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ON THE IVERSEN–TSUJI THEOREM FOR QUASIREGULAR MAPPINGS

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1. Introduction.

Quasiregular mappings have turned out to be natural generalizations of complex analytic functions to n-dimensional euclidean spaces for $n \geqq 2$. For the basic theory of quasiregular mappings, we refer to the papers by Martio, Rickman, and Väisälä [3]–[5].

A result by Martio and Rickman in [6] extends the classical Iversen–Tsuji theorem on bounded analytic functions to the case of bounded quasiregular mappings. The purpose of the paper is to improve and generalize this result.

Preliminaries are given in section 2. In section 3 we first strengthen the Iversen–Tsuji theorem by Martio and Rickman. In fact, the boundedness assumption in [6] is replaced by a much weaker assumption and to this end we need a result from [4]. After this we confine our attention to quasiregular mappings defined in the unit ball and prove a Iversen–Tsuji type result where we permit the exceptional set to be larger than in the Iversen–Tsuji theorem. The new condition on the exceptional set is formulated in terms of the continuum criterion, which was recently introduced by Martio in [2]. In section 4 the results are applied to obtain an extension of a well-known theorem by Lindelöf concerning analytic functions. Some of our results are perhaps new even for analytic functions.

2. Notation and preliminary results.

Our notation and terminology are mainly the same as in [3]–[5]. Only some basic concepts will be given here.

We consider only spaces of dimension $n \geqq 2$. The inner product $\sum x_i y_i$ of two vectors $x, y \in \mathsf{R}^n$ is denoted by $(x \mid y)$. In the spaces R^n and $\bar{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ we use the metric given by the norm $|x| = (x \mid x)^{\frac{1}{2}}$ and the spherical chordal metric q, respectively. All topological operations are performed with respect to $\bar{\mathsf{R}}^n$. For $x_0 \in \mathsf{R}^n$, $r > 0$, $B^n(x_0, r)$ denotes the ball

$$B^n(x_0,r) \;=\; \{x \in \mathsf{R}^n \;\big|\; |x-x_0|<r\}$$

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and $S^{n-1}(x_0,r)$ the sphere

$$S^{n-1}(x_0,r) = \{x \in \mathbb{R}^n \mid |x-x_0|=r\} ,$$

respectively. We use the abbreviations $B^n(r)=B^n(0,r)$, $S^{n-1}(r)=S^{n-1}(0,r)$, $B^n=B^n(1)$, and $S^{n-1}=S^{n-1}(1)$.

A continuum is a connected compact set containing at least two distinct points. A path γ is a continuous mapping $\gamma\colon \Delta \to A$, $A\subset\bar{\mathbb{R}}^n$, where Δ is an interval on the real axis. We denote $\gamma\Delta$ by $|\gamma|$ and we let $\Delta(E,F;\,G)$ denote the family of all non-constant paths $\gamma\colon[0,1)\to\bar{\mathbb{R}}^n$ such that $\gamma(0)\in E$, $\gamma(0,1)\subset G$, and $\gamma(t)\to F$ as $t\to 1$. A condenser $E=(A,C)$ is a pair where A is a domain in $\mathbb{R}^n$ and $C\subset A$ is compact. For the conformal capacity $\operatorname{cap} E$ of a condenser E we use the same definition as in [3]. We often need a result by Ziemer [14], which gives an alternative definition of the conformal capacity as follows: $\operatorname{cap} E=M(\Delta(C,\partial A;\,\mathbb{R}^n))=M(\Delta(C,\partial A;\,A))=M(\Delta(C,\partial A;\,A\backslash C))$. Here M denotes the n-modulus of a path family, see [11]. As in [4], we use the conformal capacity also to classify sets $E\subset\bar{\mathbb{R}}^n$ according to whether their capacity is zero or positive, denoted by $\operatorname{cap} E=0$ or $\operatorname{cap} E>0$, respectively.

2.1. DEFINITION. Let $E\subset\mathbb{R}^n$ be a compact set. *The continuum criterion* holds at $y\in E$ if there exists a continuum $K\subset\mathbb{R}^n$, $y\in K$ such that $M(\Gamma_K)<\infty$ where $\Gamma_K=\Delta(K,E;\,\mathbb{R}^n)$ (cf. [2]). If the continuum criterion holds at $y\in E$ we denote $M(y,E)<\infty$.

