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INNER ONE-PARAMETER GROUPS ACTING ON A FACTOR

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1. Introduction.

Let $t \to \sigma_t$ be a σ-weakly continuous one-parameter group of automorphisms on a von Neumann algebra $\mathfrak{M}$. We say that a family $(v_t)_{t \in \mathsf{R}}$ of unitaries is implementing for σ if

$$\sigma_t(x) = \mathrm{Ad}\,(v_t)(x) = v_t x v_t^* \qquad \forall\, t \in \mathsf{R},\ \forall\, x \in \mathfrak{M}\,.$$

In proposition 1 we prove that an arbitrary implementing family for σ contained in $\mathfrak{M}$ is commutative.

Let in the following $\mathfrak{M}$ be a factor and denote by $\mathfrak{M}_u$ the unitary group in $\mathfrak{M}$ and by T the group of complex numbers with modulus one. In theorem 2 we prove that a continuous projective representation that is, a homomorphism

$$\pi\colon \mathsf{R} \to \mathfrak{M}_u/\mathsf{T}$$

can be lifted to a continuous unitary representation. As a corollary we get the classical implementation result due to Bargmann [1] with a simplified proof avoiding cohomology considerations.

Let $k\colon \mathfrak{M}_u \to \mathfrak{M}_u/\mathsf{T}$ denote the quotient map. The definition

$\Phi(\dot u) = \mathrm{Ad}\,(u), \quad u \in k^{-1}(\dot u),\ \dot u \in \mathfrak{M}_u/\mathsf{T}$ gives a homomorphism Φ from the projective group $\mathfrak{M}_u/\mathsf{T}$ to the group $\mathrm{Int}\,(\mathfrak{M})$ of inner automorphisms on $\mathfrak{M}$. It is easily checked that Φ is an injection which is continuous, when $\mathfrak{M}_u/\mathsf{T}$ is equipped with the quotient topology from the σ-weak topology on $\mathfrak{M}_u$ and $\mathrm{Int}\,(\mathfrak{M})$ is equipped with the pointwise σ-weak topology.

Let $\pi\colon \mathsf{R} \to \mathfrak{M}_u/\mathsf{T}$ be a projective representation and $\sigma = \Phi \circ \pi\colon \mathsf{R} \to \mathrm{Int}\,(\mathfrak{M})$ the corresponding one-parameter group.

$$\begin{array}{ccc} \mathfrak{M}_u/\mathsf{T} & \xrightarrow{\ \Phi\ } & \mathrm{Int}\,(\mathfrak{M}) \\ & {}_{\pi}\nwarrow \quad \nearrow_{\sigma} & \\ & \mathsf{R} & \end{array}$$

If π is continuous then so is σ. In the case where the predual $\mathfrak{M}_*$ is separable, we can conclude that π is continuous if just σ is continuous, although Φ in general is not an open mapping. Indeed, if σ is continuous, we can by von

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Neumann's measurable choice theorem choose a measurable family $(v_t)_{t\in R}$ of unitaries in $\mathfrak{M}$ implementing σ. Observe that $\pi(t)=k(v_t)$, $t \in R$ which implies that π is measurable thus continuous.

The proof of theorem 2 is elementary and makes no use of measure theory, thus it extends to the non-separable case. Connes gives in [2, Corollaire 1.5.8 (c) p. 166] an example of a factor with non-separable predual, where each of the modular automorphisms is inner but no continuous unitary representation of R implementing for σ^φ is possible. This example together with theorem 2 shows the existence of a continuous one-parameter group σ, where the corresponding projective representation $\pi=\Phi^{-1}\circ\sigma$ is not continuous.

2. Reduction to the abelian case.

PROPOSITION 1. *Let $t \to \sigma_t$ be a σ-weakly continuous one-parameter group of automorphisms on a von Neumann algebra each of which is inner and let $v_t \in \mathfrak{M}$ for each $t \in R$ be a choice of a unitary implementation for σ_t. Then all v_t commute.*

PROOF. Since v_tv_s and v_sv_t both induce the automorphism σ_{t+s} on $\mathfrak{M}$ they can only differ by a unitary in the center. We can therefore define a mapping

$$c\colon R \times R \to (\text{Center}(\mathfrak{M}))_u$$

by requiring

$$\sigma_t(v_s) = v_tv_sv_t^* = c(t,s)v_s \qquad \forall t,s \in R .$$

Using the assumed properties of σ we get

$$\text{(i)} \quad c(t_1+t_2,s)v_s = \sigma_{t_1+t_2}(v_s) = \sigma_{t_1}(c(t_2,s)v_s) = c(t_1,s)c(t_2,s)v_s ,$$

$$\text{(ii)} \quad v_tv_s = c(t,s)v_sv_t = c(t,s)c(s,t)v_tv_s ,$$

$$\text{(iii)} \quad c(t,s)v_s = \sigma_t(v_s) \xrightarrow[t\to t_0]{} \sigma_{t_0}(v_s) = c(t_0,s)v_s$$

where the convergence is in the σ-weak topology.

