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IDEAL PERTURBATIONS OF ELEMENTS IN C*-ALGEBRAS

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0. Introduction.

The general class of problems considered in this paper can be described as finding a "best" approximation to a given element b of a C*-algebra A among all elements of a closed, two-sided ideal I of A. The notion of "best" needs clarification and generally it will vary with the specific problem in question. We always mean more than mere norm approximation ($\|b-c\| = \|\pi(b)\|$, where π: $A \to A/I$ is the quotient map) which can always be attained [2, p. 291]. The optimal conclusion from our point of view is to find $c \in I$ with $\|p(b+c)\| = \|\pi(p(b))\|$ for every complex polynomial p. While this cannot happen for every operator in a general C*-algebra A, it remains an open problem for von Neumann algebras. We are able to reach this goal only for special classes of operators.

One of our most useful tools is a quasi-central approximate unit which can always be found in I. This is an increasing net $\{a_\alpha\}_{\alpha \in D}$ of positive operators in I which is an approximate unit for I and which satisfies $\|a_\alpha b - b a_\alpha\| \to 0$ for every $b \in B$. This fact and its corollaries are 3.2–3.6. The same fact was discovered independently by Arveson [5] and used to extend the results of [22]. In embryonic form the quasi-central approximate unit appears in [12, 18 and 19].

The organization of the paper is as follows. The first section establishes the notation and recalls some preliminary results from the literature. In the second section we deal with theorems of the form, "if $f(x_1,\ldots,x_n) \in I$, then..."; the third section deals with the case $f(x_1,\ldots,x_n) \notin I$. Both of these sections concern general C*-algebras and generalize results of [13, 14, 15]. The fourth section specializes to the case of a von Neumann algebra A and generalizes results of [13, 14, 15, 24].

1. Notation and preliminary remarks.

Throughout the paper A will be a C*-algebra, I will be a closed two-sided ideal of A and π: $A \to A/I$ will denote the quotient map. At times we shall

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specialize A or I to get sharper results. The case where $A=B(H)$, the algebra of bounded operators on the separable Hilbert space H, and $I=C(H)$ the compact operators on H, has received the most attention from other authors but will here only serve as a motivating example.

We let A_{sa} be the self-adjoint part of A, A^+ the positive part of A, and note that if $x \in A_{sa}$ then, $x=x_+-x_-$ is the unique orthogonal decomposition of x as a difference of elements from A^+ [21, 1.4.3]. For any element $x \in A$ let $C^*(x)$ be the C*-subalgebra of A generated by x. If x is normal then $C^*(x)$ can be identified with the algebra of continuous complex-valued functions on $\sigma(x)$, the spectrum of x, which vanish at 0 (if $0 \in \sigma(x)$). This is a conclusion of the Gelfand theory as described in [7, p. 10] or [20, p. 4].

If M is a von Neumann algebra and $x \in M$, then the polar decomposition $x=u|x|$ can be performed in M [20, p. 27]. Here $|x|=(x^*x)^{\frac{1}{2}}$ and uu^* is the range projection of x (denoted by $[x]$ in section 4), whereas (with $p'=1-p$ for any projection p) $(u^*u)'$ is the null projection of x^*.

We let A^* denote the dual space of A and A^{**} its second dual. Since we do not assume that A has a unit we replace the state space by the compact set

$$Q(A) = \{f \in A^* \mid f \geqq 0,\ \|f\| \leqq 1\}\ .$$

As in [20, 1.7] and [7, § 12] we shall use the fact that A^{**} is a von Neumann algebra and identify A with its image in A^{**}, and I^{**} with its image in A^{**} under the double transposed of the inclusion map of I into A. Now $I^{**}=qA^{**}$, where q is a projection in the center of A^{**} and $(A/I)^{**}=q'A^{**}$ by a natural identification) so $\pi^{**}(b)=q'b$ for every $b \in A^{**}$.

2. C*-algebra results for $f(x_1,\ldots,x_n) \in I$.

We first note an easy fact which allows us to use the polar decomposition in a C*-algebra setting

LEMMA 2.1. *If $x \in A \subset B(H)$ and $x=u|x|$ is the polar decomposition of x in $B(H)$, then $uf(|x|) \in A$ for every continuous complex-valued function f on $\sigma(|x|)$ which vanishes at zero. (In particular we can replace $B(H)$ by any von Neumann algebra, e.g. A^{**}.)*

PROOF. For any scalars $\lambda_1,\ldots,\lambda_n$ we have

$$u\left(\sum_{1\leqq i\leqq n} \lambda_i|x|^i\right) = \sum_{1\leqq i\leqq n} \lambda_i x|x|^{i-1} \in A\ .$$

By the Stone-Weierstrass theorem, since $\sigma(|x|)$ is a compact subset of $\mathbf{R}^+$, we may uniformly approximate any continuous complex function on $\sigma(|x|)$ which vanishes at 0 by a polynomial without constant term. The lemma follows from the Gelfand theory.

The proof of next lemma is omitted. It is an easy consequence (as in the proof of Lemma 2.1) of the Stone-Weierstrass theorem and the Gelfand theory.

LEMMA 2.2. *For any $x \in A_{sa}$ and any continuous function f on $\sigma(x)$ which vanishes at 0 we have $\pi(f(x)) = f(\pi(x))$.*

PROPOSITION 2.3. *If $x, y \in A$ with $xy \in I$, then there exist $a, b \in I$ with $(x-a)(y-b) = 0$.*

PROOF. We first prove the proposition under the assumption that x and y are in A^+. Then we use Lemma 2.1 to get the general case.

Assume $x \geqq 0, y \geqq 0$. Set $x_1 = (x-y)_+$ and $y_1 = (x-y)_-$. By definition $x_1 y_1 = 0$. Note that $\pi(x)\pi(y) = \pi(xy) = 0$, so

$$\pi(x) = (\pi(x) - \pi(y))_+ = \pi((x-y)_+) = \pi(x_1)$$

by Lemma 2.2. Similarly $\pi(y) = \pi(y_1)$. Set $a = x - x_1$ and $b = y - y_1$. We have just shown that $a, b \in I$ and $(x-a)(y-b) = x_1 y_1 = 0$.

In general we use the polar decomposition in A^{**}, as mentioned in Lemma 2.1, to write $x = u|x|$ and $y^* = v|y^*|$. By assumption $xy \in I$, so by Gelfand theory and since I is closed,

$$|x|^{\frac{1}{2}}|y^*|^{\frac{1}{2}} = \lim_{\varepsilon \to 0} (\varepsilon + |x|)^{-\frac{3}{2}} (x^*x)(yy^*)(\varepsilon + |y^*|)^{-\frac{3}{2}} \in I .$$

From the first part of the proof there exist $a_1, b_1 \in I$ with

$$(|x|^{\frac{1}{2}} - a_1)(|y^*|^{\frac{1}{2}} - b_1) = 0 .$$

Let $a = u|x|^{\frac{1}{2}} a_1$ and $b = b_1 |y^*|^{\frac{1}{2}} v^*$. By Lemma 2.1 both $u|x|^{\frac{1}{2}}$ and $v|y^*|^{\frac{1}{2}}$ lie in A, so $a, b \in I$. Thus we need only show that

$$\begin{aligned} 0 &= u|x|^{\frac{1}{2}}[(|x|^{\frac{1}{2}} - a_1)(|y^*|^{\frac{1}{2}} - b_1)]\,|y^*|^{\frac{1}{2}} v^* \\ &= (x-a)(|y^*|v^* - b) = (x-a)(y-b) , \end{aligned}$$

which follows since $|y^*|v^* = (y^*)^* = y$.

COROLLARY 2.4. *If in Proposition 2.3 we have $x, y \in A^+$, then we can get $(x-a)$, $(y-b) \in I^+$; if $x, y \in A_{sa}$, we can get $(x-a)$, $(y-b) \in I_{sa}$.*

PROOF. The case $x, y \in A^+$ is already contained in the proof of Proposition 2.3. If $x, y \in A_{sa}$, then using the notation of the proof of Proposition 2.3 and the fact that $|y|=|y^*|$, we set $\alpha=\frac{1}{3}$ and define

$$x_1 = [(|x|-|y|)_+]^\alpha u|x|^\alpha(|x|-|y|)_+]^\alpha$$
$$y_1 = [(|x|-|y|)_-]^\alpha v|y|^\alpha[(|x|-|y|)_-]^\alpha .$$

Clearly $x_1 y_1 = 0$. Further, $x_1, y_1 \in A_{sa}$ since $x, y \in A_{sa}$ imply that $u|x|^\alpha = |x|^\alpha u$ and $v|y|^\alpha = |y|^\alpha v$. To finish the proof we need only show that $\pi(x_1)=\pi(x)$ and $\pi(y_1)=\pi(y)$. Using Lemma 2.2 repeatedly and the fact that $|x|^{\frac{1}{2}}|y|^{\frac{1}{2}} \in I$ implies $|x||y| \in I$ we get

$$\begin{aligned}\pi(x_1) &= [(\pi(|x|)-\pi(|y|))_+]^\alpha \pi(u|x|^\alpha)[((\pi|x|)-\pi(|y|))_+]^\alpha \\ &= [\pi(|x|)]^\alpha \pi(u|x|^\alpha)[\pi(|x|)]^\alpha \\ &= \pi(|x|^\alpha u|x|^\alpha |x|^\alpha) = \pi(x) .\end{aligned}$$

Similarly $\pi(y_1)=\pi(y)$.