The omission of a continuum K from this notation is motivated by an equivalent definition of the continuum criterion, which shows that no reference to any particular continuum is needed, cf. [7, 2.20].

Some properties of the continuum criterion and examples of sets E satisfying $M(y,E)<\infty$ for some $y\in E$ are given in [7]. All the information, which we need on the continuum criterion, is given by the next two lemmas. The lemmas follow from some results of Martio and Sarvas [7]. For details, see [13].

2.2. LEMMA. *Let $E\subset\mathbb{R}^n$ be a compact set with $y\in E$ and $M(y,E)<\infty$. Then there exists a sequence $r_1>r_2>\ldots$ of positive real numbers tending to zero such that*

(1) $S^{n-1}(y,r_j)\subset\mathbb{R}^n\backslash E$,
(2) $M(\Delta(S^{n-1}(y,r_j),E;\,\mathbb{R}^n))\to 0$ *as* $j\to\infty$.

If $e\in S^{n-1}$ and $\varphi\in(0,\pi/2)$ we denote by $K(e,\varphi)$ the open cone

$$K(e,\varphi) = \{x\in\mathbb{R}^n \mid (e\,|\,e-x)>|e-x|\cos\varphi\} .$$

Then e is the vertex of $K(e,\varphi)$. In what follows we will need the continuum criterion only for sets $E\subset\partial B^n$.

2.3. LEMMA. *Let $E\subset\partial B^n$ be a compact set with $y\in E$ and $M(y,E)<\infty$. Then for every $\varphi\in(0,\pi/2)$, $M(\Gamma_K)<\infty$ where $K=\overline{K(y,\varphi)}\cap\bar{B}^n(y,\cos\varphi)$ and*

$$\lim_{r\to 0} M(\Gamma_{K_r}) = 0\,,$$

where $K_r=\bar{B}^n(y,r)\cap K$.

2.4. REMARK. There are sets $E\subset\partial B^n$ with $M(y,E)<\infty$ for some $y\in E$ and $\operatorname{cap}(E\cap\bar{B}^n(y,r))>0$ for all $r>0$. In fact, if one modifies the examples in [7] one obtains a compact set $E\subset\partial B^n$ with $M(y,E)<\infty$ for some $y\in E$ such that $E=\{y\}\cup(\bigcup(B_i\cap\partial B^n))$, where $B_i=\bar{B}^n(x_i,r_i)$, $x_i\in\partial B^n$, and $x_i\to y$. Here the ratio $r_i/|x_i-y|$ tends to zero exponentially. Martio and Sarvas proved in [7] that $M(y,E)<\infty$ implies that E has zero capacity density at y. In this sense the condition $M(y,E)<\infty$ measures the thinness of E at y.

3. The Iversen–Tsuji theorem.

We will prove in this section two Iversen–Tsuji type results for quasiregular mappings. First we strengthen a result by Martio and Rickman in [6] and after that we prove a related result where the exceptional set is permitted to be larger.

Our Iversen–Tsuji theorem differs from the result of Martio and Rickman in two respects. Firstly, the proof given here avoids some technical difficulties and is shorter than the one in [6]. Secondly, our result is more general and perhaps new even for analytic functions. In fact, we replace the assumption in [6] that the mapping is bounded by the one that infinity is a capacity point of the omitted values.

A point y of a compact set $F\subset\bar{R}^n$ is called *a capacity point*, if $\operatorname{cap}(\bar{U}\cap F)>0$ for every neighborhood U of y. It is well-known that F contains capacity points if and only if $\operatorname{cap}F>0$. As a preliminary result we need the following lemma, which is a consequence of [4, 3.11].

3.1. LEMMA. *Let $F\subset\bar{R}^n$ be a compact set, let the origin 0 be a capacity point of F, and let $r>0$. Then for every $r_0>0$ there exists $\delta>0$ such that for every continuum $C\subset B^n(r)\setminus F$ with the euclidean diameter $d(C)\geqq r_0$*

$$M(\Gamma)\geqq\delta$$

where $\Gamma=\Delta(C,F;\bar{B}^n(r))$.