We conclude that c is a skew symmetric bihomomorphism which is σ-weakly continuous in each variable and that $c(t,t)=1$ $\forall t \in R$.

Now

$$c(2t,t) = c(t+t,t) = c(t,t)^2 = 1, \qquad \forall t \in R ,$$

$$c(q2^{-n}t,t) = c(2^{-n}t,t)^q = 1, \qquad \forall t \in R,\ \forall q,n \in N ,$$

$$c(t,s)c(-t,s) = c(o,s) = 1, \qquad \forall t,s \in R .$$

From this we conclude that $c(rt,t)=1$ for all $t \in \mathsf{R}$ and all rational dyadic number $r=q2^{-n}$, $q \in \mathsf{Z}$, $n \in \mathsf{N}$. The continuity of c in each variable finally ensures that $c(t,s)=1$, $\forall\, t,s \in \mathsf{R}$.

3. Lifting of projective representations.

THEOREM 2. Let π: $\mathsf{R} \to \mathfrak{M}_{\mathrm{u}}/\mathsf{T}$ be a continuous projective representation of R. There exists a continuous representation $t \to u_t$ of R into $\mathfrak{M}_{\mathrm{u}}$ such that $k(u_t) = \pi(t)$, $\forall\, t \in \mathsf{R}$, where $\mathfrak{M}_{\mathrm{u}}$ is equipped with the σ-weak topology and $\mathfrak{M}_{\mathrm{u}}/\mathsf{T}$ with the quotient topology.

PROOF. Let

$$E = \{(u,t) \in \mathfrak{M}_{\mathrm{u}} \times \mathsf{R} \mid k(u)=\pi(t)\} .$$

The product $(u_1,t_1)\cdot(u_2,t_2)=(u_1u_2,t_1+t_2)$ and the restriction of the product topology on $\mathfrak{M}_{\mathrm{u}} \times \mathsf{R}$ makes E a topological group which by proposition 1 is abelian. The rest of the proof is well-known. Consider the following diagram:

$$\begin{array}{ccccc} \mathsf{T} & \xrightarrow{\tilde{\imath}} & E & \xrightarrow{\tilde{k}} & \mathsf{R} \\ & & \downarrow & & \downarrow \pi \\ \mathsf{T} & \xrightarrow{i} & \mathfrak{M}_{\mathrm{u}} & \xrightarrow{k} & \mathfrak{M}_{\mathrm{u}}/\mathsf{T} \end{array}$$

where $\tilde{\imath}(z)=(z\cdot 1,0)$, $\tilde{k}(u,t)=t$ and $i(z)=z\cdot 1$, $z \in \mathsf{T}$, $u \in \mathfrak{M}_{\mathrm{u}}$, $t \in \mathsf{R}$.

Observe that the kernel $\ker(\tilde{k})=\tilde{\imath}(\mathsf{T})$ and that $\tilde{k}$ is an open map, which means that R as a topological group is the quotient of E by T. The theorem of Gleason [4, Theorem 2.2, p. 52] now tells us that E is locally compact. It should be noted that the proof of Gleason's theorem is elementary and only involves the basic concepts from the theory of locally compact groups. In particular the theorem does not require separability.

We denote by $\hat{\mathsf{T}}$ and $\hat{E}$ the dual groups of T and E, respectively, so that $\hat{\mathsf{T}}=\mathsf{Z}$. The set

$$H = \{\chi \in \hat{\mathsf{T}} \mid \exists\, \tilde{\chi} \in \hat{E}\colon \chi=\tilde{\chi}\circ\tilde{\imath}\}$$

is a subgroup of $\hat{\mathsf{T}}$. But as the characters on E separates points in E and $\hat{\mathsf{T}}$ does not contain any separating proper subgroup, we conclude that $H=\hat{\mathsf{T}}$. In particular if χ is the identity character on T, there is a character $\tilde{\chi}$ on E for which $\chi=\tilde{\chi}\circ\tilde{\imath}$. We have $\tilde{\chi}(\tilde{\imath}(z))=\chi(z)=z$, $\forall\, z \in \mathsf{T}$, which means that

$$\tilde{\imath}(\mathsf{T}) \cap \ker(\tilde{\chi}) = \{(1,0)\} .$$

Thus $\ker(\tilde{\chi})$ is a topological complement to $\tilde{\imath}(\mathsf{T})$ in E, hence there exists an isomorphism $t \to (u_t,t)$ from R to $\ker(\tilde{\chi})$. Consequently $t \to u_t$ is a continuous representation and $k(u_t)=\pi(t)$, which concludes the proof.

Corollary 3. *Let $t \to \sigma_t$ be a σ-weakly continuous one-parameter group of automorphisms each of which is inner on a factor $\mathfrak{M}$ with separable predual.*

There exists a σ-weakly continuous representation $t \to u_t$ of $\mathbf{R}$ into the unitary group in $\mathfrak{M}$ such that

$$\sigma_t(x) = u_t x u_t^*, \qquad \forall t \in \mathbf{R}, \ \forall x \in \mathfrak{M} .$$

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