Extending Proposition 2.3 to more than two factors poses a serious (unsolved) problem. If A is a von Neumann algebra, we can prove (see 4.3) that the existence of projections makes it easy. For the C*-algebra case, however, we have only the following two propositions. One should bear in mind the example immediately preceding Theorem 6 of [13] which shows that it may be necessary to perturb every factor in order to get a zero product.

PROPOSITION 2.5. *If A is abelian and $\{x_1,\ldots,x_n\}\subset A$ with $\prod_{1\le i\le n} x_i \in I$, then there are elements $\{a_1,\ldots,a_n\}\subset I$ such that $\prod_{1\le i\le n}(x_i-a_i)=0$.*

PROOF. By the Gelfand theory we can assume that $A=C_0(\Omega)$, the continuous complex-valued functions which vanish at infinity on some locally compact Hausdorff topological space Ω. In this context $I=\{a\in A : a|_K=0\}$ for some closed subset $K\subset\Omega$. Thus $\prod_{1\le i\le n} x_i \in I$ means exactly that at each point of K at least one of the function x_i must vanish. Set

$$x = \bigwedge_{1\le i\le n} |x_i|^{\frac{1}{2}} .$$

Clearly x vanishes at each point of K, so $x\in I$. Let $x_i=u_i|x_i|$ be the polar decomposition in A^{**} of each x_i. Let $a_i=u_i|x_i|^{\frac{1}{2}}x$. Then

$$\begin{aligned}\prod_{1\le i\le n}(x_i-a_i) &= \prod_{1\le i\le n}(u_i|x_i|^{\frac{1}{2}}|x_i|^{\frac{1}{2}}-u_i|x_i|^{\frac{1}{2}}x) \\ &= \left[\prod_{1\le i\le n} u_i|x_i|^{\frac{1}{2}}\right]\left[\prod_{1\le i\le n}(|x_i|^{\frac{1}{2}}-x)\right] = 0 ,\end{aligned}$$

since for each $t\in\Omega$, $|x_i|^{\frac{1}{2}}(t)-x(t)=0$ for some $i=1,\ldots,n$ by definition of x.

The last proof is impossible to generalize to the non-abelian case. The next proposition moves in a different direction, but with the same goal in mind.

PROPOSITION 2.6. *If* $\{x_1,\ldots,x_n\} \subset A_+$ *and* $x_i x_j \in I$ *for all* $i \neq j$, *then there exist* $\{a_1,\ldots,a_n\} \subset I$ *such that* $\{x_i - a_i\}_{1 \leqq i \leqq n} \subset A_+$ *and* $(x_i - a_i)(x_j - a_j) = 0$ *for all* $i \neq j$.

PROOF. By Corollary 2.4 we have it for $n=2$, so assume it is true for $(n-1)$ by induction. Set $y = \sum_{1 \leqq i \leqq n-1} x_i$. By Corollary 2.4 there exist $a_n, b \in I$ such that $(x_n - a_n)(y-b) = 0$ (since $x_n y \in I$) and $(x_n - a_n), (y-b) \in A_+$. Since $\pi(y) \geqq \pi(x_i)$ for all $i = 1,\ldots,n-1$, we may apply Proposition 5 of [16] $(n-1)$ times to obtain elements $\{b_1,\ldots,b_{n-1}\} \subset I$ such that $\{(x_i - b_i)\}_{1 \leqq i \leqq n-1} \subset A_+$ and $(x_i - b_i) \leqq (y-b)$ for $i = 1,\ldots,n-1$. Now define the C*-algebra

$$A_0 = \{x \in A : (x_n - a_n)x = x(x_n - a_n) = 0\}.$$

Since $0 \leqq (x_i - b_i) \leqq (y-b)$ and $(x_n - a_n)(y-b) = 0$, we see that $(x_i - b_i) \in A_0$ for all $i = 1,\ldots,n-1$. In A_0 the elements $\{x_i - b_i\}_{1 \leqq i \leqq n-1}$ satisfy the hypotheses of the theorem relative to the ideal $I \cap A_0$, so there are elements $\{c_1,\ldots,c_{n-1}\}$ in $A_0 \cap I$ such that $\{(x_i - b_i - c_i)\}_{1 \leqq i \leqq n-1} \subset (A_0)_+$ and

$$(x_i - b_i - c_i)(x_j - b_j - c_j) = 0 \quad \text{for } i \neq j.$$

Set $a_k = b_k + c_k$ for $k = 1,\ldots,n-1$, then check that

$$(x_k - a_k)(x_n - a_n) = (x_n - a_n)(x_k - a_k) = 0$$

for $k < n$ since $(x_k - a_k) \in A_0$.

An important question which arises naturally from [19] is the following.

QUESTION 2.7. If $x \in A$ with $x^n \in I$ for some $n = 2,3,\ldots$, can we find $a \in I$ with $(x-a)^n = 0$?

In [19] it is shown that one can make $\|(x-a)^n\|$ as small as desired by properly selecting $a \in I$. In section 4 we prove that if A is a von Neumann algebra, then even a better theorem is possible. At this stage the best we can do is the case $n=2$.

PROPOSITION 2.8. *If* $x \in A$ *with* $x^2 \in I$, *then there exist* $a \in A$ *with* $(x-a)^2 = 0$ *and* $\|x - a\| \leqq \|x\|$.

PROOF. Let $x = u|x|$ and $x^* = u^*|x^*|$ be the polar decomposition of x and x^* in A^{**}. Set $\alpha = \frac{1}{3}$ and let

$$x_1 = [(|x| - |x^*|)_-]^\alpha u|x|^\alpha [(|x| - |x^*|)_+]^\alpha .$$

Clearly $x_1^2 = 0$, so we need only show $\pi(x) = \pi(x_1)$. By using Lemma 2.2 and the fact that $|x||x^*| \in I$ (since

$$|x||x^*| = \lim_{\varepsilon \to \infty} (\varepsilon + |x|)^{-1}(x^*x(xx^*)(\varepsilon + |x^*|)^{-1}$$

just as in the proof of Proposition 2.3), we see that $\pi(x_1) = \pi(|x^*|^\alpha u|x|^\alpha|x|^\alpha)$. However, the mapping $|x| \to u|x|u^*$ defines a*-isomorphism between the C*-algebra generated by $|x|$ and the C*-algebra generated by $|x^*|$ with the image of $|x|$ being $|x^*|$. Thus one can readily show that $|x^*|^\alpha u = u|x|^\alpha$, so

$$\pi(x_1) = \pi(u|x|^\alpha|x|^\alpha|x|^\alpha) = \pi(x)\ .$$

As mentioned earlier if A is a von Neumann algebra, one can get stronger results (see 4.3). Specifically, one can replace the (implicit) polynomial $p(t) = t^n$ in Question 2.7 by any polynomial. The following commutative example shows that this is possible only because von Neumann algebras are generated by their projections.

Example 2.9. Let $A = C([0,1])$ and let $I = \{a \in A\ :\ a(0) = a(1) = 0\}$. Let $p(t) = t^2 - t$. Then for the element $x \in A$ defined by $x(t) = t$, we see that for any $a \in I$, the function $y = (x - a)$ takes on the value 0 (*at* $t = 0$) and 1 (*at* $t = 1$). Thus y defines a path Γ from 0 to 1 in $\mathbf{C}$. Since the complex polynomial $p(z) = z^2 - z$ has only two roots, we cannot have $p(x - a) = 0$ for any $a \in I$. In fact, since Γ must intersect the line $\mathrm{Re}\,(z) = \frac{1}{2}$, one can compute that $\|p(x-a)\| \geqq \frac{1}{4}$ for all $a \in I$.

This last example naturally suggests that one should find the best possible abelian case theorem before proceeding to investigate the general case. The next result is the best generally applicable result we could conjecture.

Proposition 2.10. *Let $b \in A$ be normal. Then there exists $a \in I \cap C^*(b)$ such that $\|\varphi(b+a)\| = \|\pi(\varphi(b))\|$ for all continuous functions from $\mathbf{C}$ to $\mathbf{C}$ for which $|\varphi|$ is convex and vanishes at* 0.