PROOF. Let $\Gamma_1 = \Delta(C, F \cap \bar{B}^n(r); \bar{\mathsf{R}}^n)$. By [4, 3.11] and [14] there exists $\delta > 0$ such that $M(\Gamma_1) \geqq 2\delta$. From a symmetry argument it follows that $M(\Gamma) \geqq M(\Gamma_1)/2 \geqq \delta$. The proof is complete.

3.2. THEOREM (Iversen–Tsuji). *Let $f\colon G \to \bar{\mathsf{R}}^n$ be a quasimeromorphic mapping, let $E \subset \partial G$ be a compact set of capacity zero, let $y \in E \cap \overline{(\partial G \setminus E)}$, and let ∞ be a capacity point of $\bar{\mathsf{R}}^n \setminus f(G \cap U)$ for some neighborhood U of y. Then*

$$\limsup_{x \to y} |f(x)| = \limsup_{\substack{z \to y \\ z \in \partial G \setminus E}} \left(\limsup_{x \to z} |f(x)|\right). \tag{3.3}$$

PROOF. We denote by $\tilde{a}$ and $\tilde{b}$ the left and the right hand sides of the equality (3.3), respectively. Obviously $\tilde{a} \geqq \tilde{b}$. Suppose that $\tilde{a} > \tilde{b}$. Then we may fix $a, b \in (\tilde{b}, \tilde{a})$ with $a > b$. Performing an auxiliary Möbius transformation we may assume that $y = 0$. Choose $r_1 > 0$ such that $\bar{B}^n(r_1) \subset U$ and

$$\limsup_{x \to z} |f(x)| < b \tag{3.4}$$

for $z \in (\partial G \setminus E) \cap \bar{B}^n(r_1)$. Choose a sequence (x_k) in $B^n(r_1) \cap G$ such that $|f(x_k)| > a$ and $|x_k| < 1/k$ for every $k = 1, 2, \ldots$. Since $y = 0 \in \overline{(\partial G \setminus E)}$, since E is compact, and since $\operatorname{cap} E = 0$ we find, applying [4, p. 8] as in the proof of [6, 3.3], a path $\gamma_k\colon [0,1] \to \bar{G} \setminus E$ such that $\gamma_k(0) = x_k$, $\gamma_k(1) \in \partial G \setminus E$, and

$$\gamma_k[0,1) \subset B^n(r_1/k) \cap G.$$

For each k we may choose a continuum $C_k \subset |\gamma_k| \cap G$ with

$$fC_k \subset \mathsf{R}^n \setminus \bar{B}^n(b), \quad fC_k \cap S^{n-1}((a+b)/2) \neq \varnothing,$$

and

$$fC_k \cap S^{n-1}(a) \neq \varnothing.$$

Denote $\Gamma'_k = \Delta(fC_k, F; \bar{\mathsf{R}}^n \setminus B^n(b))$, $k = 1, 2, \ldots$, where $F = \bar{\mathsf{R}}^n \setminus f(G \cap U)$. In view of the conformal invariance of capacity points there is by Lemma 3.1, $\delta > 0$ such that $M(\Gamma'_k) \geqq \delta$ for every k. Let Γ_k be the family of the maximal liftings of the elements of Γ'_k starting at C_k. For terminology, see [5, 3.11]. From the choice of r_1 and from (3.4) it follows that if $\alpha \in \Gamma_k$ then either

$$\overline{|\alpha|} \cap E \neq \varnothing \quad \text{or} \quad |\alpha| \cap S^{n-1}(r_1) \neq \varnothing;$$

see [5, 3.12]. Because $\operatorname{cap} E = 0$ the family of the former paths has zero modulus ([14]) and the modulus of the family of the latter paths tends to zero, as can be verified using a suitable modulus estimate ([11, 7.5]). By the modulus inequality in [12, 3.1]

$$0 < \delta \leqq M(\Gamma'_k) \leqq M(f\Gamma_k) \leqq K_I(f)M(\Gamma_k)$$

which leads to a contradiction, since $M(\Gamma_k) \to 0$ as $k \to \infty$. Hence $\tilde{a}=\tilde{b}$ and the proof is complete.

The maximum principle of Martio and Rickman [6, 3.9] assumes now the following more general form.

