$v = \sum_i a_i \pi(u_i) v_0 \qquad a_i \in \mathbf{C},\ u_i \in U$$

and here we may assume the elements u_i contained in an arbitrary neighborhood of e in U. In fact, if a linear form on V vanishes on $\pi(\exp tX)v_0 - v_0$ ($X \in \mathfrak{u}$, t small) it vanishes on $d\pi(\mathfrak{u})v_0 = V$. Hence

$$f(uK) = \sum_i \bar{a}_i \varphi(u_i^{-1}uK)$$

for $u \in U$ sufficiently close to e. Using now (4) we get the integral formula (1) for f with μ replaced by $-s^*\mu$. But using the characterization of the highest weights of spherical representation proved in [2bI, Chapter III, § 3] we see that μ, and therefore also $-s^*\mu$, satisfies (2).

For the converse let $\mu \in \mathfrak{a}^*$ satisfy (2) and let π be the irreducible finite-dimensional spherical representation of G on a vector space V with highest weight having restriction to $\mathfrak{a}$ given by μ. Let V^* be the dual of V, v_0 "the" unit vector fixed under $\pi(K)$ and choose $v^* \in V^*$ such that $v^*(v_0)=1$. Let $v_0^* \in V^*$ be defined by

$$v_0^*(v) = \int_K v^*(\pi(k)v)\,dk \qquad v \in V.$$

Then we can define a function φ on G/K by

$$\varphi(gK) = v_0^*(\pi(g^{-1})v_0)\,.$$

As shown in [2c, p. 34], φ is a spherical function on G, and π is equivalent to the natural representation π_φ of G on the space V_φ spanned by the translates of φ. Let $\psi \in V_\varphi$ be "the" highest weight vector for π_φ. Then

$$\psi(an\cdot 0) = e^{-\mu(\log a)}\psi(0)$$

that is, taking $\psi(0)=1$,

$$\psi(gK) = e^{-\mu(A(g))} \qquad g \in G\,.$$

But π extends to a holomorphic representation of U^c, so ψ extends to U^c as well. By holomorphic continuation this extension satisfies

$$\tilde{\psi}(uK) = e^{-\mu(A(u))} \qquad u \in U_0\,.$$

Finally ψ satisfies the functional equation

$$\int_K \psi(gkx\cdot 0)\,dk = \psi(g\cdot 0)\int_K \psi(kx\cdot 0)\,dk \qquad g, x \in G$$

which characterizes the joint eigenfunctions of $\boldsymbol{D}(G/K)$ [2a, p. 439]. By holomorphic continuation, $\tilde{\psi}$ satisfies this functional equation on U/K so is a joint eigenfunction of $\boldsymbol{D}(U/K)$. This concludes the proof.

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