Proof. Consider $A_0 = C^*(b)$ and $I_0 = A_0 \cap I$. Since for each $a \in I_0$,

$$\pi(\varphi(b+a)) = \varphi(\pi(b+a)) = \varphi(\pi(b)) = \pi(\varphi(b))\ ,$$

we have

$$\|\varphi(b+a)\| \geqq \|\pi(\varphi(b+a))\| = \|\pi(\varphi(b))\|.$$

We must find an $a \in I_0$ which makes the reverse inequality true.

Let $A_0 = C_0(\sigma(b)\backslash\{0\})$ as described in section 1 and $I_0 = \{a \in A_0 : a|_K = 0\}$ for some closed set $K \subset \sigma(b)\backslash\{0\}$. Let Q be the convex hull of $K \cup \{0\}$ (which is automatically closed), and note that the extreme points of Q lie in $K \cup \{0\}$ and that any continuous convex function from a compact convex set of **C** to **R** must assume its maximum at an extreme point.

Now let D be a disk which contains $\sigma(b) \cup Q$ and let $r:D \to Q$ be a continuous retraction. By the convexity of Q, r always can be found. Then define $a \in C_0(\sigma(b))$ by $a(z) = r(z) - z$. Clearly a is continuous, since r is continuous (and $a(0) = 0$). Further $a(z) = 0$ for $z \in Q$, so $a \in I$. Thus

$$\begin{aligned} \|\varphi(b+a)\| &= \|\varphi(r(z))\| \leqq \sup\{|\varphi(w)| : w \in Q\} \\ &= \sup\{|\varphi(w)| : w \in \text{extreme points of } Q\} \\ &\leqq \sup\{|\varphi(w)| : w \in K\} = \|\varphi(\pi(b))\| = \|\pi(\varphi(b))\| \end{aligned}$$

by Lemma 2.2.

If b is not normal, $\varphi(b)$ is only defined when φ is an entire function, unless restrictions are made on the spectrum of b. As long as we are dealing with general C*-algebras and general (non-normal) elements, the last result explains our strong interest in the polynomials $\varphi(t) = t^n$.

3. C*-algebra results for $f(x_1, \ldots, x_n) \notin I$.

The methods of this section are quite different from those of section 2. Our results generalize those in [13] and the basic ideas of the proofs are closely related to those in [13]. The key lies in Theorem 3.2 on the existence of special approximate units.

LEMMA 3.1. *For a positive, increasing net* $\{u_\lambda \mid \lambda \in \Lambda\}$ *in* I *the following are equivalent.*

(1) $\{u_\lambda\}$ *is an approximate unit for* I.

(2) $u_\lambda \to q$ *in the* $\sigma(A^{**}, A^*)$ *topology. (See section* 1.)

(3) $f(u_\lambda) \to 1$ *for every pure state* f *of* I.

PROOF. That (3) implies (1) follows from [2, 5.1] and that (2) implies (3) is immediate. Assuming (1), to prove (2) it suffices to show that $f(u_\lambda - q) \to 0$ for every state f of I since every element of A^* is a linear combination of four states. By definition of $q, f(q) = 1$ for every state of I, so we need only show that $f(u_\lambda) \to 1$ for every state of I. Since

$$u_{\lambda_0}^3 \leqq u_{\lambda_0} u_\lambda u_{\lambda_0} \to u_{\lambda_0}^2,$$

we get $\|u_\lambda\| \leqq 1$. For a state f of I and a positive element $a \in I$, $\|a\| \leqq 1$,

$$|f(u_\lambda a)|^2 \leqq f(a^2)f(u_\lambda^2) \leqq f(u_\lambda) .$$

Since the left side can approach 1, the right hand side must converge to 1.

DEFINITION. We say that a positive, increasing approximate unit $\{u_\lambda \mid \lambda \in \Lambda\}$ for I is *quasi-central* for A if for each $x \in A$

$$\|u_\lambda x - x u_\lambda\| \to 0 .$$

THEOREM 3.2. *Let $\{v_i \mid i \in Q\}$ be a positive, increasing approximate unit for I contained in an ideal I_0 of A which is dense in I. There is then a positive, increasing approximate unit for I contained in* Conv $\{v_i \mid i \in Q\}$, *which is quasi-central for A.* ("Conv" *means* "*convex hull of*".)

PROOF. Let Λ denote the collection of all (non-empty) finite subsets of A and for each λ in Λ let $|\lambda|$ denote the cardinality of λ. Given i and λ let $M_{i\lambda}$ denote the set of elements u in

$$\mathrm{Conv}\,\{v_j \mid j \succ i\}$$

such that $\|ux - xu\| < |\lambda|^{-1}$ for all x in λ. We claim that $M_{i\lambda} \neq \emptyset$.

To see this fix $x_1, x_2, \ldots, x_n$ in A and let $C = \oplus_{k=1}^n A$. Consider the net $\{c_j \mid j \succ i\}$ in C where

$$c_{jk} = v_j x_k - x_k v_j, \qquad 1 \leqq k \leqq n .$$

Working in A^{**} we know from section 1 that $v_j \to q$ where q is the open central projection in A^{**} for which $I = qA^{**} \cap A$. It follows that $c_j \to 0$ σ-weakly in C^{**}. But since C in the σ-weak topology and the norm topology has the same continuous functionals, we conclude from the Hahn–Banach theorem that Conv $\{c_j\}$ contains zero as a limit point in norm. Consequently, there is a convex combination $u = \sum_j \gamma_j v_j$, $j \succ i$, such that

$$\|u x_k - x_k u\| < 1/n \qquad \text{for every } k \leqq n .$$

With $\lambda = \{x_1, \ldots, x_n\}$ we see that $u \in M_{i\lambda}$.

Invoking the axiom of choice we select an element $u_{i\lambda}$ from each $M_{i\lambda}$, $i \in Q$, $\lambda \in \Lambda$. We define a partial order in the set U of these elements by $u_{i\lambda} \prec u_{j\mu}$ if $i \prec j$, $\lambda \subset \mu$ and $u_{i\lambda} \leqq u_{j\mu}$. To show that U is a directed set take $u_{i\lambda}$ and $u_{j\mu}$ in U; say $u_{i\lambda} = \sum \gamma_n v_n$ and $u_{j\mu} = \sum \gamma_m v_m$. Find k in Q such that $k \succ i$, $k \succ j$ and $k \succ n$, $k \succ m$ for all n and m occurring in the expressions for $u_{i\lambda}$ and $u_{j\mu}$, respectively. Take $\nu \supset \mu \cup \lambda$ and consider the element $u_{k\nu}$ in U. We have $i \prec k$, $\lambda \subset \nu$ and also $u_{i\lambda} \leqq u_{k\nu}$; because if $u_{k\nu} = \sum \gamma_j v_j$ then $j \succ k \succ n$ for all j and n, whence $v_j \geqq v_n$ for all

j and n, so that finally $u_{k\nu} \geqq u_{i\lambda}$. Consequently $u_{k\nu} \succ u_{i\lambda}$ and similarly $u_{k\nu} \succ u_{j\mu}$, whence U is directed.

By construction the net U is contained in Conv $\{v_i\}$ and $u_{i\lambda} \geqq v_i$ for all i and λ. By Lemma 3.1 U is an approximate unit for I. Moreover, U is quasi-central, since $\|u_{i\lambda}x - xu_{i\lambda}\| \leqq |\lambda|^{-1}$ for each x in λ.

We note that this last result was proved independently by W. Arveson in [4] as were several of the Corollaries which follow.

COROLLARY 3.3. *If I contains a strictly positive element b, then $C^*(b)$ contains a quasi-central approximate unit for I relative to A.*

PROOF. We need only to show that $C^*(b)$ contains an increasing positive approximate unit for I. However, this follows from [1] (see also [17, 3.10.6]). Specifically one may take $u_n = b^{1/n}$ and $\{u_n\}$ will be an approximate unit for I.

Since every separable C*-algebra contains a strictly positive element, the next corollary is immediate from Corollary 3.3.

COROLLARY 3.4. *If I is separable, then I contains an abelian quasi-central approximate unit relative to A.*

COROLLARY 3.5. *If $A = B(H)$ and $I = C(H)$ for a separable Hilbert space H, then for each orthonormal basis $\{\eta_n\}_{n \geqq 1}$ of H there is a quasi-central approximate unit $\{u_\lambda\}$ for I relative to A with all $\{u_\lambda\}$ diagonal relative to $\{\eta_n\}_{n \geqq 1}$ and of finite rank.*

PROOF. Let P_k be the orthogonal projection onto the span of $\{\eta_1, \ldots, \eta_k\}$. Then $\{P_k\}_{k \geqq 1}$ is an approximate unit for I, each element of which is diagonal for $\{\eta_n\}_{n \geqq 1}$ and of finite rank. Thus the theorem gives the conclusion immediately.