3.5. COROLLARY. *Let f: $G \to \bar{\mathsf{R}}^n$ be a quasimeromorphic mapping of a bounded domain G, let ∞ be a capacity point of $\bar{\mathsf{R}}^n \backslash fG$, and let $E \subset \partial G$ be a compact set of capacity zero. If* $\limsup_{x\to y} |f(x)| \leqq M$ *for every point $y \in \partial G \backslash E$, then $|f| \leqq M$.*

PROOF. The proof follows from hte proof of [6, 3.9] except that we apply Theorem 3.2 instead of [6, 3.3]. In fact, if we use the notation in the proof of [6, 3.9], it suffices to show that ∞ is a capacity point of $\bar{\mathsf{R}}^n \backslash f_1 G_1$. This is obvious if we show that $|f_1(x)| \leqq M$ for $x \in E \backslash F$. This follows if we use the same method as in the proof of Theorem 3.2 and apply the modulus inequality in [12, 3.1].

3.6. REMARK. For a recent proof of the Iversen–Tsuji theorem in the case of analytic functions, see Garnett [1]. The proof of Theorem 3.2 seems to simplify the known proofs even in the case of analytic functions, see for instance [1] and [9, pp. 17–19].

In the above Iversen–Tsuji theorem the compact exceptional set E on the boundary of the domain G was assumed to be of capacity zero. In the remaining part of the section we will replace this assumption by $M(y, E) < \infty$ for some $y \in E$, which guarantees that E is very thin at y but y may still be a capacity point of E, see Remark 2.4. In order to obtain a counterpart of the Iversen–Tsuji theorem under such a weaker condition on E, it seems to be necessary to restrict the tangential behavior of ∂G at y. This restriction is needed to guarantee that there exists in $G \cup \{y\}$ a continuum $K \ni y$ with the property $M(\Delta(K, E; G)) < \infty$. For the sake of simplicity we consider only the case $G = B^n$ although an extension to somewhat more general domains is possible. These domains include for instance those which are locally quasiconformally collared on the boundary in the sense of [11, 17.5].

3.7. THEOREM. *Let f: $B^n \to \bar{\mathsf{R}}^n$ be a quasimeromorphic mapping, let $E \subset \partial B^n$ be a compact set with $M(y, E) < \infty$ for some $y \in E$, and let ∞ be a capacity point of $\bar{\mathsf{R}}^n \backslash f(B^n \cap U)$ for some neighborhood U of y. Then*

$$\limsup_{\substack{x \to y \\ x \in K(y,\varphi)}} |f(x)| \leqq \limsup_{\substack{z \to y \\ z \in \partial B^n \setminus E}} \left(\limsup_{x \to z} |f(x)|\right) \tag{3.8}$$

for every $\varphi \in (0, \pi/2)$.

PROOF. We first remark that $M(y, E) < \infty$ implies by [2, 3.8] $y \in \overline{(\partial B^n \setminus E)}$ and hence the limit (3.8) makes sense. Fix $\varphi \in (0, \pi/2)$. Denote by $\tilde{a}$ and $\tilde{b}$ the left and the right hand sides of (3.8), respectively. Suppose that $\tilde{a} > \tilde{b}$. Then we may choose real numbers $a, b \in (\tilde{b}, \tilde{a})$ with $a > b$. It follows from Lemmas 2.2 and 2.3 that there exists a decreasing sequence (r_k), $r_k \in (0, 1/k)$ of positive real numbers with

$$S^{n-1}(y, r_k) \subset \mathsf{R}^n \setminus E, \quad k = 1, 2, \ldots \tag{3.9}$$

$$M(\tilde{\Gamma}_k) \to 0 \quad \text{as } k \to \infty \tag{3.10}$$

where $\tilde{\Gamma}_k = \Delta(D_k, E; B^n)$ and $D_k = S^{n-1}(y, r_k) \cup (\overline{K(y,\varphi)} \cap \bar{B}^n(y, 1/k))$. Choose $x_k \in B^n(y, 1/k) \cap K(y, \varphi)$ such that $|f(x_k)| > a$, $k = 1, 2, \ldots$. After relabeling we may assume that $\bar{B}^n(y, r_1) \subset U$ and

$$\limsup_{x \to z} |f(x)| < b \tag{3.11}$$

for every $z \in (\partial B^n \setminus E) \cap \bar{B}^n(y, r_1)$. By (3.9) and (3.11) we may choose $y_k \in S^{n-1}(y, r_k) \cap B^n$ such that $|f(y_k)| < b$ for every integer k. From the choice of the sequences (x_k) and (y_k) it follows that there is for each k a continuum $C_k \subset D_k \cap B^n$ with $fC_k \subset \mathsf{R}^n \setminus \bar{B}^n(b)$ and

$$fC_k \cap S^{n-1}((a+b)/2) \neq \emptyset, \quad fC_k \cap S^{n-1}(a) \neq \emptyset\,.$$