COROLLARY 3.6. *If A is separable, then I contains an abelian sequential, quasi-central approximate unit for I relative to A.*

PROOF. Let $\{x_n\}$ be a dense sequence in A, $\{y_n\}$ a dense sequence in I and let b be a strictly positive element in I (since I is also separable). Then by Corollary 3.3, $C^*(b)$ contains a quasi-central approximate unit $\{u_\lambda\}$ for I relative to A. By induction we can choose $u_{\lambda_1} \leqq \mu_{\lambda_2} \leqq \ldots$ such that

(1) $$\|u_{\lambda_k}x_n - x_n u_{\lambda_k}\| < 2^{-k} \quad \text{for all } n \leqq k, \text{ and}$$

(2) $$\|u_{\lambda_k}y_n - y_n\| < 2^{-k} \quad \text{for all } n \leqq k\,.$$

A simple triangle inequality argument using (1) and (2) shows that we have a quasi-central approximate unit.

This last corollary will be quite useful in the rest of this section. We shall use it to extend and improve several results in [13]. We are certainly grateful to C.L. Olsen for making available the results in [13] prior to publication.

THEOREM 3.7. *Given* $\{x_1,\ldots,x_l\}\subset I$ *with* $\prod_{1\leqq i\leqq l} x_i \notin I$ *and a fixed positive integer* m *with* $2\leqq m\leqq l$, *then there exists* $c\in I$ *with* $0\leqq c\leqq 1$ *and, if* $d_i=x_i(1-c)$, *then for every* $m\leqq k\leqq l$,

$$(*)\qquad \left\|\left(\prod_{1\leqq i\leqq m} d_i\right)\left(\prod_{m<i\leqq k} x_i\right)\right\| = \left\|\prod_{1\leqq i\leqq k} \pi(x_i)\right\|.$$

(If $m=k$, $\prod_{m<i\leqq k} x_i=1$ *by convention.)*

PROOF. Let $A_0=C^*(x_1,\ldots,x_l)$, $I_0=A_0\cap I$, $\pi_0: A_0\to A_0/I_0$. Note that for $x\in A_0$, $\|\pi_0(x)\|=\|\pi(x)\|$, since the induced map φ in the commutative diagram below is a $*$ isomorphism, hence isometric.

$$\begin{array}{ccc} A_0 & \xrightarrow{\;i\;} & A \\ {\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ A_0/I_0 & \xrightarrow{\;\varphi\;} & A/I \end{array}$$

Since the left hand side of (*) is greater than or equal to the right hand side, for any $c\in I$, we can prove the theorem by showing that there is a $c\in I_0$ with $0\leqq c\leqq 1$ which gives the reverse inequality. We can avoid some subscripts by identifying A with A_0, and we can thus assume A is separable. This allows us to apply Corollary 3.4 to get an abelian, quasi-central approximate unit $\{u_\lambda\}_{\lambda\in D}$ for I relative to A.

Let $\alpha_k=\|\prod_{1\leqq i\leqq k}\pi(x_i)\|$, let $\alpha=\min\{\alpha_k : 1\leqq k\leqq l\}$, take a strictly decreasing sequence $\{\delta_n\}_{n\geqq 1}$ of positive real numbers with $\delta_1=1$ and $\sum_{n\geqq 1}\delta_n<\infty$, and set $\varepsilon_n=\alpha\sum_{j\geqq n}\delta_{j+1}$. We shall construct a sequence $\{u_n\}_{n\geqq 1}\subset\{u_\lambda\}_{\lambda\in D}$ such that if $a_0=1$ and $a_n=\prod_{1\leqq j\leqq n}(1-\delta_j u_j)$ for $n\geqq 1$, then

$$(**)\qquad \left\|\left[\prod_{1\leqq i\leqq m}(x_i a_n)\prod_{m+1\leqq i\leqq k} x_i\right]\right\| < \alpha_k+\varepsilon_n \quad \text{for all } m\leqq k\leqq l\,.$$

Assume we have defined u_j so that (**) holds for all $j<n$. (If $n=1$, we have not defined any u_j as yet.) For fixed k, $m \leqq k \leqq l$, set

$$b_k = \left[\prod_{1\leqq i\leqq m} (x_i a_{n-1})\right]\left[\prod_{m+1\leqq i\leqq k} x_i\right]$$

and

$$F_k(\lambda) = \{f \in Q(A) : f(b_k(1-\delta_n u_\lambda)^2 b_k^*) \geqq (\alpha_k+\varepsilon_n)^2\} .$$

Since $\{u_\lambda\}_{\lambda\in D}$ is increasing and abelian, $\{b_k(1-\delta_n u_\lambda)^2 b_k^*\}_{\lambda\in D}$ is decreasing, so $\{F_k(\lambda)\}_{\lambda\in D}$ is a "decreasing net" of compact sets in $Q(A)$ (i.e., if $\lambda_1 \geqq \lambda_2$, then $F_k(\lambda_1) \subset F_k(\lambda_2)$. Since D is directed, $\{F_k(\lambda)\}_{\lambda\in D}$ have the finite intersection property. There are two cases.

CASE 1. $\bigcap_{\lambda\in D} F_k(\lambda) \neq \varnothing$. Let $f \in \bigcap_{\lambda\in D} F_k(\lambda)$. Thus

$$f(b_k(1-\delta_n u_\lambda)^2 b_k^*) \geqq (\alpha_k+\varepsilon_n)^2 \quad \text{for all } \lambda \in D .$$

Taking strong limits in A^{**} we get $u_\lambda \to q$, where q is the central cover of I^{**} in A^{**} (see section 1). Thus

$$f(b_k(1-\delta_n q)^2 b_k^*) \geqq (\alpha_k+\varepsilon_n)^2 .$$

Since $\|f\| \leqq 1$,

$$\|b_k(1-\delta_n q)^2 b_k^*\| \geqq (\alpha_k+\varepsilon_n)^2 \quad \text{or} \quad \|b_k(1-\delta_n q)\| \geqq \alpha_k+\varepsilon_n .$$

Now since q is central,

$$\begin{aligned}\|b_k(1-\delta_n q)\| &= \operatorname{Max}\{\|q(b_k(1-\delta_n q))\|, \|q'(b_k(1-\delta_n q))\|\} \\ &= \operatorname{Max}\{\|b_k q\|(1-\delta_n), \|q' b_k\|\} \geqq \alpha_k+\varepsilon_n\end{aligned}$$

As described in section 1, qA^{**} is exactly the kernel of π^{**}, so

$$\|q' b_k\| = \|\pi(b_k)\| = \left\|\pi\left(\prod_{1\leqq i\leqq k} x_i\right)\right\| = \alpha_k .$$

This means (since $\varepsilon_n \neq 0$) that $\|q' b_k\| \geqq \alpha_k+\varepsilon_n$ is impossible. We must instead have

$$\|b_k q\|(1-\delta_n) \geqq \alpha_k+\varepsilon_n .$$

But for $n=1$, $(1-\delta_n)=0$, so this leads out of Case 1. For $n>1$ more calculation is needed to reach a similar contradiction as follows. (Remember that by induction, $\|b_k\| < \alpha_k+\varepsilon_{n-1}$.)

$$\begin{aligned}\alpha_k+\varepsilon_n \leqq \|b_k q\|(1-\delta_n) \leqq \|b_k\|(1-\delta_n) &< (\alpha_k+\varepsilon_{n-1})(1-\delta_n) \\ &= \alpha_k+\varepsilon_{n-1}-\alpha_k\delta_n-\varepsilon_{n-1}\delta_n .\end{aligned}$$

Thus $\delta_n(\alpha_k+\varepsilon_{n-1})<(\varepsilon_{n-1}-\varepsilon_n)=\alpha\delta_n\leqq\alpha_k\delta_n$. This contradiction shows that Case 1 can never hold for any k. Hence we must always have

CASE 2. For each $m\leqq k\leqq l$, $\bigcap_{\lambda\in D}F_k(\lambda)=\emptyset$. Since (for each k) the sets $\{F_k(\lambda)\}_{\lambda\in D}$ form a decreasing net of compact subsets of $Q(A)$ as described above, there is for each k some $\lambda_k\in D$ such that $F_k(\lambda)=\emptyset$ for all $\lambda\geqq\lambda_k$. Let $\lambda_0\geqq\lambda_k$ for all $m\leqq k\leqq l$. Then for $\lambda\geqq\lambda_0$, $F_k(\lambda)=\emptyset$ for all $m\leqq k\leqq l$, so

$$f(b_k(1-\delta_nu_\lambda)^2b_k^*)\ <\ (\alpha_k+\varepsilon_n)^2\qquad\text{for all } f\in Q(A)\,.$$

Since $(b_k(1-\delta_nu_\lambda)^2b_k^*)\geqq 0$, the compactness of $Q(A)$ yields some $f\in Q(A)$ such that

$$\|b_k(1-\delta_nu_\lambda)^2b_k^*\|\ =\ f(b_k(1-\delta_nu_\lambda)b_k^*)\ <\ (\alpha_k+\varepsilon_n)^2\,.$$

Note that this was true for all $\lambda\geqq\lambda_0$ and $m\leqq k\leqq l$, so $\|b_k(1-\delta_nu_\lambda)\|<\alpha_k+\varepsilon_n$. Since $\|1-\delta_nu_\lambda\|\leqq 1$, we also have

$$\|b_k(1-\delta_nu_\lambda)^m\|\ <\ \alpha_k+\varepsilon_n\qquad\text{for all } \lambda\geqq\lambda^0 \text{ and } m\leqq k\leqq l\,.$$

Writing this out we get

$$\left\|\left[\prod_{1\leqq i\leqq m}x_ia_{n-1}\right]\left[\prod_{m<i\leqq k}x_i\right](1-\delta_nu_\lambda)^m\right\|\ <\ \alpha_k+\varepsilon_n$$

for all $\lambda\geqq\lambda_0$ and $m\leqq k\leqq l$. Since $\{u_\lambda\}_{\lambda\in D}$ is a quasi-central approximate unit for I relative to A, we have some $\lambda_n\in D$ with $\lambda_n\geqq\lambda_0$ such that if $\lambda\geqq\lambda_n$, then

$$\left\|\left[\prod_{1\leqq i\leqq m}(x_ia_{n-1}(1-\delta_nu_\lambda))\right]\left[\prod_{m<i\leqq k}x_i\right]\right\|\ <\ \alpha_k+\varepsilon_n\qquad\text{for all } m\leqq k\leqq l\,.$$

Thus if we use $u_{\lambda_n}=u_n$ then $a_n=a_{n-1}\,(1-\delta_nu_n)$ and the last inequality is exactly (**). This proves the induction step so the sequence $\{u_n\}_{n\geqq 1}\subset\{u_\lambda\}_{\lambda\in D}$ exists and satisfies (**) for all $m\leqq k\leqq l$.

Note that $0\leqq a_n\leqq 1$, so, if $c_n=1-a_n$, we get $c_n\in I$ and $0\leqq c_n\leqq 1$. Also

$$\|c_n-c_{n-1}\|\ =\ \|a_n-a_{n-1}\|\ =\ \|a_{n-1}\delta_nu_n\|\ \leqq\ \delta_n\,.$$

Since $\sum_{n\geqq 1}\delta_n<\infty$, we see that $c_n\to c\in I$ in norm and $0\leqq c\leqq 1$. Now for any $m\leqq k\leqq l$, if $d_i=x_i(1-c)$ for $1\leqq i\leqq m$,

$$\left\|\left[\prod_{1\leqq i\leqq m}d_i\right]\left[\prod_{m<i\leqq k}x_i\right]\right\|\ =\ \lim_{n\to\infty}\left\|\left[\prod_{1\leqq i\leqq m}x_i(1-c_n)\right]\left[\prod_{m<i\leqq k}x_i\right]\right\|$$
$$\leqq\ \lim_{n\to\infty}(\alpha_k+\varepsilon_n)\ =\ \alpha_k\,.$$

Since the reverse inequality follows from the definition of α_k, we have (*) and the theorem is proved.

Note that if b is a strictly positive element in I, then the element c in 3.7 can be chosen as a continuous function of b. Indeed, choosing the quasi-central approximate unit $\{u_\lambda\}$ in $C^*(b)$, it follows from the construction that $c \in C^*(b)$.

In the next theorem we show that if $x \in A$ and $x^n \notin I$, then we can get a simultaneous best approximation in I to x for all powers less than or equal to n. Further, if $\pi(x)$ is not quasi-nilpotent, we can get one best approximation in I to x for all powers of x.

THEOREM 3.8. *If $x \in A$ and m is a positive integer with $x^m \notin I$, then there exists an element $c \in I$ with $0 \leqq c \leqq 1$ and $\|(x(1-c))^k\| = \|\pi(x^k)\|$ for all $1 \leqq k \leqq m$. If the spectral radius of $\pi(x)$ is not zero, then c can be chosen independently of m.*

PROOF. The first assertion of the theorem is a direct corollary of Theorem 3.7, so we need only prove the second. Assume that $\beta > 0$ is the spectral radius of $\pi(x)$ and set $\beta_m = \|\pi(x^m)\|^{1/m}$ for all $m \geqq 1$. Assume that A is separable as at the start of Theorem 3.7. For simplicity we assume $\|x\| = \frac{1}{2}$. Define $\delta_n = 3^{-n+1}$ and $\varepsilon_n = \sum_{k \geqq n} \delta_{k+1}$ for $n = 1, 2, \ldots$. Set $\varepsilon_0 = \frac{1}{4} = \varepsilon_1/2$. Let $\{u_\lambda\}_{\lambda \in D} \subset I$ be an abelian quasi-central approximate unit for I relative to A (by Corollary 3.6 since A is assumed to be separable). We propose to choose a subsequence $\{u_n\}_{n \geqq 1} \subset \{u_\lambda\}_{\lambda \in D}$ such that if $a_n = \prod_{1 \leqq j \leqq n} (1 - \delta_j u_j)$, then for each $n = 1, 2, \ldots$ we have

$$(*) \qquad \|(xa_n)^n\|^{1/n} < \beta_m(1+\varepsilon_n) \quad \text{for all } m = 1, 2, \ldots .$$

The proof of the existence of $\{u_n\}_{n \geqq 1}$ will go by induction on n. To get started set $a_0 = 1$.

Let n be a natural number and suppose u_j has been defined to satisfy (*) for all natural numbers $j < n$. (For $n = 1$ we have defined no u_j at all.) For each $n = 1, 2, \ldots$ define $b_m = (xa_{n-1})^m$ and define for each $\lambda \in D$,

$$F_m(\lambda) = \{f \in Q(A) : f(b_m(1-\delta_n u_\lambda)^{2m} b_m^*) \geqq [\beta_m(1+\varepsilon_n - \varepsilon_{n-1}\delta_n)]^{2m}\} .$$

In a manner similar to that in the proof of Theorem 3.7 we can show that there is a $\lambda_m \in D$ with $F_m(\lambda_m) = \emptyset$ (The details are omitted.)

Since $\{\beta_n\}$ is a decreasing sequence with limit β, we may choose m_0 such that

$$\beta_{m0} < \beta(1 + \varepsilon_{n-1}\delta_n(1+\varepsilon_n)^{-1}) .$$

We can also choose an integer q_0 such that $q \geqq q_0$ gives

$$[\beta(1+\varepsilon_n - (\varepsilon_{n-1}\delta_n)^2(1+\varepsilon_n)^{-1})]^{q/1+q} \leqq \beta(1+\varepsilon_n) .$$

Now $F_m(\lambda_m)=\emptyset$ means that for all $\lambda \geqq \lambda_m$,

$$\|b_m(1-\delta_n u_\lambda)^m\| < [\beta_m(1+\varepsilon_n-\varepsilon_{n-1}\delta_n)]^m .$$

Since $\{u_\lambda\}_{\lambda\in D}$ form a abelian quasi-central approximate unit for I relative to A we can find $\bar{\lambda} \in D$ such that for $\lambda \geqq \bar{\lambda}$ and $m \leqq m_0 q_0$, we have

$$(**) \qquad \|(xa_n)^m\|^{1/m} < \beta_m(1+\varepsilon_n-\varepsilon_{n-1}\delta_n) .$$

Set $u_n=u_{\bar{\lambda}}$ and note that (*) is satisfied for all $m \leqq m_0 q_0$. Suppose $m > m_0 q_0$ so that $m=qm_0+r$ with q and r integers and $q \geqq q_0$, $0 \leqq r < m_0$. Now a sequence of calculations will verify (*) as follows.

$$\begin{aligned}
\|(xa_n)^m\|^{1/m} &= \|[(xa_n)^{m_0}]^q(xa_n)^r\|^{1/m} \leqq \|(xa_n)^{m_0}\|^{q/m} \\
&\leqq [\|(xa_n)^{m_0}\|^{1/m_0}]^{q/1+q} \leqq [\beta_m(1+\varepsilon-\varepsilon_{n-1}\delta_n)]^{q/1+q} \qquad \text{(by (**))} \\
&\leqq [\beta(1+\varepsilon_{n-1}\delta_n(1+\varepsilon_n)^{-1})(1+\varepsilon_n-\varepsilon_{n-1}\delta_n)]^{q/1+q} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by choice of } m_0) \\
&= [\beta(1+\varepsilon_n-(\varepsilon_{n-1}\delta_n)^2(1+\varepsilon_n)^{-1})]^{q/1+q} \qquad \text{(by algebra)} \\
&\leqq \beta(1+\varepsilon_n) \qquad \text{(by choice of } q \geqq q_0) \\
&\leqq \beta_m(1+\varepsilon_n) \qquad \text{(since } \{\beta_m\} \text{ is decreasing)} .
\end{aligned}$$

Thus (*) is verified.