Let $\Gamma'_k = \Delta(fC_k, F;\ \bar{\mathsf{R}}^n \setminus B^n(b))$, $k = 1, 2, \ldots$, where $F = \bar{\mathsf{R}}^n \setminus f(B^n \cap U)$. As in the proof of Theorem 3.2, it follows from Lemma 3.1 that there exists $\delta > 0$ such that $M(\Gamma'_k) \geqq \delta$ for every integer k. Let Γ_k be the family of the maximal liftings of the elements of Γ'_k starting at C_k, see [5, 3.11]. Applying [5, 3.12] it follows from (3.10), (3.11), and [11, 7.5] that $M(\Gamma_k) \to 0$ as $k \to \infty$. By the modulus inequality in [12, 3.1]

$$0 < \delta \leqq M(\Gamma'_k) \leqq M(f\Gamma_k) \leqq K_I(f) M(\Gamma_k)$$

which leads to a contradiction as $k \to \infty$. Hence $\tilde{a} \leqq \tilde{b}$ and the proof is complete.

4. Some applications.

In this section we will give various applications of the above results. The main application, Theorem 4.1, extends a well-known theorem by Lindelöf [8,

p. 75]. Lindelöf's theorem states that a bounded analytic function of the unit disc, which is continuous on the boundary except at one boundary point b, and which has a limit along *the boundary* at b, has actually the same limit along *the closure* at b.

4.1. THEOREM. *Let $f\colon G \to \bar{R}^n$ be a quasimeromorphic mapping with $\operatorname{cap}(\bar{R}^n \setminus fG) > 0$, let $E \subset \partial G$ be a compact set of capacity zero, and let $y \in E \cap \overline{(\partial G \setminus E)}$. Suppose that there is a continuous mapping $\bar{f}\colon \bar{G} \setminus E \to \bar{R}^n$ such that $\bar{f} | G = f$ and such that the limit*

$$\lim_{\substack{x \to y \\ x \in \partial G \setminus E}} \bar{f}(x) = \alpha$$

exists. Then also the limit

$$\lim_{\substack{x \to y \\ x \in \bar{G} \setminus E}} \bar{f}(x) \tag{4.2}$$

exists and equals α.

PROOF. Performing an auxiliary Möbius transformation we may assume that $\alpha = 0$. Suppose that the limit (4.2) does not exist. Then there is a sequence (x_k) in $\bar{G} \setminus E$ tending to y with

$$\lim |\bar{f}(x_k)| = \bar{b} > 0 .$$

Denote $F = \bar{R}^n \setminus fG$. Since $\operatorname{cap} F > 0$, there is $\tilde{a} \in (0, \bar{b})$ with $\operatorname{cap}(F \setminus B^n(\tilde{a})) > 0$. Choose $\varrho > 0$ such that

$$\bar{f}((\bar{G} \setminus E) \cap \bar{B}^n(y, \varrho)) \subset B^n(\tilde{a}/2) .$$

If we modify Lemma 3.1 and the proof of Theorem 3.2, we obtain a contradiction. The proof is complete.

4.3. THEOREM. *Let $f\colon B^n \to \bar{R}^n$ be a quasimeromorphic mapping with $\operatorname{cap}(\bar{R}^n \setminus fB^n) > 0$ and let $E \subset \partial B^n$ be a compact set with $M(y, E) < \infty$ for some $y \in E$. Suppose that there is a continuous mapping $\bar{f}\colon \bar{B}^n \setminus E \to \bar{R}^n$ such that $\bar{f} \mid B^n = f$ and such that the limit*

$$\lim_{\substack{x \to y \\ x \in \partial B^n \setminus E}} \bar{f}(x) = \alpha \tag{4.4}$$

exists. Then the angular limit

$$\lim_{\substack{x \to y \\ x \in K(y,\varphi)}} f(x)$$

exists and equals α *for every* $\varphi \in (0, \pi/2)$.