As in 3.7 we write $c_n=1-a_n$ and note that $c=\lim_{n\to\infty} c_n$ exists such that $c \in I$, $0 \leqq c \leqq 1$. By (*) we have for each $m=1,2,\ldots$

$$\|[x(1-c)]^m\|^{1/m} = \lim_{n\to\infty} \|(xa_n)^m\|^{1/m} = \beta_m ,$$

as desired.

In [4] the problem of best approximation by nilpotent operators is solved for $B(H)$. The next theorem deals with general C*-algebras and gives an estimate of how well an operator can be approximated by nilpotents of a given fixed order.

THEOREM 3.9. *Define a sequence of positive real numbers* $\{\delta_n\}_{n\geqq 1}$ *by*

$$\delta_1 = 1 \quad \text{and} \quad \delta_{n+1} = (5n)^{(2^{-n})}\delta_n + n^{-\frac{1}{2}} \quad \text{for } n \geqq 1 .$$

If $x \in A$ with $\|x\| \leqq 1$ and n is a positive integer with $\|x^n\| \leqq \varepsilon$ for some $\varepsilon \leqq 1$, then there is some $y \in A$ with $y^n=0$ such that $\|x-y\| < \delta_n \varepsilon^\alpha$ where $\alpha=2^{1-n}$. Moreover, if L is a closed left ideal of A and $x \in L$, then we can choose $y \in L$.

PROOF. For $n=1$ take $y=0$ and it works trivially. Assume we have proved the theorem for some n and we shall establish it for $(n+1)$. Let $x \in L$ for some closed left ideal L of A with $\|x\| \leqq 1$ and $\|x^{n+1}\| < \varepsilon \leqq 1$. Let α, β, γ be positive numbers with $0 < \alpha < \beta < \gamma < 1$. Let f and g be piecewise-linear, non-decreasing functions from $\mathbf{R}$ to $\mathbf{R}$ such that

$$f(t) = \begin{cases} 0 & \text{for} \quad t \leqq \alpha \\ 1 & \text{for} \quad t \geqq \beta \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 0 & \text{for} \quad t \leqq \beta \\ t & \text{for} \quad t \geqq \gamma \end{cases}.$$

(While f and g obviously depend on α, β and γ we shall suppress this dependence to retain notational simplicity.) If we let $x = u|x|$ be the polar decomposition of x in A^{**} (see Lemma 2.1), we can put $e = f(|x|)$ and $x_1 = ug(|x|)$. Just as in the proof of Lemma 2.1 we see that e and x_1 are in L. Further they satisfy $\|x_1\| \leqq 1$, $\|e\| \leqq 1$, $\|x - x_1\| \leqq \gamma$ and $[x_1(1-e)] = 0$.

Since $\alpha f(t) \leqq t$, we have $\alpha^2 e^2 \leqq |x|^2$. Thus

$$\begin{aligned} \alpha^2 \|ex^n\|^2 &= \|\alpha ex^n\|^2 = \|(x^*)^n \alpha^2 e^2 x^n\| \leqq \|(x^*)^n |x|^2 x^n\| \\ &= \|\,|x| x^n\|^2 = \|x^{n+1}\|^2 < \varepsilon^2, \end{aligned}$$

thus $\|ex^n\| < \alpha^{-1}\varepsilon$ and moreover,

$$\|x^n - x_1^n\| \leqq \left\| \sum_{0 \leqq k \leqq n-1} [x^k (x - x_1) x_1^{n-k-1}] \right\| \leqq n\|x - x_1\| \leqq n\gamma .$$

Since $x_1 e = x_1$, $(ex_1)^n = ex_1^n$. Hence

$$\|(ex_1)^n\| = \|ex_1^n\| \leqq n\gamma + \|ex^n\| < n\gamma + \alpha^{-1}\varepsilon .$$

We may now choose and fix α, β and γ to be smaller than $\varepsilon^{\frac{1}{2}} n^{-\frac{1}{2}}$ but close enough to it to insure that

$$(n\gamma + \alpha^{-1}\varepsilon) < (5n\varepsilon)^{\frac{1}{2}} .$$

(This is clear since $(n\gamma + \alpha^{-1}\gamma) \to (4n\varepsilon)^{\frac{1}{2}}$ as $\alpha, \gamma \to \varepsilon^{\frac{1}{2}} n^{-\frac{1}{2}}$.) Let

$$L_0 = \{ y \in A \,:\, y(1-e) = 0 \} .$$

Clearly L_0 is a closed left ideal in A and so is $L_1 = L \cap L_0$. Further $ex_1(1-e) = 0$, so $ex_1 \in L_1$. By the induction hypothesis, since

$$\|(ex_1)^n\| < n\gamma + \alpha^{-1}\varepsilon < (5n\varepsilon)^{\frac{1}{2}},$$

there is a $y_1 \in L_1$ with $y_1^n = 0$ and

$$\|ex_1 - y_1\| < \delta_n [(5n\varepsilon)^{\frac{1}{2}}]^{(2^{1-n})} = \delta_n (5n\varepsilon)^{(2^{-n})} .$$

Put $y = (1-e)x_1 + y_1$. Then $y \in L$ and, since $y_1(1-e) = 0$,

$$y^{n+1} = (1-e)x_1 y_1^n + y_1^{n+1} = 0 .$$

Finally,

$$\begin{aligned}\|x-y\| &= \|x-(1-e)x_1-y_1\| = \|(x-x_1)+(ex_1-y_1)\| \\ &\leqq \|x-x_1\|+\|ex_1-y_1\| < \||x|-g(|x|)\|+\delta_n(5n\varepsilon)^{(2^{-n})} \\ &\leqq \varepsilon^{\frac{1}{2}}n^{-\frac{1}{2}}+\delta_n(5n\varepsilon)^{(2^{-n})} \leqq \delta_{n+1}\varepsilon^{(2^{-n})}.\end{aligned}$$

This completes the proof for $(n+1)$, hence the theorem follows by induction.

Recall that one of our (unanswered) question is whether if $x \in A$ with $x^n \in I$, then there is an $a \in I$ with $(x-a)^n=0$. The next corollary answers an approximate version of this question.

COROLLARY 3.10. *If $x^n \in I$ and $\varepsilon>0$, there exist $a \in I$ and $y \in A$ such that*

$$y^n=0, \qquad \|x-a\|. \leqq \|x\| \qquad \text{and} \qquad \|y-(x-a)\| < \varepsilon .$$

PROOF. Let $\{u_\lambda\}_{\lambda\in D}$ be an abelian quasi-central approximate unit for $I \cap C^*(x)$ relative to $C^*(x)$ and choose $\varepsilon_1>0$ such that

$$\varepsilon_1 \leqq (\varepsilon\delta_n^{-1})^{(2^{n-1})}, \qquad \text{with } \delta_n \text{ as in 3.9 .}$$

Now

$$\lim_{\lambda\to\infty} \|(x(1-u_\lambda))^n\| = \lim_{\lambda\to\infty} \|x^n(1-u_\lambda)^n\| = 0 ,$$

(since $\{u_\lambda\}_{\lambda\in D}$ is an abelian quasi-central approximate unit), so we can take $a=xu_\lambda$ to get (eventually) $\|(x-a)^n\|<\varepsilon_1$. By Theorem 3.9 we may choose the desired $y \in A$.

4. Results for von Neumann algebras.

Many of the results in this section are generalizations of theorems of [13, 14 and 15]. In all cases the results are valid for some C*-algebras A which are not von Neumann algebras, but the crucial element is always the existence of projections. Thus A could be an AW*-algebra [9] and often even less [10]. To avoid laborious formulations we shall make the following assumption for this section.

ASSUMPTION FOR SECTION 4. A is a von Neumann algebra acting on a separable Hilbert space H with I weakly dense in A.

Our first result is the generalization of Theorem 2.3 of [14] found in [25]. The follow-up is a generalization of Theorem 6 of [13] with a shorter proof.

PROPOSITION 4.1. *If $x, y \in A$ with $xy \in I$, then there exists a projection $e \in A$ such that $xe', ey \in I$.*

PROOF. By Proposition 2.3 there exist $a, b \in I$ with $(x-a)(y-b)=0$. If $e' = [y-b]$, then $e(y-b)=ey-eb=0$, so $ey=eb \in I$. Also $(x-a)e'=0$, so $xe' = ae' \in I$.

THEOREM 4.2. *If $\{x_1,\ldots,x_n\} \subset A$ with $\prod_{1\leqq i\leqq n} x_i \in I$, then there exist $\{a_1,\ldots,a_n\} \subset I$ such that $\prod_{1\leqq i\leqq n}(x_i-a_i)=0$.*

PROOF. By Proposition 2.3 the theorem holds for $n=2$. Suppose by induction it holds for $(n-1)$ factors. By Proposition 4.1 there is a projection $e \in A$ such that $(\prod_{1\leqq i<n} x_i)e' \in I$ and $ex_n \in I$. By the induction hypothesis there exist $\{a_1,\ldots,a_{n-2}, b\} \subset I$ such that

$$\left(\prod_{1\leqq i\leqq n-2}(x_i-a_i)\right)(x_{n-1}e'-b) = 0 .$$

Set $a_{n-1}=be'$ and $a_n=ex_n$, so $(x_n-a_n)=e'x_n$. Then

$$\prod_{1\leqq i\leqq n}(x_i-a_i) = \left(\prod_{1\leqq i<n-2}(x_i-a_i)\right)(x_{n-1}-a_{n-1})(e'x_n)$$

$$= \left(\prod_{1\leqq i<n-2}(x_i-a_i)\right)(x_{n-1}e'-b)(e'x_n) = 0 .$$

Using Proposition 4.1 and mimicking the proof of Theorem 2.4 of [14] one can get the following theorem. See also [10].

THEOREM 4.3. *Let f be a complex polynomial. If $x \in A$ with $f(x) \in I$, then $f(x-a)=0$ for some $a \in I$.*

It is reasonable to conjecture that for any $x \in A$ there exists $a \in I$ such that $\|f(x-a)\| = \|\pi(f(x))\|$ for every complex polynomial f. Many special cases have been proved with restriction being put on the element x under consideration [14, 15, 24]. The next result, which contains Proposition 2.1 of [24], Theorem 1 of [10] and Theorem 12 of [13], puts restrictions on A and I but not (directly) on x. This point of view reduces to some topological manipulations of compact, totally disconnected spaces, and the algebraic content of the theorem vanishes entirely.

THEOREM 4.4. *Let Γ be a compact, totally disconnected topological space, X a Banach space, $\Gamma_1 \subset \Gamma$ a closed subset. Let A be the Banach space of all* *continuous functions from Γ to X with sup norm and let I be the closed subspace of A consisting of all functions in A which vanish on Γ_1. For any $b \in A$ there exists $a \in I$ such that $(b+a)(\Gamma)=b(\Gamma_1)$.*

Before proving the theorem note that if X is a C*-algebra, then so is A and I becomes a closed two-sided ideal. Zsidó [24, Proposition 2.1] essentially does the case $X=\mathbf{C}$, while for Olsen [13, Theorem 12] X is a finite dimensional matrix algebra and Γ is the Stone-Čech compactification of the integers.

PROOF OF THEOREM 4.4. We first need a procedure for approximating our goal; then we shall use this procedure successively to obtain the result. For any $\varepsilon>0$, $b_1 \in A$ and $\Gamma_0 \subset \Gamma$ an open-and-closed subset such that

$$\sup\{\operatorname{dist}(b_1(\gamma), b(\Gamma_1)) \; : \; \gamma \in \Gamma_0\} < \varepsilon$$

we shall find $c \in A$ such that $\|c\|<\varepsilon$, $c|_{\Gamma\setminus\Gamma_0}=0$ and

$$\sup\{\operatorname{dist}((b_1+c)(\gamma), b(\Gamma_1)) \; : \; \gamma \in \Gamma_0\} < \varepsilon/2 .$$

To do this, find an open-and-closed neighborhood U_γ of each γ in Γ_0 such that if $\gamma_1 \in U_\gamma$, then

$$\|b_1(\gamma_1)-b_1(\gamma)\| < \varepsilon/4 .$$

By the compactness of Γ_0 we may cover it with a finite number of $\{U_\gamma\}$, say $U_{\gamma_1},\ldots,U_{\gamma_n}$. Set

$$V_1 = U_{\gamma_1} \quad \text{and} \quad V_j = U_{\gamma_j}\setminus\bigcup_{i<j} U_{\gamma_i} \quad \text{for } 2\leqq j\leqq n .$$

Then $\{V_1,\ldots,V_n\}$ are disjoint open-and-closed sets and for any $\gamma,\gamma' \in V_i$,

$$\|b_1(\gamma)-b_1(\gamma')\| < \varepsilon/2 .$$

Let $\omega_i \in V_i$ for $1\leqq i\leqq n$ and choose by hypothesis $x_1,\ldots,x_n$ in $b(\Gamma_1)$ such that

$$\|b_1(\omega_i)-x_i\| < \varepsilon \quad \text{for all } 1\leqq i\leqq n .$$

Define $c \in A$ by $c(\gamma)=0$ if $\gamma \notin \Gamma_0$, $c(\gamma)=x_i-b_1(\omega_i)$ if $\gamma \in V_i$ for some $1\leqq i\leqq n$. Since the $\{V_i\}_{1\leqq i\leqq n}$ are disjoint and cover Γ_0, this uniquely defines a continuous function c. Clearly $\|c\|<\varepsilon$ and for any $\gamma \in V_i$,

$$\begin{aligned}\operatorname{dist}((b_1+c)(\gamma), b(\Gamma_1)) &\leqq \|b_1(\gamma)+x_i-b_1(\omega_i)-x_i\| \\ &= \|b_1(\gamma)-b_1(\omega_i)\| < \varepsilon/2 ,\end{aligned}$$

Since the last inequality is independent of $i=1,\ldots,n$ and $\gamma \in V_i$, we get the conclusion of the first paragraph of the proof.

Now suppose $\Gamma_0 \subset \Gamma$ is an open-and-closed subset and that $\varepsilon > 0$ with

$$\sup\{\operatorname{dist}(b(\gamma), b(\Gamma_1)) : \gamma \in \Gamma_0\} < \varepsilon .$$

We shall find $d \in A$ with $d(\gamma)=0$ outside Γ_0, $\|d\| < 2\varepsilon$ and $(b+d)(\Gamma_0) \subset b(\Gamma_1)$. We do this by induction, using the estimation in the first paragraph. Define $d_1 \in A$ so that $\|d_1\| < \varepsilon$ and

$$\sup\{\operatorname{dist}((b+d_1)(\gamma), b(\Gamma_1)) : \gamma \in \Gamma_0\} < \varepsilon 2^{-1} .$$

Suppose we have defined $d_1, \ldots, d_n$ so that $\|d_k\| < \varepsilon 2^{-(k-1)}$ and

$$\sup\left\{\operatorname{dist}\left(\left(b + \sum_{1 \leqq k \leqq n} d_k\right)(\gamma), b(\Gamma_1)\right) : \gamma \in \Gamma_0\right\} < \varepsilon 2^{-n} .$$

Apply the first paragraph to the element $(b + \sum_{1 \leqq k \leqq n} d_k)$ (instead of b_1) to get d_{n+1} with $\|d_{n+1}\| < \varepsilon 2^{-n}$ and

$$\sup\left\{\operatorname{dist}\left(\left(b + \sum_{1 \leqq k \leqq n+1} d_k\right)(\gamma), b(\Gamma_1)\right) : \gamma \in \Gamma_0\right\} < \varepsilon 2^{-n-1} .$$

Set $d = \sum_{n \geqq 1} d_n$. Clearly $d \in A$, $\|d\| < 2\varepsilon$. Further, if $\gamma \in \Gamma_0$ and

$$\operatorname{dist}((b+d)(\gamma), b(\Gamma_1)) = \delta > 0 ,$$

then we shall reach a contradiction as follows. By the construction there exist $\{x_n\}_{n \geqq 1} \subset b(\Gamma_1)$ such that

$$\left\|\left(b + \sum_{1 \leqq i \leqq n} d_i\right)(\gamma) - x_n\right\| < \varepsilon 2^{-n} .$$

Thus

$$\|(b+d)(\gamma) - x_n\| \leqq \left\|\left(\sum_{i > n} d_i\right)(\gamma)\right\| + \varepsilon 2^{-n} \xrightarrow[n \to \infty]{} 0 .$$

Now we complete the proof of the theorem. Let

$$E_n = \{x \in X : 2^{-n} \leqq \operatorname{dist}(x, b(\Gamma_1))\} \quad \text{for all } n \geqq 1$$

and let V_1 be an open-and-closed subset of Γ, disjoint from Γ_1, such that $V_1 \supset b^{-1}(E_1)$. Suppose we have defined open-and-closed sets $V_1, \ldots, V_n$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{1 \leqq i \leqq n} V_i \supset b^{-1}(E_n)$. Let

$$U = b^{-1}(E_{n+1}) \setminus \bigcup_{1 \leqq i \leqq n} V_i .$$

Then U and Γ_1 are disjoint closed subsets of the totally disconnected, compact, open subset $[\Gamma \setminus \bigcup_{1 \leqq i \leqq n} V_i]$ of Γ, so V_{n+1} can be an open-and-closed subset of $[\Gamma \setminus \bigcup_{1 \leqq i \leqq n} V_i]$ containing U and disjoint from Γ_1. This completes the induction. By the previous part of the proof we can define $a_n \in I$ with $(b+a_n)(V_n) \subset b(\Gamma_1)$ and, for $n>1$, $\|a_n\| \leqq 2^{-n+1}$. (This last follows since $V_n \cap b^{-1}(E_{n-1}) = \emptyset$ for $n>1$.) Set $a = \sum_{n\geqq 1} a_n$, so $a \in I$ and

$$(b+a)(V_n) = (b+a_n)(V_n) \subset b(\Gamma_1)\,.$$

Since $\bigcup_{n\geqq 1} V_n \supset \bigcup_{n\geqq 1} E_n$, $(b+a)(\Gamma) \subset b(\Gamma_1)$.