PROOF. Follows from Theorem 3.7 and from the proof of Theorem 4.1.

4.5. REMARK. It is interesting to compare Theorems 4.3 and 3.7 with some results of Ohtsuka in [10]. The author is grateful to Professor Ohtsuka for this reference.

Let $B = \{(x,y) \mid 0 < y < 1\}$, let F be a compact set on the real axis, and let $f\colon B \cup F \to \mathsf{R}^2$ be a bounded continuous function with $f(x) \to 0$ as $x \to \infty$ and $x \in F$ and let f be analytic in B. It follows from [10, Theorem 4] that f tends to 0 in any strictly narrower substrip of B if F has positive average linear measure near $x = +\infty$. For terminology, see [10, p. 149]. If one maps B by $h\colon z \mapsto -e^{-\pi z}$ onto the upper half plane, this result gives a sufficient condition for the fact that $f \circ h^{-1}$ has 0 as the angular limit at the origin. The condition on F is the best possible; see [10, Theorem 5].

In the final application of the results we discuss asymptotic values of quasimeromorphic mappings. A point $z \in \bar{\mathsf{R}}^n$ is *an asymptotic value* of $f\colon G \to \bar{\mathsf{R}}^n$ at $y \in \partial G$ if there exists a path $\gamma\colon [0,1) \to G$ with $\gamma(t) \to y, f(\gamma(t)) \to z$ as $t \to 1$.

4.6. COROLLARY. *Let us consider the situation of Theorem* 4.3. *If* K_1 *and* K_2 *are continua in* $B^n \cup \{y\}$, $y \in K_i$, $i = 1,2$, *such that* $M(\Delta(K_i, E; B^n)) < \infty$, $i = 1,2$, *then the limits*

$$\lim_{\substack{x \to y \\ x \in K_1}} f(x) \quad \textit{and} \quad \lim_{\substack{x \to y \\ x \in K_2}} f(x)$$

both exist and equal the value α *in* (4.4). *Thus* f *has a well-defined asymptotic value along a path* $\gamma\colon [0,1) \to B^n$, $\gamma(t) \to y$ *as* $t \to 1$ *if* $M(\Delta(|\gamma|, E; B^n)) < \infty$.

PROOF. From the proof of Theorem 4.3 it follows that the cone $K(y, \varphi)$ can be replaced by any continuum $K \subset B^n \cup \{y\}$, $y \in K$ with $M(\Delta(K, E; B^n)) < \infty$. Hence the proof follows.

REFERENCES

1. J. B. Garnett, *On the Iversen–Tsuji theorem*, Indiana Math. J. 24 (1974), 285–299.
2. O. Martio, *Equicontinuity theorem with an application to variational integrals*, Duke Math. J. 42 (1975), 569–581.
3. O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 448 (1969), 1–40.
4. O. Martio, S. Rickman, and J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 465 (1970), 1–13.
5. O. Martio, S. Rickman, and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 488 (1971), 1–31.
6. O. Martio and S. Rickman, *Boundary behavior of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 507 (1972), 1–17.
7. O. Martio and J. Sarvas, *Density conditions in the n-capacity*, Indiana Math. J. (To appear.).
8. R. Nevanlinna, *Analytic functions* (Grundlehren Math. Wissensch. 162), Springer-Verlag, Berlin · Heidelberg · New York, 1970.
9. K. Noshiro, *Cluster sets* (Ergebnisse Math. 28), Springer-Verlag, Berlin · Heidelberg · New York, 1960.
10. M. Ohtsuka, *Generalizations of Montel–Lindelöf's theorem on asymptotic values*, Nagoya Math. J. 10 (1956), 129–163.
11. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings* (Lecture Notes in Mathematics 229), Springer-Verlag, Berlin · Heidelberg · New York, 1971.
12. J. Väisälä, *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 509 (1972), 1–14.
13. M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A I, Dissertationes 11 (1976), 1–44.
14. W. Ziemer, *Extremal length and p-capacity*, Michigan Math. J. 16 (1969), 43–51.

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