We conclude this paper with a few observations on the paper [24] of Zsidó. First it is worth noting that the II_∞-factor case is noticeably different from the $B(H)$ case in one important aspect. The following proof was suggested by George Elliott.

Proposition 4.5. *If A is a type II_∞ factor and I is the closure of the set of elements of finite trace, then I contains no strictly positive element.*

Proof. Let τ be the trace on M and assume that h is strictly positive in I. Since $h \in I$, $0 \in \sigma(h)$ (since I has no unit). Further, 0 cannot be an isolated point of $\sigma(h)$, for, if q is the spectral projection of h corresponding to $\sigma(h) - \{0\}$, then $q'I = 0$. (This follows since h is strictly positive and $q'h = 0$.) Since I is weakly dense in $A, q'I = 0$ implies $q' = 0$ or $q = 1$. Thus 0 is not isolated in $\sigma(h)$. Therefore we can choose $\{t_n\}_{n\geqq 1} \subset (0,1)$ such that $t_1 > t_2 > \ldots > t_n \to 0$ and the spectral projection p_n of h corresponding to the interval $(t_{n+1}, t_n]$ is non-zero. Choose for each n a projection $e_n \neq 0$ in I with $e_n \leqq p_n$ and $\tau(e_n) \leqq 2^{-n}$. Set $e = \sum_{n\geqq 1} e_n$, so $e \in I$ since $\tau(e) \leqq 1$.

Normalizing h so that $\|h\| = 1$, we have that the sequence $\{h^{1/n}\}$ is an approximate unit for $C^*(h)$ and hence (as in the proof of Corollary 3.3) an approximate unit for I. However, for any $\varepsilon > 0$ and fixed n, there is a k with $\|h^{1/n} p_k\| < \varepsilon$. Then

$$\begin{aligned}\|e(1-h^{1/n})e\| &\geqq \|e_k(1-h^{1/n})e_k\| = \|e_k p_k (1-h^{1/n}) p_k e_k\| \\ &\geqq \|e_k - e_k p_k h^{1/n} p_k e_k\| \geqq \|e_k\| - \|h^{1/n} p_k\| > 1-\varepsilon\,.\end{aligned}$$

Thus $\{h^{1/n}\}$ is not an approximate unit for e, a contradiction.

Inspection of the proof of Corollary 1.3 of [24] reveals that the author could as well have used the norm estimate of his Proposition 1.1 and proved the following.

Proposition 4.6 (Zsidó). *If $a \in A_{sa}$ and $\varepsilon > 0$, there is an orthogonal sequence of projections $\{p_n\} \subset I$ such that $\sum_{n\geqq 1} p_n = 1$, $(a - \sum_{n\geqq 1} p_n a p_n) \in I$ and $\|a - \sum_{n\geqq 1} p_n a p_n\| < \varepsilon$.*

In order to have a notion of "diagonal algebra" which is valid for any A, we note that a maximal abelian C*-subalgebras of $C(H)$ is exactly the algebra of all operators diagonal with respect to a fixed basis. Therefore the maximal abelian C*-subalgebras of I are the correct algebras to use in generalizing Weyl's theorem to arbitrary A. For the record we note an easy fact.

PROPOSITION 4.7. *If C is a maximal abelian C*-subalgebra of I, then C'' is a maximal abelian C*-subalgebra of A.*

PROOF. Let $\{b_\alpha\} \subset C$ be a positive increasing approximate unit for C and let $b_\alpha \to p$ strongly in A. Then p is a projection. Further $p'Ip'$ is a C*-algebra which is contained in the annihilator of C, since p being a unit for C means $p'C = Cp' = \{0\}$. The maximality of C implies that $p'Ip' = \{0\}$. Since I is weakly dense in A, $p'Ap' = 0$, i.e., $p = 1$. Suppose that $a \in A \cap C'$, then $b_\alpha a \to a$ strongly and $\{b_\alpha a\} \subset I \cap C' = C$, so a lies in the strong closure of C which is C''.

Now we can state our generalization of Weyl-von Neumann's theorem [6, 11, 23]. (See also [5, 22, 24].)

THEOREM 4.8. *For any normal operator $y \in A$ and $\varepsilon > 0$ there is some $c \in A$ with*

$$(y-c) \in I, \qquad \|y-c\| < \varepsilon \qquad \text{and} \qquad c = \sum_{n \geq 1} \alpha_n e_n ,$$

where $\{e_n\} \subset I$ are orthogonal projections and $\{\alpha_n\} \subset \mathbf{C}$. Further we may take $\{e_n\}$ to lie in maximal abelian C-subalgebra of I.*

PROOF. As in the proof of Theorem 3.1 of [24] we may consider $C^*(y) \subset C^*(x)$ for $x \in A_{\text{sa}}$. Suppose the theorem were true for self-adjoint elements x and (in this case) c could also be chosen self-adjoint. Then we could find $\{c_n\} \subset A$ by the theorem with $(x-c_n) \in I$, $\|x-c_n\| < n^{-1}$, $c_n = \sum_{m \geq 1} \alpha_{nm} e_{nm}$ etc. Let $y = f(x)$, where f is a continuous complex-valued function on $\sigma(x)$. Let $\{p_k\}$ be polynomials without constant term such that $p_k \to f$ uniformly on $\sigma(x)$. For each p_k we have $p_k(c_n) \to p_k(x)$, so $f(c_n) \to f(x)$ by uniform convergence. Further

$$f(c_n) = \lim_{k \to \infty} p_k(c_n) ,$$

so $f(c_n) = \sum_{m \geq 1} f(\alpha_{nm}) e_{nm}$. Thus for sufficiently large j we could put $c = f(c_j)$, $e_n = e_{jn}$, $\alpha_n = f(\alpha_{jn})$ and the theorem would follow for the normal operator y. We are thus reduced to proving the theorem for the case $y \in A_{\text{sa}}$ under the additional restriction that c must be in A_{sa}.

By Proposition 4.6 we can choose $\{p_n\} \subset I$ with $(a - \sum_{n\geq 1} p_n a p_n) \subset I$ and $\|y - \sum_{n\geq 1} p_n y p_n\| < \varepsilon/2$. By the spectral theorem we may choose spectral projections $\{q_{nm}\}_{1\leq m\leq k_n}$ of each $p_n y p_n$ and real scalars $\{\beta_{nm}\}_{1\leq m\leq k_n}$ such that

$$\left\| p_n y p_n - \sum_{1\leq m\leq k_n} \beta_{nm} q_{nm} \right\| < \varepsilon 2^{-n-2}.$$

Clearly $q_{nm} q_{ij} = \delta_{ni} \delta_{mj} q_{nm}$, and we may arrange $\{q_{nm}\}$ and $\{\beta_{nm}\}$ into sequences $\{e_n\}_{n\geq 1}$ and $\{\alpha_n\}_{n\geq 1}$ so that

$$\sum_{n\geq 1} \sum_{1\leq m\leq k_n} \beta_{nm} q_{nm} = \sum_{n\geq 1} \alpha_n e_n .$$

Let $c = \sum_{n\geq 1} \alpha_n e_n$. Now

$$(*) \qquad (y-c) = \left(y - \sum_{n\geq 1} p_n y p_n \right) + \left(\sum_{n\geq 1} \left(p_n y p_n - \sum_{1\leq m\leq k^n} \beta_{nm} q_{nm} \right) \right).$$

Since the first term on the right of (*) and each term in the absolutely convergent series of the second term on the right of (*) all lie in I, $(y-c) \in I$. Further (*) shows that $\|y-c\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, and the theorem follows.

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